Radar Network Scanning Coordination Based on Ensemble Transform Kalman Filtering Variance Optimization

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Abstract

In this work the variance of the error of analyzed wind fields obtained from an ensemble Kalman filter is used as a criterion with which to optimize radar network scanning strategies. The measurement equation in the Kalman filter approach is obtained from variational wind retrieval and, thus, is a function of the retrieval scanning parameters. It is shown that the mapping from radar parameters to the variance of the error is differentiable. The ensemble transform is introduced to facilitate the computational effort. The approach presented in principle may be used to optimize the scanning strategy in a network with any number of radars. Numerical examples are presented with networks consisting of two, four and nine radars using a quasi monte carlo optimization scheme. Error estimates for the approximation of the optimal strategies are discussed.

1. Introduction.

The objective of this work is to determine criteria by which cooperative measurements may be made from a radar network that in some way maximizes information. The approach is to use Kalman-based filtering procedures that produce not only estimated wind fields but also estimates of the variance of the error of the estimated wind fields. Mathematically, we use the fact that wind retrieval procedures depend on the measurement parameters. The measurement parameters are radar scanning parameters. We consider the radar locations to be fixed and known as well as the scanning angles. Radar scanning is time dependent. Temporal errors in traditional multiple-Doppler wind analysis arise when data gathered by scanning Doppler radars over an observational time window are treated as valid at a common analysis time. Such errors become significant when horizontal propagation or intrinsic evolution patterns become important [2, 6, 3]. One way these errors can be minimized is to adopt a rapid-scan strategy in which the scan time (time for the beam to re-visit locations) is less than the time scale for the evolution or propagation of the observed features. However, hardware limitations impose upper bounds on the angular velocity of scanning radars. Thus, in order for volumetric data of features of meteorological interest to be gathered sufficiently rapidly, it may be desirable to restrict the scans to only partial azimuthal sectors, that is azimuthal sectors of generally less than than 360 degrees. In the following, will consider scan strategies in which azimuthal sectors are of fixed azimuthal extent (less than 360 degrees), and seek to determine their optimal orientation (central beam angle or direction).

Key to our approach is that the rotational period (time for a full 360 degree scanning rotation) is partitioned into time intervals equal length N_s. This determines a fixed scanning angle of $360/N_s$ degrees. Actually, to fix ideas in numerical computations, we assume scanning angles of 90 degrees. It is the directions of the scans that are to be chosen in order to observe phenomena most efficiently. Alternatively, in a "sit and spin" mode, radar viewing strategies are fixed with a full 360 degree scan and have no capability to adapt. With the partitioned case, one is led to the issue of how to choose a scanning direction when it is desired to obtain the most information. Thus, the focus here is to determine the direction each radar in the network should be pointed in order to maximize the information on wind fields. The result coordinates the network radar directions to reduce the variance of Kalman estimate errors.

There are two components in the filtering procedure. The first is a state model describing the evolution of system states as a function of time. The state consists of a time-dependent vector field defined over the domain of interest, and thus, may satisfy various assumed models depending on the accuracy of the desired the modelled physics. We assume a simple model advancement in terms of an estimate of the time rate of change of a given test wind velocity vector field. The second part of the filter describes the measurement information. This amounts to the output of a retrieval procedure yielding an estimate of the vector wind field from radial data.

The wind field retrieval problem seeks to estimate three dimensional vector wind within a three dimensional domain Ω from radar data consisting of measurements of radial velocities from N radar sites. Admissible wind fields are often constrained by physical laws and regularization. The retrieval problem may be formulated as a minimization problem on a Hilbert space of admissible vector fields by weakly constraining the physical laws and regularization. The existence of a unique solution to this problem is then a consequence of classical Hilbert space theory. Typically, the retrieval problem depends on parameters describing the radar measurement model that must be specified in its formulation. Consequently, the solution of the retrieval problem depends on these parameters, and they may be considered as controls around which optimization problems may be designed. In this study, the location of the radars is considered fixed as well as the sweep angles. The sweep angle is the scanning sector angle, of the radar measurement. The parameters of interest are given it terms of an $n-$ tuple associated with the actual scanning directions of the radars. The problem is to determine directions minimizing the variance of the estimated error.

The synthesis of three-dimensional vector wind wind fields from Doppler radar data is an important part of mesoscale research and operational meteorology, with particularly vital applications in hazard warning and nowcasting (e.g., tornado detection and prediction), and in numerical weather prediction. Techniques of single-Doppler velocity retrieval vary in complexity from the simple Velocity Azimuth Display (VAD), in which the imposed model is a wind field that varies linearly with the spatial coordinates, to the full model adjoint techniques in which the radial wind obtained from time integration of the complete dynamical equation set of a numerical weather prediction model is fit to radial wind observations over a window of time. Dual-Doppler wind retrieval techniques may also be couched in an adjoint or other variational framework. Key developments in the history of single- and multiple-Doppler wind retrievals, and some of the remaining problems are summarized in Shapiro et al [13]. The application here is motivated by the design and operation of radar networks composed of multiple radars with relatively short range, in this case 30 km. The issue of coordinating the direction at which the radar are making measurements is a central concern in the collection of data.

Our efforts provide a general framework for coordinating a radar network in the following general steps.

1. Input an estimated wind field.

2. Generate an ensemble of perturbed wind fields about the estimated wind field.

3. Advance the model for each perturbed wind field to obtain an ensemble of predicted fields.

4. Generate an admissible scanning parameter $q \in \mathcal{Q}_{ad}$.

5. Evaluate criterion $J(q)$

6. Select the scanning parameter q_o minimizing $J(q)$

7. Using the "optimal" q_o to set radar parameters, make measurements to obtain a retrieved wind field estimate.

8. Return to 1. and repeat the procedure.

Generation of the ensemble, fitting the retrieval procedure and underlying model into the Kalman filter approach, definition and analysis of the criterion, and the choice and implications of the optimization sampling are addressed in subsequent sections of this work. The scheme is given more precisely in the discussion of the numerical examples.

In Section 2 we describe the classical Kalman filter and introduce certain of its generalizations. Of interest is dependence of Kalman filters on parameters in measurements models and generally how the Kalman filter may be used to determine optimal scanning parameters. Roughly speaking, scanning parameters are determined so as to minimize the analyzed error variance that would result from using those parameters. The advantage is that this scheme may be carried out without actually making measurements. It is based on minimizing the variance of the error that is the result of the measurement process and the state evolution model. The Kalman filter itself is not practical for applications since error variances are not well-known and must be estimated in practice. Hence, the ensemble Kalman filter [5] is introduced in which forecast covariances are calculated from an ensemble of forecasts given the current state of knowledge and the state evolution model. Continuity and differentiability properties of the approximated error variance based on the ensemble formulation facilitate the treatment of optimization problems. However, because of the dimensionality problems that arise in the application of these methods, the ensemble Kalman filter is not computationally practical and the ensemble transform Kalman filter [1] is introduced. This approach uses a lower dimensional ensemble space to calculate inverses more efficiently. Although we focus on the Kalman filter applied to wind retrieval based on radar data, our development may be easily carried over to other data assimilation applications.

In Section 3 the basic general Hilbert space formulation that was originally presented in [14] is given in sufficient detail for the applications here. The retrieved wind velocity is obtained as the solution of an elliptic variational boundary value problem depending on scanning direction parameters. Properties of the retrieved velocity as a function of the scanning parameters are examined, and it is shown that the retrieved wind field is differentiable with respect to the scanning directions.

