Practice Exam 1 Solutions

1.) $f(x) = 3 + x^2 + \tan\left(\frac{\pi x}{2}\right)$ then $(f^{-1})(3) = 0$ since $f(0) = 3$

then $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ so $f'(x) = 2x + \frac{\pi}{2}\sec^2\left(\frac{\pi x}{2}\right)$

so $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(0)} = \dfrac{1}{2(0) + \frac{\pi}{2}\sec^2(0)} = \dfrac{2}{\pi}$

2.) $f(x) = \sqrt{x^5 + x^2 + x + 1}$ then $(f^{-1})(2) = 1$ since $f(1) = \sqrt{1+1+1+1} = \sqrt{4} = 2$

then $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ so $f'(x) = \dfrac{5x^4 + 2x + 1}{\sqrt{x^5 + x^2 + x + 1}}$

so $(f^{-1})'(2) = \dfrac{1}{f'(1)} = \dfrac{\sqrt{1+1+1+1}}{5+2+1} = \dfrac{2}{8} = \dfrac{1}{4}$

3.) $y = \dfrac{2^x \sin x}{x^3 - x + 4}$ so $\ln y = \ln\left(\dfrac{2^x \sin x}{x^3 - x + 4}\right) = \ln(2^x \sin x) - \ln(x^3 - x + 4)$

$= x\ln 2 + \ln(\sin x) - \ln(x^3 - x + 4)$

then $\dfrac{y'}{y} = \ln 2 + \dfrac{\cos x}{\sin x} - \dfrac{3x^2 - 1}{x^3 - x + 4}$ so $y' = \dfrac{2^x \sin x}{x^3 - x + 4}\left(\ln 2 + \dfrac{\cos x}{\sin x} - \dfrac{3x^2 - 1}{x^3 - x + 4}\right)$

4.) $y = \sqrt{\dfrac{\cos x}{x^4 + 1}}$ then $\ln y = \ln\sqrt{\dfrac{\cos x}{x^4 + 1}} = \dfrac{1}{2}\ln\left(\dfrac{\cos x}{x^4 + 1}\right)$

$= \dfrac{1}{2}\left(\ln(\cos x) - \ln(x^4 + 1)\right)$

then $\dfrac{y'}{y} = \dfrac{1}{2}\left(-\dfrac{\sin x}{\cos x} - \dfrac{4x^3}{x^4 + 1}\right)$ so $y' = \dfrac{1}{2}\sqrt{\dfrac{\cos x}{x^4 + 1}}\left(-\dfrac{\sin x}{\cos x} - \dfrac{4x^3}{x^4 + 1}\right)$

5.) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ is $\frac{0}{0}$ type so l'hôpital gives

$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}$ still $\frac{0}{0}$ type l'hôpital again

hence $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

6.) $\lim_{x\to 0^+} \sin x \ln x$ is $0\cdot\infty$ type, write $\sin x \ln x = \frac{\ln x}{\csc x}$ now have $\frac{\infty}{\infty}$ type

use l'hôpital $\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x\to 0^+} \underbrace{\frac{-1}{x}\sin x}_{\to 1 \text{ as } x\to 0^+} \underbrace{\tan x}_{\to 0} = 0$

7.) $\int e^{\cos x}\sin(2x)\,dx = \int 2e^{\cos x}\sin x\cos x\,dx$ $\quad w = \cos x$, $dw = -\sin x\,dx$

$\int = -2\int e^w w\,dw$ $\quad u = w$, $du = dw$, $dv = e^w\,dw$, $v = e^w$

so $\int = -2\left(we^w - \int e^w\,dw\right) = -2\left(we^w - e^w\right) + C = 2e^{\cos x} - \cos x\, e^{\cos x} + C$

8.) $\int x^3\sin(x^2)\,dx = \int x\cdot x^2\sin(x^2)\,dx$ $\quad w = x^2$ $\quad dw = 2x\,dx$

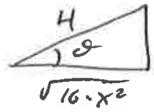
$\int = \frac{1}{2}\int w\sin w\,dw$ $\quad u = w$, $du = dw$, $dv = \sin w\,dw$, $v = -\cos w$

so $\int = \frac{1}{2}\left(-w\cos w + \int \cos w\,dw\right) = \frac{1}{2}\left(-w\cos w + \sin w\right) + C$

