Practice Exam 2 Solutions

1.) $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx = \int_{-\infty}^{a} x^3 e^{-x^4}\,dx + \int_a^{\infty} x^3 e^{-x^4}\,dx$ for some $a \in (-\infty, \infty)$

1^{st} $\int_a^{\infty} x^3 e^{-x^4}\,dx = \lim_{t\to\infty} \int_a^t x^3 e^{-x^4}\,dx$ $\quad u = x^4 \quad du = 4x^3dx \quad x = t \Rightarrow u = t^4 \quad x = a \Rightarrow u = a^4$

$\Rightarrow \int = \frac{1}{4}\lim_{t\to\infty} \int_{a^4}^{t^4} e^{-u}\,du = -\frac{1}{4}\lim_{t\to\infty} e^{-u}\Big|_{a^4}^{t^4} = \frac{1}{4}\lim_{t\to\infty}\left(e^{-a^4} - e^{-t^4}\right) = \frac{1}{4}e^{-a^4}$ converges

2^{nd} $\int_{-\infty}^{a} x^3 e^{-x^4}\,dx = \lim_{t\to-\infty} \int_t^a x^3 e^{-x^4}\,dx$ $\quad u = x^4 \quad du = 4x^3dx \quad x = t \Rightarrow u = t^4 \quad x = a \Rightarrow u = a^4$

$\Rightarrow \int = \frac{1}{4}\lim_{t\to-\infty} \int_{t^4}^{a^4} e^{-u}\,du = \frac{1}{4}\lim_{t\to-\infty}\left(e^{-t^4} - e^{-a^4}\right) = -\frac{1}{4}e^{-a^4}$ converges

So $\int_{-\infty}^{\infty} = \frac{1}{4}\left(e^{-a^4} - e^{-a^4}\right) = 0$

2.) $\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+3}\,dx$ $\quad u = e^x \quad du = e^x dx \quad x = t \Rightarrow u = e^t \quad x = 0 \Rightarrow u = 1$

So $\int = \lim_{t\to\infty}\int_1^{e^t} \frac{1}{u^2+3}\,du = \lim_{t\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big|_1^{e^t} = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{e^t}{\sqrt{3}}\right) - \frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

3.) $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$ $\quad$ can see $x - x^2 \geq 0$ for $0 \leq x \leq 1 \Leftarrow$ limits

then $y' = \frac{1-2x}{\sqrt{x-x^2}} + \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} = \frac{1-2x}{\sqrt{x-x^2}} + \frac{1}{2\sqrt{x-x^2}} = \frac{2-4x+1}{2\sqrt{x-x^2}} = \frac{3-4x}{2\sqrt{x-x^2}}$

so $\sqrt{1+(y')^2}\,dx = \sqrt{1 + \frac{(3-4x)^2}{4(x-x^2)}}\,dx = \sqrt{1 + \frac{9-24x+16x^2}{4(x-x^2)}}\,dx$

$= \sqrt{\frac{4x-4x^2+9-24x+16x^2}{4(x-x^2)}}\,dx = \sqrt{\frac{9-20x+12x^2}{4(x-x^2)}}\,dx$

then $L = \int_0^1 \sqrt{\frac{9-20x+12x^2}{4(x-x^2)}}\,dx$

4.) $y = 3 + \frac{1}{2}\cosh(2x)$ on $0 \le x \le 1$

then $y' = \sinh(2x)$ so $\sqrt{1+(y')^2}\,dx = \sqrt{1+\sinh^2(2x)}\,dx = \sqrt{\cosh^2(2x)}\,dx = \cosh(2x)\,dx$

then $L = \int_0^1 \cosh(2x)\,dx$

5.) $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$ the sum is telescoping

so $s_n = \sum_{j=1}^{n} \left(e^{\frac{1}{j}} - e^{\frac{1}{j+1}}\right) = e - e^{\frac{1}{2}} + e^{\frac{1}{2}} - e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$

$= e - e^{\frac{1}{n+1}}$

then $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(e - e^{\frac{1}{n+1}}\right) = e - e^0 = e - 1$

6.) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ consider $\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$

so $2 = A(n+1) + B(n-1)$

$n=1 \Rightarrow 2 = 2A \Rightarrow A = 1$

$n=-1 \Rightarrow 2 = -2B \Rightarrow B = -1$

so $\sum_{n=2}^{\infty} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$ the sum is telescoping

So $s_n = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right) = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n+1}$

$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$

then $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = 1 + \frac{1}{2} = \frac{3}{2}$