In Section 4 the scanning optimization procedure is formulated by specializing the treatment in Section 2 to the application in Section 3. As an event of interest evolves in time, a scanning strategy is developed by minimizing the error variance generated through the application of the Kalman filter. The scanning parameter optimizing relative error over the generated ensemble is then used to collect data to be implemented in the next step to obtain a new estimate of the wind field. Results of a numerical study are discussed for examples with networks containing two, four, and nine radars tracking wind fields associated with a prototypical meteorological flow field in which a vortex traverses an observational domain. The set of admissible scanning parameters consists of directions for each radar. Optimization is carried out on a subcollection of N-tuples of admissible scanning directions that are obtained through the generation of an equi-distributed set of directions. Differentiability properties are useful in providing an estimate of the optimizers over the subset of directions.

2. Kalman filtering procedures and dependence on measurement parameters.

In this section we summarize filtering procedures whose observational equations depend on a parameter. The objective is to use the variance of the estimation error as a criterion with which to select parameters. It is assumed that there is a prediction model or state equation expressing a column n-vector x_{k+1} as the state at the k+1 st time step in terms of the state at the kth time step. For ease a linear form

$$x_{k+1} \;=\; F_{k+1}x_k \;+\; \mu_k \tag{2.1}$$

is assumed where F_{k+1} is an $n \times n$ matrix of real numbers. The subscript k indicates a time stepping index for $k = 0, 1, 2, ...$ The μ_k models errors in the state equation. An observational equation is given by

$$y_k \;=\; H_k\; x_k \;+\; \nu_k. \tag{2.2}$$

where y_k denotes an m-vector of observations at the time step index k. The $m \times n$ matrix H_k is a function of a measurement parameter designated by q that is eventually to be chosen in some optimal way from an admissible subset $\mathcal{Q}_{ad}$ of a Hilbert space $\mathcal{Q}$. We write $H_k(q)$ to emphasize dependence on the parameter q. To discuss the mapping $q \mapsto H(q)$ where $H(q)$ is an $n \times n$ real valued matrix, we introduce the Hilbert space $\mathbf{H}(n)$ of real valued $n \times m$ matrices with the norm

$$\|H\|_{\mathbf{H}(n)} \;=\; [trace(H^T H)]^{1/2}$$

see [7]. Hence, the mapping $q \mapsto H(q)$ is a function from $\mathcal{Q}$ into $\mathbf{H}(n)$. The column m-vector ν_k is a random vector describing the errors in the measurement model. It is assumed that measurement error and state error are independent. The $n \times n$ and $m \times m$ matrices

$$Q_k \;=\; cov(\mu_k, \mu_k) \tag{2.3}$$

$$R_k = cov(\nu_k, \nu_k) \tag{2.4}$$

describe the covariances of the state and the observational error. For the Kalman filter [10] the matrices F_k and $H_k(q)$ are given. The covariance

$$Q_k \text{ and } R_k$$

are known symmetric positive definite matrices. In computing the error for the k+1 st step, the covariance of the error at the k th step is assumed known by the $n \times n$ matrix

$$P_{k|k}.$$

A predicted or forecast covariance is then calculated by the expression

$$P_{k+1|k} = F_{k+1} P_{k|k} F_{k+1}^T + Q_{k+1}. \tag{2.5}$$

The superscript T denotes vector or matrix transposition. Defining the Kalman gain matrix at the k + 1 st step by

$$\mathcal{K}_{k+1}(q) = P_{k+1|k} H_{k+1}(q)[R_{k+1} + H_{k+1}(q) P_{k+1|k} H_{k+1}(q)^T]^{-1} \tag{2.6}$$

the updated covariance is expressed by the formula

$$P_{k+1|k+1}(q) = P_{k+1|k} - P_{k+1|k} H_{k+1}(q)^T \mathcal{K}_{k+1} H_{k+1}(q) P_{k+1|k}. \tag{2.7}$$

The forecast error variance at the k+1 st time step associated with the parameter q is expressed as

$$J(q) = trace[P_{k+1|k+1}(q)] \tag{2.8}$$

The objective is to determine a parameter q_o from within a prescribed set of admissible parameters $\mathcal{Q}$ that minimizes $J(q)$. This is formulated in terms of a minimization problem given formally by

$$\text{Find } q_o \in \mathcal{Q}_{ad} \text{ such that } J(q_o) = inf\{J(q) : q \in \mathcal{Q}\} \tag{2.9}$$

Consider the filter at the k + 1 st step. To simplify notation and following [5], we write $R = R_{k+1}$, $H(q) = H_{k+1}(q)$, the forecast covariance $P_a = P_{k+1|k+1}$, the updated covariance $P_f = P_{k+1|k}$, and the gain matrix $\mathcal{K}(q) = \mathcal{K}_{k+1}(q)$ so that equations (2.6)-(2.8) become

$$\mathcal{K}(q) = P_f H(q)^T [R + H(q) P_f H(q)^T])^{-1} \tag{2.6(i)}$$

$$P_a(q) = P_f - \mathcal{K}(q) H(q) P_f. \tag{2.7(i)}$$

$$J(q) = trace[P_a(q)] \tag{2.8(i)}$$

Remark 2.1. Interest in this work centers around the functional dependence $q \mapsto H(q)$. To treat this we assume that $q \in Q_{ad} \subset Q$ where Q is a Banach space. The matrices $H(q)$ we view as belonging to the Hilber space of $n \times n$ matrices of real numbers denoted by $\mathbf{H}$ with norm $\|H\| \ = \ [trace(H^T H)]^{1/2}$, [7].

The following is a consequence of equations (2.6)(i)-(2.8)(i).

Theorem 2.2 If $q \mapsto H(q)$ is differentiable, then the mapping $q \mapsto P_a(q)$ is differentiable.

Corollary 2.3 If $q \mapsto H(q)$ is differentiable, then the mapping $q \mapsto J(q)$ is differentiable as well.

Proof. Let e_i be the column unit vector defined by

$$(e_i)_j \ = \ 1 \text{ if } i = j \text{ and } 0 \text{ otherwise.}$$

Observe that

$$J(q) \ = \ \sum_{i=1}^{n} e_i^T P_a(q) e_i \tag{2.10}$$

and differentiability follows.

In the ensemble Kalman filter method[5], the prediction model covariance is calculated directly from an ensemble of model predictions from equation (2.1). Thus, assume that x_k is given as the (estimated) state t_k. From this estimate, an ensemble of K states is generated for the time t_{k+1} by

$$x_{k+1}^i \ = \ F_{k+1} x_k \ + \ \mu_k^i. \tag{2.11}$$

where we denote the members generated in the ensemble of $n-$ vectors by

$$x_{k+1}^1, \ x_{k+1}^2, \ \dots \ , \ x_{k+1}^K.$$

The $n-$ vector of means is denoted by $\overline{x}_{k+1}$. The forecast covariance is approximated from the ensemble of states.

$$\widetilde{P}_f \ = \ \frac{1}{K-1} \sum_{i=1}^{K} \{x_{k+1}^i \ - \overline{x}_{k+1}\} \ \{x_{k+1}^i \ - \overline{x}_{k+1}\}^T. \tag{2.12}$$

It is convenient to define the matrix

$$X \ = \ \frac{1}{\sqrt{K-1}} [x_{k+1}^1 - \overline{x}_{k+1} \ x_{k+1}^2 - \overline{x}_{k+1} \ \dots \ x_{k+1}^K - \overline{x}_{k+1}] \tag{2.13}$$

At the k+1 st step $\widetilde{P}_f$ is the ensemble approximation of the forecast error covariance. The approximation of the updated covariance at the k+1 st step is expressed by equation (2.7).

