Name: Key

Math 166 Section 19061 — *Exam 2* — *September 25, 2011*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! There are 10 problems all of them each worth 5 points for a total of 50 points. There is also a bonus problem at the end worth 3 points.

For problems 1 and 2, evaluate the integrals.

1. $\displaystyle\int \frac{x-1}{x^2+3x+2}\,dx = \int \frac{x-1}{(x+2)(x+1)}\,dx = I$

$\displaystyle\frac{x-1}{(x+2)(x+1)} = \frac{A_1}{x+2} + \frac{A_2}{x+1} \Rightarrow x-1 = A_1(x+1) + A_2(x+2)$

$x=-1 \Rightarrow -2 = A_2$

$x=-2 \Rightarrow -3 = -A_1 \Rightarrow A_1 = 3$

$\displaystyle I = 3\int \frac{1}{x+2}\,dx - 2\int \frac{1}{x+1}\,dx = 3\ln|x+2| - 2\ln|x+1| + C$

2. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{e^x \cdot e^x}{1+e^x}\,dx = I$ $\qquad u = e^x$, $du = e^x\,dx$

so $\displaystyle I = \int \frac{u}{1+u}\,du = \int \frac{u+1-1}{u+1}\,du = \int du - \int \frac{1}{1+u}\,du$

$= u - \ln|1+u| + C$

$= e^x - \ln|1+e^x| + C$

3. Determine if the following integral converges or diverges: $\int_0^1 \ln x \, dx$

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \lim_{t\to 0^+} \left(x\ln x\Big|_t^1 - \int_t^1 dx \right) = \lim_{t\to 0^+} \left(-t\ln t - (1-t)\right)$$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

$$= \lim_{t\to 0^+} \left(-t\ln t + t - 1\right)$$

$$= -1 - \lim_{t\to 0^+} t\ln t \qquad 0\cdot\infty \text{ type}$$

$$= -1 - \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = -1 + \lim_{t\to 0^+} t = -1$$

so it converges

4. Find the length of the curve $y = (x+4)^{3/2}$ on $0 \le x \le 4$.

$$y' = \frac{3}{2}(x+4)^{\frac{1}{2}} \qquad 1 + (y')^2 = \frac{9}{4}(x+4) = 9 + \frac{9}{4}x + 1$$

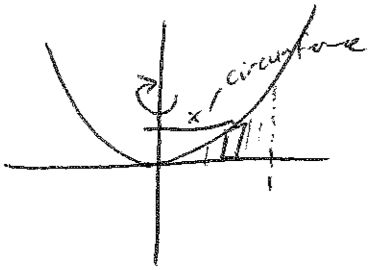
$$ds = \sqrt{10 + \frac{9}{4}x}\,dx$$

$$L = \int_0^4 \sqrt{10 + \frac{9}{4}x}\,dx = \frac{4}{9}\int_{10}^{19} \sqrt{u}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{10}^{19} = \frac{8}{27}\left(19\sqrt{19} - 10\sqrt{10}\right)$$

$u = 10 + \frac{9}{4}x$

$du = \frac{9}{4}dx$

5. Find the area of the surface obtained by rotating the curve $y = x^2$ about the y-axis on $0 \le x \le 1$.



$$y' = 2x \qquad ds = \sqrt{1+4x^2}\,dx$$

$$SA = 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx = \frac{2\pi}{8}\int_1^5 \sqrt{u}\,du$$

$u = 1+4x^2$

$du = 8x\,dx$

$$= \frac{\pi}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^5$$

$$= \frac{\pi}{6}(5\sqrt{5}-1)$$

6. Find the centroid of the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

$$A = \int_0^1 e^x\,dx = e^x\Big|_0^1 = e - 1$$

$$M_y = \int_0^1 xe^x\,dx = xe^x\Big|_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$M_x = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{4}(e^2-1) = \frac{1}{4}(e-1)(e+1)$$

$$\bar{x} = \frac{M_y}{A} = \frac{1}{e-1}, \quad \bar{y} = \frac{M_x}{A} = \frac{\frac{1}{4}(e-1)(e+1)}{e-1} = \frac{1}{4}(e+1)$$

7. Find $\frac{dy}{dx}$ if $x = t + \sin t$ and $y = t - \cos t$.

$$\frac{dx}{dt} = 1 + \cos t \quad , \quad \frac{dy}{dt} = 1 + \sin t$$

$$\frac{dy}{dx} = \frac{1+\sin t}{1+\cos t}$$

8. Set up the integral that represents the arc length of $x = \ln t$, $y = \sqrt{1+t}$ for $1 \le t \le 5$. DO NOT EVALUATE THE INTEGRAL.

$$\frac{dx}{dt} = \frac{1}{t} \quad , \quad \frac{dy}{dt} = \frac{1}{2\sqrt{1+t}} \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{t^2} + \frac{1}{4(1+t)} = \frac{4+4t+t^2}{4t^2(1+t)}$$

$$= \frac{(t+2)^2}{4t^2(1+t)}$$

$$L = \int_1^5 \sqrt{\frac{(t+2)^2}{4t^2(1+t)}}\, dt = \frac{1}{2}\int_1^5 \frac{t+2}{t\sqrt{1+t}}\, dt$$

or $\quad L = \int_1^5 \sqrt{\frac{1}{t^2} + \frac{1}{4(1+t)}}\, dt \quad$ (without simplification)

9. Find a polar equation for the curve represented by the following Cartesian equation: $x^2 + 3xy + y^2 = 5$.

$x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta + 3r^2\cos\theta\sin\theta + r^2\sin^2\theta = 5$

$\Rightarrow r^2 + 3r^2\cos\theta\sin\theta = 5$

$\Rightarrow r^2\left(1 + \frac{3}{2}\sin(2\theta)\right) = 5$

$\Rightarrow r^2 = \dfrac{5}{1 + \frac{3}{2}\sin(2\theta)}$

10. Find the values of θ on $0 \leq \theta \leq 2\pi$ of the curve $r = \sin\theta$ for which the tangent line is vertical.

$x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$, $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$

$r = f(\theta) = \sin\theta$ $\quad f'(\theta) = \cos\theta$

$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$, $\frac{dy}{d\theta} = \cos\theta\sin\theta + \cos\theta\sin\theta = \sin(2\theta)$

vert is when $\frac{dx}{d\theta} = 0$ & $\frac{dy}{d\theta} \neq 0$

in $0 \leq \theta \leq 2\pi$ $\quad \frac{dx}{d\theta} = 0 \Leftrightarrow \cos(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\frac{dy}{d\theta} \neq 0 \Leftrightarrow \sin(2\theta) \neq 0 \quad 2\theta \neq 0, \pi, 2\pi \Rightarrow \theta \neq 0, \frac{\pi}{2}, \pi$

so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

11. (BONUS) Let f be a continuous function such that

$$\int_0^1 f(x)dx = 1.$$

Define

$$y(x) = \int_0^x \sqrt{(f(t))^2 - 1}\, dt.$$

Find the arc length of $y(x)$ on $0 \le x \le 1$.

$y' = \sqrt{(f(x))^2 - 1}$ $\quad 1+(y')^2 = 1+(f(x))^2 - 1 = (f(x))^2$

$ds = \sqrt{1+(y')^2}\,dx = \sqrt{f(x)^2}\,dx = f(x)dx$

$L = \int_0^1 f(x)dx = 1$