7.) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, let $f(x) = \frac{1}{x(\ln x)^2}$ then $f(x) > 0$ on $[2, \infty)$

and $f'(x) = -\frac{((\ln x)^2 + 2\ln x)}{(x(\ln x)^2)^2} > 0$ on $[2, \infty)$ so $f \searrow$

and $\lim_{x\to\infty} f(x) = 0$ so $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2} dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

so $\int = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du = -\lim_{t\to\infty} \frac{1}{u}\Big|_{\ln 2}^{\ln t} = \frac{1}{\ln 2} < \infty$ so series converges

by integral test

8.) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$, let $f(x) = \frac{\ln x}{x^3}$ then $f(x) \geq 0$ on $[1, \infty)$

$\lim_{x\to\infty} \frac{\ln x}{x^3} = 0$ by L'Hôpital, and $f'(x) = \frac{1 - 3\ln x}{x^4} \leq 0$ on $[1, \infty)$ so $f \searrow$

then $\int_1^{\infty} \frac{\ln x}{x^3} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3} dx$ $\quad u = \ln x$, $x = e^u$, $du = \frac{1}{x}dx$

so $\int = \lim_{t\to\infty} \int_0^{\ln t} u e^{-2u} du$ $\quad v = u$, $dv = du$, $dw = e^{-2u} du$, $w = -\frac{1}{2}e^{-2u}$

$\Rightarrow \int = \lim_{t\to\infty} \left(-\frac{1}{2}ue^{-2u}\Big|_0^{\ln t} + \frac{1}{2}\int_0^{\ln t} e^{-2u} du\right) = \lim_{t\to\infty} \left(-\frac{1}{2}\ln t\, e^{-2\ln t} - \frac{1}{4}e^{-2u}\Big|_0^{\ln t}\right)$

$= \lim_{t\to\infty} \left(-\frac{1}{2}\ln t\, e^{-2\ln t} - \frac{1}{4}e^{-2\ln t} + \frac{1}{4}\right) = \frac{1}{4}$ so series converges by integral-test

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9.) $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ know $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} (\frac{1}{e})^n$ converges as it's a geometric

series w/ $r = \frac{1}{e} < 1$ then consider $\lim_{n\to\infty} \frac{(1+\frac{1}{n})^2 e^{-n}}{e^{-n}} = \lim_{n\to\infty} (1+\frac{1}{n})^2 = 1 < \infty$

so by limit comparison test $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ also converges

10.) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as it's the harmonic series

consider $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^{1+\frac{1}{n}}}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}\cdot\frac{1}{n^{\frac{1}{n}}}} = \lim_{n\to\infty} n^{\frac{1}{n}}$

study $\lim_{x\to\infty} x^{\frac{1}{x}}$ ∞^0-type, let $y = x^{\frac{1}{x}}$ then $\ln y = \frac{1}{x}\ln x$

then $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ by l'hopital $\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = e$ so $\lim_{n\to\infty} n^{\frac{1}{n}} = e < \infty$

so $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges by limit comparison test

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11.) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$, this series is alternating but $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = 1 \neq 0$

therefore the series diverges by A.S.T.

12.) $\sum_{n=1}^{\infty} \cos(n\pi)\sin\left(\frac{\pi}{n}\right) = \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$, this series is alternating

and $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = 0$ and $\sin\left(\frac{\pi}{n}\right) < \sin\left(\frac{\pi}{n+1}\right)$ so ~~the~~ the terms

are decreasing. Thus the series converges by A.S.T.

13.) $\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)\,dx$ $\quad u = f(x)$ $\quad dv = \cos(nx)\,dx$

$du = f'(x)\,dx$ $\quad v = \frac{1}{n}\sin(nx)$

so $L = \lim_{n\to\infty} \left(\frac{1}{n} f(x)\sin(nx) - \frac{1}{n}\int_0^1 f'(x)\sin(nx)\,dx \right)$

since f is cont. on $[0,1]$ it is bnded, i.e. $|f(x)| < M_1$ for some M_1,

similarly $|f'(x)| < M_2$ and $|\sin(nx)| < 1$

so $-\left(\underbrace{\lim_{n\to\infty}\left(\frac{1}{n}M_1 - \frac{1}{n}M_2\right)}_{=0}\right) < L < \lim_{n\to\infty}\left(\frac{1}{n}M_1 - \underbrace{\frac{1}{n}M_2}_{=0}\right)$

so by Squeeze Thm the limit is 0.