The focus of the ensemble Kalman filter is to approximate $\widetilde{P}_f$ and thereby the gain

$$(2.6)(ii) \qquad \widetilde{\mathcal{K}}(q) \;=\; \widetilde{P}_f H(q)^T[H(q)\widetilde{P}_f H(q)^T + R]^{-1}$$

as well as the updated covariance approximation and functional

$$(2.7)(ii) \qquad \widetilde{P}_a(q) \;=\; \widetilde{P}_f \;-\; \widetilde{\mathcal{K}}(q)H(q)\widetilde{P}_f.$$

$$(2.8)(ii) \qquad \widetilde{J}(q) = trace[\widetilde{P}_a(q)]$$

to generate an approximation to the analyzed error covariance $\widetilde{P}_a$.

Because the dimension n can be very large, calculation of the gain matrix $\widetilde{K}(q)$ is currently computationally not practical. The ensemble transform method [1] seeks to remedy this difficulty by inverting operators in an ensemble space of lower dimension determined by the number of elements in the ensemble. Given the forecast covariance is

$$(2.13) \qquad \widetilde{P}_f = X\ X^T,$$

the ensemble transform seeks an easily computable matrix $\mathcal{T}$ such that

$$(2.14) \qquad \widetilde{P}_a = X\ \mathcal{T}\mathcal{T}^T X^T.$$

The ensemble transform Kalman filter is introduced and developed in [1]. An orthonormal matrix C and a diagonal matrix Γ both of which depend on the parameter q are constructed such that

$$(2.15) \qquad [X^T H(q)^T R^{-1} H(q) X]C(q) \;=\; C(q)\Gamma(q).$$

The matrix $\mathcal{T}$ depends on q and is given by

$$(2.16) \qquad \mathcal{T} \;=\; \mathcal{T}(q) \;=\; C(q)[\Gamma(q) + I^{K\times K}]^{-\frac{1}{2}}.$$

Since $\Gamma(q)$ is diagonal, the inversion and square root are easily calculated. The ensemble transform, thus, provides a more efficient method to calculate the analysis covariance $\widetilde{P}_a(q)$.

Remark 2.5. The ensemble transform Kalman filter is introduced to facilitate the computation of the ensemble approximation updated covariance $\widetilde{P}(q)$. Differentiability properties depend only on $\widetilde{P}(q)$ and not on the ensemble transform expressions.

3. Retrieval of Wind Fields from Radar Data.

In this section we describe the observational model corresponding to equation (2.2) for our application. The retrieval of wind velocities from radar data forms the underlying equations for the observational model. The retrieval of wind fields from radar data is posed as a minimization problem seeking wind fields matching data under various physical and regularizing constraints [12, 13, 14]. The problem of estimating wind field information from radar data requires the specification of a retrieval functional that includes terms involving the data model (radar measurement model), the physics-based model, and the regularization for well-posedness. The model describing the relation between observed radar data and the vector-valued function constituting the actual wind field defines a mapping whose output is the radar data resulting from that wind field. This mapping depends on the specifics of the measurement process and is the point at which radar scanning parameters become involved in the problem formulation. The physics-based model is used to constrain the wind field. It aids in the interpolation between radar sites in a physically reasonable way. Finally, a regularization term is included in the retrieval functional to assure that the associated minimization problem has a unique solution.

To obtain the analogue of equation (2.2), we assume data is obtained from the application of the radar measurement operator on a given wind field $\mathbf{x} \mapsto \mathbf{w}(\mathbf{x})$ defined on a set Ω. The observational equation is then the retrieval equation mapping $\mathbf{w}$ to the retrieved wind velocity estimate. This equation defines a retrieval operator that depends on the retrieval scheme and the measurement operator. The retrieval operator depends on the measurement parameter through its dependence on the measurement model. Hence, with different measurement models, one obtains different retrieval operators. The common feature is the dependence on the measurement model parameters.

The formulation of the retrieval problem is given in [12, 14]. We give enough detail here for the completeness of our discussion. Let Ω denote an observational volume that, for ease, is a rectangular volume of points $\mathbf{x} = (x, y, z)^T$ in $\Re^3$ such that

$$\Omega = \{\mathbf{x} : 0 < x < L_x, 0 < y < L_y, 0 < z < L_z\}$$

and with its base denoted by

$$\Omega_0 = \{(x, y, 0)^T : 0 < x < L_x, 0 < y < L_y\}.$$

Assume there are N radar site locations $\mathbf{x}_1, ..., \mathbf{x}_N$ in Ω_0. In general vectors are understood to be column vectors.

Define the following vector-valued functions from Ω into $\Re^3$ that are used to describe the wind field within the set Ω :

$$\mathbf{v}_s(\mathbf{x}) \;=\; \text{velocity of scattering particles in the sample volume } \Omega$$

$$\mathbf{v}(\mathbf{x}) \;=\; \text{air velocity} \;:\; \mathbf{v}(\mathbf{x}) \;=\; v_1(\mathbf{x})\mathbf{i} + v_2(\mathbf{x})\mathbf{j} + v_3(\mathbf{x})\mathbf{k}.$$

Let

$$\mathbf{v}_t(\mathbf{x}) \;=\; \text{terminal velocity of the scatterers}$$

where $\mathbf{v_t}$ is defined to be positive. In practice, $\mathbf{v}_t$ can be parameterized in terms of the radar reflectivity field [4]. The vectors $\mathbf{v},\ \mathbf{v}_t,$ and $\mathbf{v}_s$ are related by

$$\mathbf{v}_s(\mathbf{x}) \;=\; \mathbf{v}(\mathbf{x}) - \mathbf{v}_t(\mathbf{x}). \tag{3.1}$$

We also use $\mathbf{u}$ and $\mathbf{w}$ to indicate wind fields. Usually, $\mathbf{w}$ is an input wind field in the retrieval scheme. Define the vector-valued function pointing from the ith radar location $\mathbf{x}_i$ toward the point $\mathbf{x} \in \Omega$

$$\mathbf{r}_i(\mathbf{x}) = \mathbf{r}(\mathbf{x}, \mathbf{x}_i) \;=\; \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|} \text{ for } \mathbf{x} \neq \mathbf{x}_i \text{ and } \mathbf{0} \text{ for } \mathbf{x} = \mathbf{x}_i. \tag{3.2}$$

The radial velocity observed at the ith radar within the coverage set is then expressed in terms of the product

$$v_r(\mathbf{x}, \mathbf{x}_i) = \mathbf{r}_i(\mathbf{x})^T\ \mathbf{v}_s(\mathbf{x}). \tag{3.3}$$

To model the coverage of the ith radar, we specify a real-valued function

$$\phi_i(\mathbf{x})$$

taking the value one over the coverage set of fixed angular extent β (less than 2π) of the ith radar and zero elsewhere. The coverage set is determined over a sector whose vertex is located at $\mathbf{x}_i$ and is centered on a direction indicated by the unit vector

$$\mu(\alpha) \;=\; [cos(\alpha)\ sin(\alpha)\ 0]^T \tag{3.4}$$

where $\alpha \in [-\pi, \pi)$. For the purposes of this work, a coverage function is defined for each radar location $\mathbf{x}_i$ for $i = 1, ..., N$ in terms of a characteristic function defined over a conical sector C with vertex at $\mathbf{x}_i$, centered on the vector $\widehat{\mathbf{u}}(\alpha_i)$, with aperture angle β and with radius R. Thus, for each $i = 1, ..., N$

$$\Xi_{C_i}(\mathbf{x}) \;=\; 1 \text{ if } x \in C_i \text{ and } \;=\; 0 \text{ otherwise.} \tag{3.5}$$