$= \frac{1}{2}\left(\sin(x^2) - x^2\cos(x^2)\right) + C$

9.) $\int \frac{dx}{x^2\sqrt{16-x^2}}$ pick $x = 4\sin\theta$ so $dx = 4\cos\theta\, d\theta$ & $16-x^2 = 16-16\sin^2\theta$
$= 16\cos^2\theta$

& $\frac{x}{4} = \sin\theta$

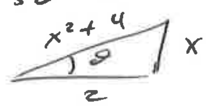


so $\int = \int \frac{4\cos\theta\, d\theta}{16\sin^2\theta\sqrt{16\cos^2\theta}} = \frac{1}{16}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{16}\int \csc^2\theta\, d\theta = -\frac{1}{16}\cot\theta + C$

$= -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + C$

10.) $\int \frac{x^5}{\sqrt{x^2+4}}\,dx$ pick $x = 2\tan\theta$ so $dx = 2\sec^2\theta\, d\theta$ & $x^2+4 = 4\tan^2\theta + 4$
$= 4\sec^2\theta$

& $\frac{x}{2} = \tan\theta$



So $\int = \int \frac{32\tan^5\theta \cdot 2\sec^2\theta\, d\theta}{2\sec\theta} = 32\int \tan^5\theta \sec\theta\, d\theta$

$= 32\int \tan^2\theta \tan^2\theta \tan\theta\sec\theta\, d\theta = 32\int (\sec^2\theta - 1)^2 \sec\theta\tan\theta\, d\theta$ $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$= 32\int (u^2-1)^2\,du = 32\int (u^4 - 2u^2 + 1)\,du = 32\left(\frac{u^5}{5} - \frac{2}{3}u^3 + u\right) + C$

$= 32\left(\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right) + C = 32\left(\frac{1}{5}\left(\frac{x^2+4}{2}\right)^5 - \frac{2}{3}\left(\frac{x^2+4}{2}\right)^3 + \frac{x^2+4}{2}\right) + C$

11.) $\int \frac{4x^2-7x-12}{x^3-x^2-6x}\,dx$ consider $\frac{4x^2-7x-12}{x^3-x^2-6x} = \frac{4x^2-7x-12}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$

$\Rightarrow 4x^2-7x-12 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$

$x=0 \Rightarrow -12 = -6A$ so $A = 2$

$x=3 \Rightarrow 3 = 15B$ so $B = \frac{1}{5}$

$x=-2 \Rightarrow 18 = 10C$ so $C = \frac{9}{5}$

thus $\int = \int \frac{2}{x}\,dx + \int \frac{1}{5}\frac{1}{x-3}\,dx + \int \frac{9}{5}\frac{1}{x+2}\,dx = 2\ln|x| + \frac{1}{5}\ln|x-3| + \frac{9}{5}\ln|x+2| + C$

12.) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$ consider $\displaystyle\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$\Rightarrow \quad x^2+2x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$x=0 \Rightarrow -1 = -A$ so $A=1$

$x=1 \Rightarrow 2 = 2B$ so $B=1$

$x=-1 \Rightarrow -2 = 2C$ so $C=-1$

then $\displaystyle\int = \int \frac{1}{x}dx + \int \frac{1}{x-1}dx - \int \frac{1}{x+1}dx = \ln|x| + \ln|x-1| - \ln|x+1| + C$

13.) since $f'(x)>0 \Rightarrow f \nearrow$ and so f is 1-1

so $\displaystyle\int_0^1 f^{-1}(y)\,dy$ let $y=f(x)$, $dy = f'(x)dx$ with $y=0 \Rightarrow x=0$, $y=1 \Rightarrow x=1$ since $f(0)=0$, $f(1)=1$

then $\displaystyle\int_0^1 f^{-1}(y)\,dy = \int_0^1 f^{-1}(f(x))\,f'(x)\,dx = \int_0^1 x\,f'(x)\,dx$ $\quad u=x \quad dv = f'(x)dx$, $du = dx \quad v = f(x)$

so $\displaystyle\int_0^1 = x f(x)\Big|_0^1 dx - \int_0^1 f(x)dx = 1 - 0 - \frac{1}{3} = \frac{2}{3}$