Of particular interest are sets

$$C_i(\alpha) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_i| \leq R,\ \mathbf{r}(\mathbf{x}, \mathbf{x}_i)^T\ \mu(\alpha) \geq \gamma = cos(\beta)\} \tag{3.6}$$

so that

$$\phi_i(\mathbf{x}) \;=\; \phi_i(\mathbf{x}, \alpha_i) \;=\; \Xi_{C_i(\alpha_i)}(\mathbf{x}) \tag{3.7}$$

The set $C_i(\alpha)$ is the scanning set for the radar located at the point $\mathbf{x}_i$ in the direction of $\widehat{\mathbf{u}}(\alpha)$. The locations $\mathbf{x}_i,\ i = 1, ..., N$ are fixed and the admissible set are the coverage functions determined by the direction unit vectors $\mu(\alpha_i)$ for $\alpha_i \in [-\pi, \pi)$ and the fixed aperture angle β. The observation from the ith radar of the wind velocity $\mathbf{v}_s$ at the point $\mathbf{x}$ may be expressed as

$$\mathbf{v_r}(\mathbf{x}, \mathbf{x_i}) \ = \ \phi_\mathbf{i}(\mathbf{x}, \alpha_\mathbf{i})\mathbf{r_i}(\mathbf{x})\mathbf{v_s}(\mathbf{x}).$$

Remaining terms in the model include the divergence free condition embodied in the continuity equation and a regularization term included so that the problem is well-posed. Discussion of these terms is given in [8, 12, 14] For the present application the important feature is that the functional to minimized is quadratic and is associated with an inner product over a suitable Hilbert space.

To give a Hilbert space formulation see [14], introduce the Hilbert spaces

$$\mathbf{H} \ = \ L^2(\Omega, \Re^3) \tag{3.8}$$

with the inner product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}^T \mathbf{v} d\mathbf{x} \tag{3.9}$$

and

$$\mathbf{V} = H^1(\Omega, \Re^3) \tag{3.10}$$

with bilinear form

$$((\mathbf{u}, \mathbf{v})) = \int_\Omega \{\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2 + \nabla u_3 \cdot \nabla v_3\} d\mathbf{x}$$

and inner product

$$(\mathbf{u}, \mathbf{v})_V = ((\mathbf{u}, \mathbf{v})) \ + \ (\mathbf{u}, \mathbf{v}). \tag{3.11}$$

and norm

$$\|\mathbf{v}\|_V = (\mathbf{v}, \mathbf{v})_V^{\frac{1}{2}}. \tag{3.12}$$

Conservation of mass is enforced as a weak constraint. Hence, the bilinear form on $\mathbf{V}$ is defined by

$$(\mathbf{u}, \mathbf{v})_1 = \int_\Omega [\nabla \cdot \mathbf{u}][\nabla \cdot \mathbf{v}] d\mathbf{x}. \tag{3.13}$$

The weak formulation of the retrieval problem is posed as a minimization problem using the space $\mathbf{V}$ as defined by (3.10) with (3.11) cf. [12]. The objective functional is given as

$$\mathcal{V}(\mathbf{v}) = \frac{\epsilon}{2}((\mathbf{v},\mathbf{v})) \ + \ \frac{K}{2}(\mathbf{v},\mathbf{v})_1 \ +$$

$$+\frac{K_1}{2}\int_\Omega \{\sum_{i=1}^{N} \phi_i^2(\mathbf{x},\alpha_i)[\mathbf{r}_i^T[(\mathbf{v}(\mathbf{x})-\mathbf{v}_t)-(\mathbf{w}(\mathbf{x})-\mathbf{v}_t)]^2\}d\mathbf{x} \tag{3.14}$$

over the space of functions $\mathbf{V}$ where $\epsilon, K,$ and K_1 are positive constants. Note that a function $\mathbf{w}$ is specified in the functional instead of the observed data and represents the "true" wind field. The retrieval problem is thus

$$\text{Find } \mathbf{u} \in \mathbf{V} \text{ such that } \mathcal{V}(\mathbf{u}) \ = \ \text{infimum } \{\mathcal{V}(\mathbf{v}) : \mathbf{v} \in \mathbf{V}\} \tag{3.15}$$

Time is not explicitly included. In this formulation the retrieval problem is solved over a sequence of times. It is assumed that the radial velocity is known at each point $\mathbf{x}$ within a given radar's scanning set at each time.

Our interest here will focus on the dependence of solutions of the retrieval problem on the collection of radar scanning directions. Hence, we view the vector

$$q = [\alpha_1 \ \alpha_2 \ . \ . \ . \ \alpha_N]$$

as a parameter to be determined. The vector q determines N unit vectors given by $\widehat{\mathbf{u}}(\alpha_i)$ for $i = 1, ..., N$ from (3.4). Define the functions

$$\Phi(q)(\mathbf{x}) \ = \ \sum_{i=1}^{N} \phi_i(\mathbf{x},\alpha_i)\mathbf{r}_i(\mathbf{x})\mathbf{r}_i^T(\mathbf{x}), \tag{3.16}$$

It is also convenient to define the bilinear forms on $\mathbf{H}$

$$(\mathbf{u},\mathbf{v})_{\Phi(q)} = \int_\Omega \mathbf{u}(\mathbf{x})^T\Phi(q)(\mathbf{x})\mathbf{v}(\mathbf{x})d\mathbf{x}. \tag{3.17}$$

With the above definitions, we may write the criterion $\mathcal{V}$ as

$$\mathcal{V}(q)(\mathbf{v}) = \frac{\epsilon}{2}((\mathbf{v},\mathbf{v})) + \frac{K}{2}(\mathbf{v},\mathbf{v})_1 + \frac{K_1}{2}[(\mathbf{v},\mathbf{v})_{\Phi(q)} \ -$$

$$-2(\mathbf{w},\mathbf{v})_{\Phi(q)} \ + (\mathbf{w},\mathbf{w})_{\Phi(q)}]. \tag{3.18}$$

Existence of a unique solution to the minimization problem follows from the discussions in [14] and the Hilbert space formulation given therein.

The solution of the minimization problem (3.15) and (3.18) is characterized by the optimality conditions. The solution $\mathbf{u} \ = \ \mathbf{u}(q)$ of the minimization problem satisfies the equation

$$\epsilon((\mathbf{u},\mathbf{v})) + K(\mathbf{u},\mathbf{v})_1 + K_1(\mathbf{u},\mathbf{v})_{\Phi(q)} = K_1(\mathbf{w},\mathbf{v})_{\Phi(q)} \tag{3.19}$$

for all $\mathbf{v} \in \mathbf{V}$.

For the application to the Kalman filter described in Section 2, we approximate (3.19) by finite elements. The approximation of the retrieval problem numerically follows the classical finite element arguments [11]. Approximations may be based on finite elements obtained as tensor products of piecewise linear splines defined on partitions of the intervals $(0, L_x)$, $(0, L_y)$, and $(0, L_z)$ into n_x, n_y, and n_z subintervals, respectively. Hence, the number of elements in the x, y, and z directions are $\widehat{m}_x = n_x + 1, \widehat{m}_y = n_y + 1, \widehat{m}_z = n_z + 1$, respectively. The number of basis elements for the 3 spatial dimensional problem is given by $\widehat{m} = \widehat{m}_x \times \widehat{m}_y \times \widehat{m}_z$. We denote the basis elements as

$$b_1(\mathbf{x}), \; \ldots, \; b_{\widehat{m}}(\mathbf{x})$$

spanning a subspace, $\mathbf{V}^{\widehat{m}}$, of the space $\mathbf{V}$ and define the column $\widehat{m}$ vector-valued function on Ω by

$$\mathbf{x} \mapsto \underline{b}(\mathbf{x}) = [b_1(\mathbf{x}), ..., b_{\widehat{m}}(\mathbf{x})]^T$$

and with $n = 3\widehat{m}$ the $3 \times n$ matrix-valued function on Ω by

$$\mathbf{x} \mapsto B(\mathbf{x}) = \begin{bmatrix} \underline{b}(\mathbf{x})^T & \underline{0} & \underline{0} \\ \underline{0} & \underline{b}(\mathbf{x})^T & \underline{0} \\ \underline{0} & \underline{0} & \underline{b}(\mathbf{x})^T \end{bmatrix}$$

where $\underline{0}$ represents an $\widehat{m}$-row vector of zeros. We also define the column $\widehat{m}$-vectors $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ as well as the n-column vector $\widetilde{c} = [\underline{c}_1^T, \underline{c}_2^T, \underline{c}_3^T]^T$.

The components of the wind velocity are represented as

$$v_{1\widehat{m}}(\mathbf{x}) = \underline{b}(\mathbf{x})^T \underline{c}_1$$

$$v_{2\widehat{m}}(\mathbf{x}) = \underline{b}(\mathbf{x})^T \underline{c}_2$$

$$v_{3\widehat{m}}(\mathbf{x}) = \underline{b}(\mathbf{x})^T \underline{c}_3.$$

The approximating wind velocity vector is expressed as

$$\mathbf{v}_n(\mathbf{x}) = [v_{1\widehat{m}}(\mathbf{x}) \; v_{2\widehat{m}}(\mathbf{x}) \; v_{3\widehat{m}}(\mathbf{x})]^T = B(\mathbf{x})\widetilde{c}.$$

In a similar way we approximate the input wind field by

$$\mathbf{w}_{\widehat{m}}(\mathbf{x}) \; = \; B(\mathbf{x})\widetilde{\omega}. \tag{3.20}$$

To approximate the objective functional, define the $n \times n$ matrices

$$G_0 = \int_\Omega B(\mathbf{x})^T B(\mathbf{x}) d\mathbf{x}$$

$$G_1 = \int_\Omega [\nabla^T B(\mathbf{x})]^T [\nabla^T B(\mathbf{x})] d\mathbf{x}.$$

Define the $\widehat{m} \times \widehat{m}$ matrix $[g_2]$ by setting entries

$$[g_2]_{ij} = \int_\Omega \nabla b_i(\mathbf{x}) \cdot \nabla b_j(\mathbf{x}) d\mathbf{x}$$

for $i, j = 1, ..., \widehat{m}$. Let the $n \times n$ matrices be given by

$$G_2 = \begin{bmatrix} g_2 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_2 \end{bmatrix}.$$

$$G(q) = \int_\Omega B(\mathbf{x})^T \Phi(q) B(\mathbf{x}) d\mathbf{x}.$$

With the above definitions the objective functional evaluated at the finite element approximations of the wind velocity is given by

$$\mathcal{V}(q)(\widetilde{c}) \;=\; \mathcal{V}(q)(\mathbf{v}_n) \;= \tag{3.21}$$

$$= \frac{1}{2}\widetilde{c}^T[\epsilon G_2 + KG_1 + K_1 G(q)]\widetilde{c} \;-\; K_1 \widetilde{\omega}^T G(q)\widetilde{c} + K_1 \widetilde{\omega}^T G(q)\widetilde{\omega}.$$

The equation

$$[\epsilon G_2 + KG_1 + K_1 G(q)]\widetilde{c}(q) \;=\; K_1 G(q)\widetilde{\omega} \tag{3.22}$$

is obtained as the optimality condition and

$$\widetilde{c}(q) \;=\; K_1[\epsilon G_2 + KG_1 + K_1 G(q)]^{-1} G(q)\widetilde{\omega} \tag{3.23}$$

Define the matrices

$$H_0(q) \;=\; K_1[\epsilon G_2 + KG_1 + K_1 G(q)]^{-1} \tag{3.24}$$

and

$$H(q) \;=\; H_0(q) G(q) \tag{3.25}$$

The matrix $H(q)$ is the retrieval operator for our application corresponding that the measurement operator discussed in Section 2.

We now consider the differentiability with respect to the direction angle parameter α. Towards this end, let the function where $\mathbf{x} \mapsto g(\mathbf{x})$ be a continuous real-valued function defined on Ω. Define the real-valued function $\mathcal{F} : \Re \mapsto \Re$ in terms of the integral given by

$$\mathcal{F}(\alpha) \;=\; \int_\Omega \Xi(\alpha)(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}$$

where $\Xi(\alpha)$ is the indicator function of the set

$$C(\alpha) \;=\; \{\mathbf{x} : x_1^2 + x_2^2 \;\leq R^2, \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \mathbf{u}(\alpha) \;\geq\; cos(\beta)\}.$$

It follows that

$$\mathcal{F}(\alpha) \;=\; \int_{C(\alpha)} g(\mathbf{x})d\mathbf{x}$$

Introducing polar coordinates, we see that

$$\mathcal{F}(\alpha) \;=\; \int_{\alpha-\beta}^{\alpha+\beta} \int_0^{L_z} \int_0^R \widehat{g}(r,\theta,z) r dr dz d\theta$$

where $\widehat{g}(r,\theta,z) = g(rcos(\theta), rsin(\theta), z)$. It follows that

$$\frac{d}{d\alpha}\mathcal{F}(\alpha) \;=\; \int_0^{L_z} \int_0^R [\widehat{g}(r,\alpha+\beta,z) - \widehat{g}(r,\alpha-\beta,z)] r dr dz.$$

For the case of N radar sites, we write

$$\mathcal{F}(\mathbf{u},\mathbf{v})(q) \;=\; \mathcal{F}(\mathbf{u},\mathbf{v})(\alpha_1,...,\alpha_N) \;=\; (\mathbf{u},\mathbf{v})_{\Phi(q)}$$

$$= \int_\Omega [\sum_{i=1}^N \phi_i(\mathbf{x},\alpha_i)\mathbf{u}(\mathbf{x})^T \mathbf{r}_i(\mathbf{x})\mathbf{r}_i^T(\mathbf{x})\mathbf{v}(\mathbf{x})d\mathbf{x}$$

Set

$$g_l(\mathbf{x}) = \phi_l(\mathbf{x},\alpha_l)\mathbf{u}(\mathbf{x})^T[\mathbf{r}_l(\mathbf{x})\mathbf{r}_l^T(\mathbf{x})]\mathbf{v}(\mathbf{x})$$

and in terms of polar coordinates set

$$g_l(\mathbf{x}) \;=\; \widehat{g}_l(r,\theta,z)$$

and state the following.

Proposition 3.1. For each $\mathbf{u}$ and $\mathbf{v} \in \mathbf{V}$ the function $q \;=\; (\alpha_1,...,\alpha_N) \mapsto \mathcal{F}(\mathbf{u},\mathbf{v})(\alpha_1,...,\alpha_N)$ is differentiable and the partial derivatives of $\mathcal{F}(\mathbf{u},\mathbf{v})$ with respect to α_l is given by

$$\frac{\partial}{\partial\alpha_l}\mathcal{F}(\mathbf{u},\mathbf{v})(q) \;=\; \frac{\partial}{\partial\alpha_l}\mathcal{F}(\alpha_1,...,\alpha_n) \;=\; \int_0^{R_z}\int_0^R [\widehat{g}_l(r,\alpha_l+\beta,z) - \widehat{g}_l(r,\alpha_l-\beta,z)] r dr dz$$

Hence, the derivative of $\mathcal{F}$ is expressed as

$$D\mathcal{F}(\mathbf{u},\mathbf{v})(q)q' \;=\; \sum_{l=1}^N \frac{\partial}{\partial\alpha_l}\mathcal{F}(\mathbf{u},\mathbf{v})(q)\alpha_l'$$

The differentiability of $\mathbf{u}(q)$ follows.

Theorem 3.2. The solution $\mathbf{u}(q)$ of (3.19) is Frechét differentiable with respect to q and the Frechét differential of $\mathbf{u}(q)$ with increment q' satisfies the variational equation

$$\epsilon((D\mathbf{u}(q)q',\mathbf{v})) + K(D\mathbf{u}(q)q',\mathbf{v})_1 + K_1(D\mathbf{u}(q)q',\mathbf{v})_{\Phi(q)} \; = \; = \; K_1(D\mathcal{F}(\mathbf{w},\mathbf{v})(q)q' \; - \; (D\mathcal{F}(\mathbf{u}(q),\mathbf{v})(q)q'$$

The finite dimensional version (3.22) satisfies

Corollary 3.3. The solution $\widetilde{c}(q)$satisfying the equation

$$[\epsilon G_2 + KG_1 + K_1G(q)]D\widetilde{c}(q)q' = K_1[DG(q)q'][\widetilde{\omega} - \widetilde{c}(q)]$$

The derivative of the matrix $G(q)$ is obtained as follows. Define the matrix value function

$$\mathcal{G}_l(\mathbf{x}) \; = \; B(\mathbf{x})^T\mathbf{r}_l(\mathbf{x})\mathbf{r}_l^T(\mathbf{x})B(\mathbf{x})$$

so that

$$G(q) \; = \; \int_\Omega \sum_{k=1}^N \phi_k(\mathbf{x},\alpha_k)\mathcal{G}_k(\mathbf{x})d\mathbf{x}.$$

Proposition 3.4. The partial derivative $D_{\alpha_l}G(q)$ is given by

$$D_{\alpha_l}G(q) \; = \; \int_0^{L_z}\int_0^R [\widehat{\mathcal{G}}(r,\alpha_l+\beta,z) - \widehat{\mathcal{G}}(r,\alpha_l-\beta,z)]rdrdz$$

where $\widehat{\mathcal{G}}(r,\theta,z)$ denotes $\mathcal{G}(\mathbf{x})$ expressed in terms of polar coordinates.

4. Retrieval-based Ensemble Transform Kalman Filter.

In this section the filter equations are summarized and the formulas expressing derivatives are given. In the application to our problem we assume that the state model is of the form from (3.23)-(3.25).

$$\widetilde{\omega}_{k+1} \; = \; F_{k+1}\widetilde{\omega}_k + \widetilde{\mu}_k \tag{4.1}$$

where $\widetilde{\omega}_k$ is an n-vector, F_{k+1} is an $n\times n$ matrix and μ_k is a random n-vector. An ensemble of K solutions $\widetilde{\omega}_{k+1}^i$ for $i = 1,...,K$ is generated. Denoting the mean of the vectors by $\widetilde{\omega}_{k+1}^i$ by $\overline{\widetilde{\omega}}$, the forecast error covariance is given by

$$\widetilde{P}_f \; = \; \frac{1}{K-1}\sum_{i=1}^K \{\widetilde{\omega}_{k+1}^i \; - \overline{\widetilde{\omega}}_{k+1}\}\,\{\widetilde{\omega}_{k+1}^i \; - \overline{\widetilde{\omega}}_{k+1}\}^T \tag{4.2}$$

The observational equation is given by

$$(4.3) \qquad c_k(q) \; = \; H_k(q)\widetilde{\omega}_k \; + \; \nu_k$$

where $H_k(q)$ is given in

$$(3.24) \qquad H_0(q) \; = \; K_1[\epsilon G_2 + KG_1 + K_1 G(q)]^{-1}$$

and

$$(3.25) \qquad H(q) \; = \; H_0(q)G(q)$$

where the time stepping subscript k has been suppressed. The ensemble approximation of the Kalman gain is given by

$$(2.6)(ii) \qquad \mathcal{K}(q) \; = \; \widetilde{P}_f H(q)^T (R + H(q)\widetilde{P}_f H(q)^T)^{-1}$$

and the updated error covariance is

$$(2.7)(ii) \qquad \widetilde{P}_a(q) \; = \; \widetilde{P}_f \; - \; \mathcal{K}(q)H(q)\widetilde{P}_f.$$

$$(2.8)(ii) \qquad J(q) \; = \; trace[\widetilde{P}_a(q)] = \sum_{i=1}^{n} e_i^T \widetilde{P}_a(q) e_i$$

The derivatives of these operators are based on the discussion in the previous section. Hence, the partial derivative with respect to the parameter q_l of these matrices are again matrices.

Proposition 4.1. The Kalman gain and covariance are differentiable functions from $\Re^N$ into the Hilbert space of $\mathbf{H}(n)$ of $n \times n$ real-valued matrices, and accordingly, the variance is a differentiable function from $\Re^N$ into $\Re$. The derivatives are expressed as

$$D_l H_0(q) \; = \; -H_0(q)^2 D_l G(q)$$

The partial derivatives with respect to q_l are given by

$$(4.4) \qquad D_l H(q) \; = \; H_0(q)[I \; - \; H_0(q)] D_l G(q)$$

$$(4.5) \qquad D_l\mathcal{K}(q) \; = \; -\widetilde{P}_f \; D_l H(q)^T \; [R \; + \; H(q)\widetilde{P}_f H(q)]^{-1}$$
$$- \; \widetilde{P}_f \; H(q)^T [R \; + \; H(q)\widetilde{P}_f H(q)]^{-2} \; [H(q)\widetilde{P}_f D_l H(q)^T \; + \; D_l H(q)\widetilde{P}_f H(q)^T]$$

$$(4.6) \qquad D_l P_a(q) \; = \; -[D_l\mathcal{K}(q) \; H(q) \; + \; \mathcal{K}(q) \; D_l H(q)] P_f$$

$$(4.7) \qquad D_l J(q) \; = \; e_l D_l P_a(q) e_l$$

Remark 4.2. The differentiability is used in the next section with quasi-Monte Carlo method to estimate minimality of the function with respect to the family of test admissible parameters generated.

5. The Model and Numerical Study.

For the purposes of constructing numerical tests with simulated observational data, we consider the vector wind field defined on the domain

$$\Omega \;=\; [0,150] \times [0,150] \times [0,3]$$

with

$$\Omega_0 \;=\; [0,150] \times [0,150]$$

containing a cylindrical vortex traversing diagonally across Ω_0. Here all numbers are in units of kilometers. It is assumed that the components of the background wind field are zero. To this background is added the generated vortex that evolves with time. To construct the example, the center of the est vortex is given by

$$\mathbf{x}_0(t) \;=\; [10t,\; x_2(t) = 10t + 10,\; 0]^T \tag{5.1}$$

The portion of the wind field at time t contributed by the vortex is obtained from

$$\mathbf{u}(\mathbf{x},t) \;=\; \begin{cases} [-\frac{\partial\psi}{\partial x_2},\; \frac{\partial\psi}{\partial x_1},\; 0] & \text{when } \mathbf{x} < R \\ \qquad 0 & \text{when } \mathbf{x} \geq R \end{cases} \tag{5.2}$$

where

$$\psi(\mathbf{x},t) \;=\; -exp[([\frac{|\mathbf{x} - \mathbf{x}_0(t)|}{R}]^2 - 1)^{-1}] \tag{5.3}$$

For the purpose of the numerical experiment, however, we use the underlying model

$$\mathbf{u}(\mathbf{x}, t+dt) \;=\; \mathbf{u}(\mathbf{x},t) \;+\; dt\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} \tag{5.4}$$

to advance the wind vector field. In fact, using the notation introduced previously, the model is given in terms of a column n-vector $\widetilde{\omega}$ arising from the expression for the wind field given by (3.20). The components of of $\widetilde{\omega}$ are obtained from value of $\mathbf{u}$ at nodal points arranged as the column n vector $\widetilde{\omega}$. The model equations are given by

$$\widetilde{\omega}_{k+1} \;=\; \widetilde{\omega}_k \;+\; \widetilde{D}_k \tag{5.5}$$

where the vector $\widetilde{D}_k$ expresses the product of the time step and the time derivative of $\widetilde{\omega}$ with error. In the application of the model equations $\widetilde{\omega}_k$ is taken to be the best estimate of the wind field at time $t = k$. To generate an ensemble of predicted wind fields

$$\widetilde{\omega}^i_{k+1} \text{ for } i \;=\; 1, 2, ..., K,$$

uniformly distributed random errors are added to $\widetilde{\omega}_k$ as well as $\widetilde{D}_k$ to provide uncertainty not only in the wind fields but in their changes with respect to time as well. This ensemble is then use to construct the forecast covariance for the k+1 step $(\widetilde{P}_f)_{k+1}$ as well as the matrix $\mathcal{T}$ and the analyzed covariance $(\widetilde{P}_a)_{k+1}$ as outlined in equations (2.12)-(2.16). For the purposes of the numerical experiment ensembles of size 50 are used.

We note that $\mathcal{T}$ and thus $(\widetilde{P}_a)_{k+1}$ depend on q through the observation operator $H(q)$. A search is conducted for q_o from among an admissible set $\mathcal{Q}_{ad}$ that minimizes the

$$trace[(\widetilde{P}_a(q))_{k+1}]$$

with respect to the admissible set. As indicated then in the previous sections, the observational equation is then applied to construct an estimate $\widetilde{c}(q_o)$ using data collected based on the scanning parameter q_o. This vector is then used in the model equation to to predict fields at the next time step. The algorithm is then repeated.

The general algorithm for scanning optimization is summarized as follows.

1. Input estimated wind field.

2. Generate an ensemble of perturbed wind fields about the estimated.

3. Advance the model for each perturbed wind field to obtain an ensemble of predicted fields.

4. Generate an admissible scan parameter $q \in \mathcal{Q}_{ad}$.

5. Calculate the associated $C(q)$ and $\Gamma(q)$ matrices.

6. Calculate the associated $\mathcal{T}(q)$ matrix.

7. Calculate the analysis error covariance matrix $P_{k+1|k+1}(q)$.

8. Compute the value of the functional $J(q) = trace(P_{k+1|k+1}(q)$.

9. Select the scanning parameter q_o minimizing $J(q)$

10. Using the "optimal" q_o to set radar parameters, make measurements to obtain a retrieved wind field estimate.

11. Return to 1. and repeat the procedure.

For the purposes of the example we use a mesh with 10 subintervals of length $15km$ in each of the x and y directions and $1km$ in the z direction.

Using piecewise linear basis elements, it follows that the basis contains 242 basis functions and that $n = 726$. The radius of the test vortices is $R = 15km$ for Figures 1-12 and, for comparison $R = 30km$ in Figures 13-18. In the case of $R = 15$ the vortex radius equals the mesh dimension. Tests are conducted with $N = 2$, 4, and 9 radars arranged as indicated in the figures by black stars. The figures portray the position of the vortex and the shaded coverage of the radars for the network for a scanning configuration that minimizes the variance with respect to an admissible set of directions as has been described above. It should be pointed out that the unshaded areas interior to the shaded regions are also in the coverage areas. It is assumed that the range of each radar is $30km$ and the scan width is 45 degrees. Each radar is pointed in the direction of a unit vector

$$\mathbf{u}(\alpha) = [cos(\alpha)\ sin(\alpha)\ 0]^T$$

generated for an angle $\alpha \in [0, 2\pi)$.

The collection of admissible vectors $q = [\alpha_1, \alpha_2, ..., \alpha_N]$ of directions are generated as equidistributed sequences of vectors [9] as follows. For $i = 1, ..., N$ let components of a vector of length N be given by $p_i = (i^2 + 1)^{1/2}$. The ith component of the nth term in the sequence of vectors is generated by

$$(\alpha_i)^n = 2(np_i - [np_i])\pi$$

where $[\cdot]$ denotes the greatest integer function. The vector

$$\mathbf{u}((\alpha_i)^n) = [cos((\alpha_i)^n)sin((\alpha_i)^n)0]^T$$

is the nth direction vector for the ith radar in a sequence indexed by n. It can be shown [9] that the sequence of vectors generated in this way fills the N-cube in a regular way. The admissible set for the numerical study consists of L matrices of directions generated in this manner $\mathcal{Q}_{ad}^L$. For the purposes of the numerical experiment we take $L = 20$.

Remark 5.1. The estimate

$$|minJ(q) - min_L J(q)| = O(L^{-1/N}(logL)^{1/N} \tag{5.6}$$

follows from the differentiability of the criterion demonstrated in previous sections and the boundedness of the derivative [9].

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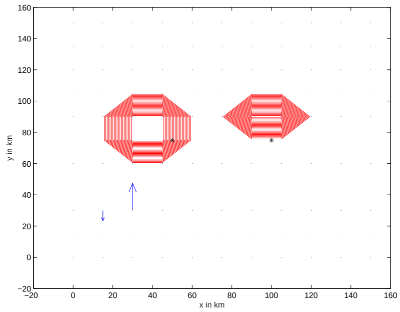

Figure 1: Two Radar Coverage at t=3, R=15

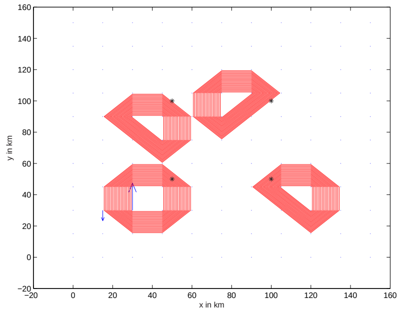

Figure 2: Four Radar Coverage at t=3, R=15

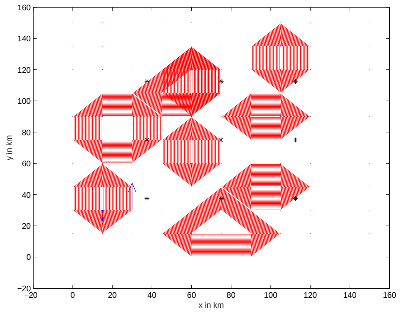

Figure 3: Nine Radar Coverage at t=3, R=15

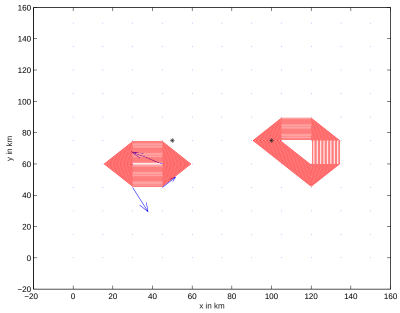

Figure 4: Two Radar Coverage at t=5, R=15

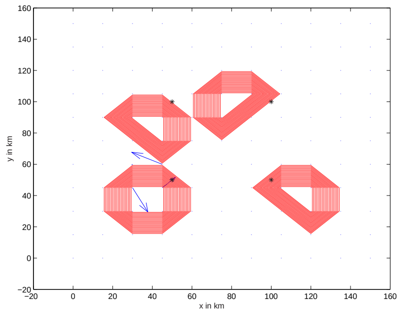

Figure 5: Four Radar Coverage at t=5, R=15

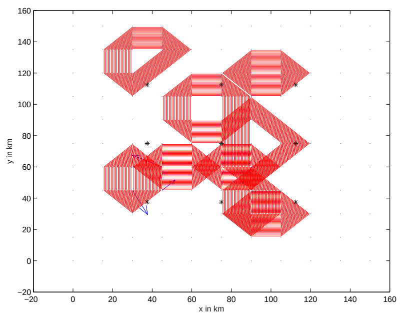

Figure 6: Nine Radar Coverage at t=5, R=15

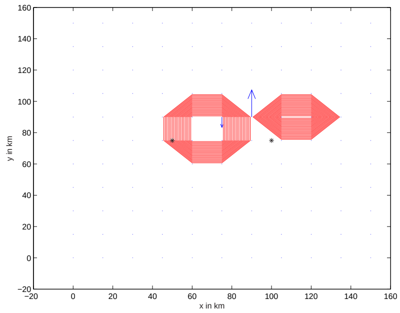

Figure 7: Two Radar Coverage at t=9, R=15

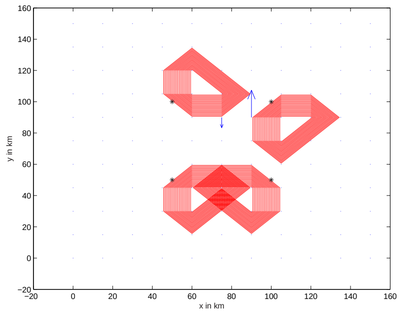

Figure 8: Four Radar Coverage at t=9, R=15

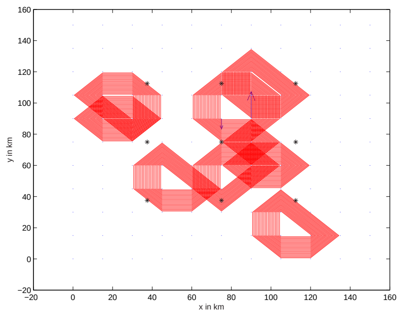

Figure 9: Nine Radar Coverage at t=9, R=15

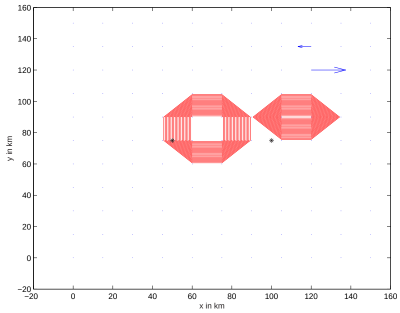

Figure 10: Two Radar Coverage at t=13, R=15

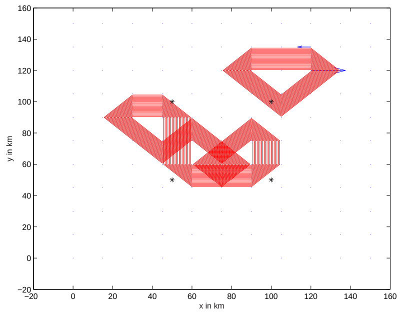

Figure 11: Four Radar Coverage at t=13, R=15

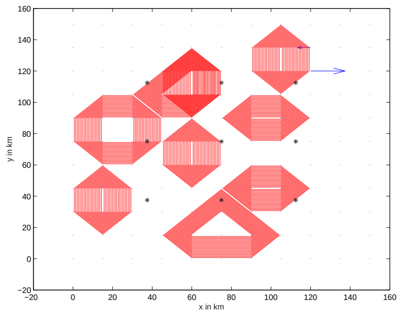

Figure 12: Nine Radar Coverage at t=13, R=15

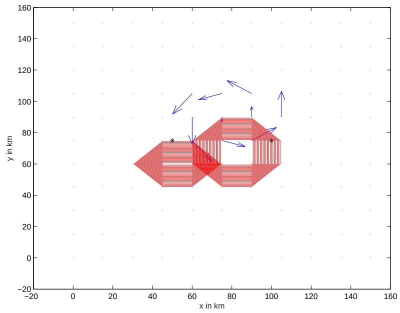

Figure 13: Two Radar Coverage at t=9, R=30

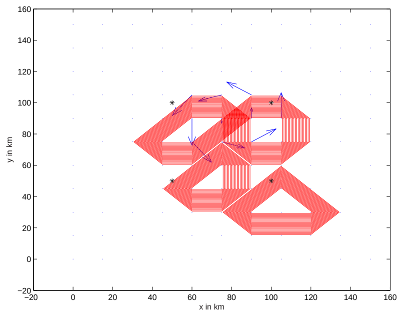

Figure 14: Four Radar Coverage at t=9, R=30

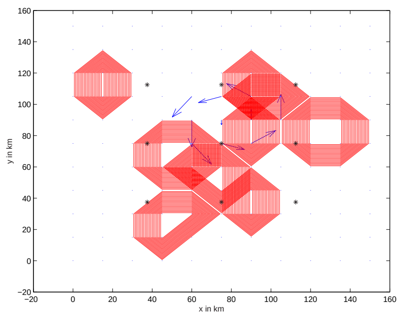

Figure 15: Nine Radar Coverage at t=9, R=30

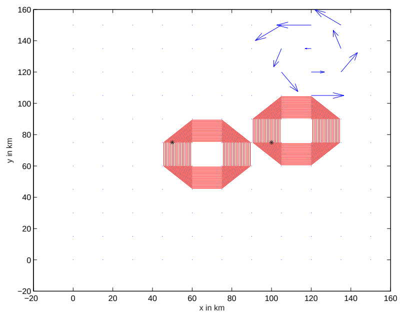

Figure 16: Two Radar Coverage at t=13, R=30

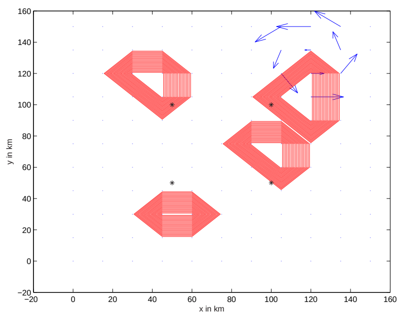

Figure 17: Four Radar Coverage at t=13, R=30

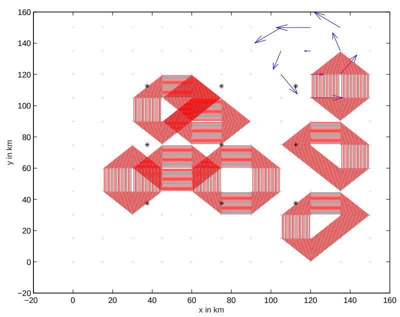

Figure 18: Nine Radar Coverage at t=13, R=30

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