Introduction to Differential Equations: Exam 2.

Name____________________________________

ID#____________________________________

Find the general solution of the following equations.

1.(10pnts) $y'' + 8y' + 25y = 0$

$r^2 + 8r + 25 = 0$

$r = \frac{-8 \pm \sqrt{64 - 100}}{2}$

$r = \frac{-8 \pm \sqrt{-36}}{2}$

$r = -4 \pm 3i$

$y = c_1 e^{-4x} \cos(3x) + c_2 e^{-4x} \sin(3x)$

2. (10pnts) $2y'' - 4y' + 2y = 0$

$2r^2 - 4r + 2 = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$y = c_1 e^{-x} + c_2 x e^{-x}$

3. (10pnts) $y'' + y' - 6y = 0$

$r^2 + r - 6 = 0$

$(r-2)(r+3) = 0$

$r = 2 \quad r = -3$

$y = c_1 e^{2x} + c_2 e^{-3x}$

4. (15pnts) Use the method of **undetermined coefficients** to find a particular solution of the equation $y'' + 2y' + 3y = 2x^2 - 1$.

$y = Ax^2 + Bx + C$

$y' = 2Ax + B$

$y'' = 2A$

$2A + 2(2Ax + B) + 3(Ax^2 + Bx + C) = 2x^2 - 1$

$3Ax^2 + (4A + 3B)x + 2A + 2B + 3C = 2x^2 - 1$

$3A = 2$

$A = \frac{2}{3}$

$4A + 3B = 0$

$3B = -4A$

$B = -\frac{4}{3}A$

$B = \frac{-8}{9}$

$2A + 2B + 3C = -1$

$3C = -1 - 2A - 2B$

$C = -\frac{1}{3} - \frac{2}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{-8}{9}\right)$

$= -\frac{1}{3} - \frac{4}{9} + \frac{16}{27}$

$= \frac{-9 - 12 + 16}{27}$

$C = -\frac{5}{27}$

$y = \frac{2}{3}x^2 - \frac{8}{9}x - \frac{5}{27}$

5. (15 pnts) Use the method of **variation of parameters** to find a particular solution of the equation $y'' - 3y' + 2y = 10e^{-2x}$.

$$r^2 - 3r + 2 = 0$$
$$(r-2)(r-1) = 0$$

$$y = Ae^{2x} + Be^{x}$$
$$y' = 2Ae^{2x} + Be^{x} \qquad A'e^{2x} + B'e^{x} = 0$$
$$y'' = 2A'e^{2x} + B'e^{x} + 4Ae^{2x} + Be^{x}$$
$$2A'e^{2x} + B'e^{x} + \cancel{4Ae^{2x}} + \cancel{Be^{x}} - 3(\cancel{2Ae^{2x}} + \cancel{Be^{x}})$$
$$+ 2(\cancel{Ae^{2x}} + \cancel{Be^{x}}) = 10e^{-2x}$$

$$A'e^{2x} + B'e^{x} = 0$$
$$2A'e^{2x} + B'e^{x} = 10e^{-2x}$$

$$A' = \frac{\begin{vmatrix} 0 & e^{x} \\ 10e^{-2x} & e^{x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix}} = \frac{-10e^{-x}}{-e^{3x}} = 10e^{-4x}; \quad A = -\frac{5}{2}e^{-4x}$$

$$B' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 10e^{-2x} \end{vmatrix}}{-e^{3x}} = \frac{10}{-e^{3x}} = -10e^{-3x}, \quad B = \frac{10}{3}e^{-3x}$$

$$y = \left(-\frac{5}{2}e^{-4x}\right)e^{2x} + \frac{10}{3}e^{-3x}e^{x}$$

$$y = -\frac{5}{2}e^{-2x} + \frac{10}{3}e^{-2x} = \frac{5}{6}e^{-2x}$$

6. (15pnts) Solve the initial value problem $y'' + 4y = 4$, $y(0) = 1$, $y'(0) = 1$

$y(x) = 1 + \sin x$

$y_h = A\cos x + B\sin x$

$y_p = A \Rightarrow 4A = 1$

$A = \frac{4}{4} = 1$

$y = 1 + A\cos x + B\sin x$

$y(0) = 1 + A = 1$

$A = 0$

$y'(x) = -\cancel{A\sin x} + B\cos x$

$y'(0) = B = 1$

7. (15pnts) Find a particular solution of $y'' - 4y = 4e^{2x}$.

$r^2 - 4 = 0$

$r = 2,\ r = -2$

$y = Axe^{2x}$

$y' = Ae^{2x} + 2Axe^{2x} = A(1+2x)e^{2x}$

$y'' = 2A(1+2x)e^{2x} + 2Ae^{2x}$

$= A(4 + 4x)e^{2x}$

$A(4 + \cancel{4x})e^{2x} - 4(\cancel{Axe^{2x}}) = 4e^{2x}$

$4A = 4$

$A = 1$

$y = xe^{2x}$

8. (10pnts) Find all **positive** eigenvalues λ and associated eigenfunctions of the eigenvalue problem $9y'' + \lambda y = 0$, y(0)=y(π)=0.

$$y'' + \frac{\lambda}{9} y = 0$$

$$y = A\cos\left(\frac{\sqrt{\lambda}}{3}x\right) + B\sin\left(\frac{\sqrt{\lambda}}{3}x\right)$$

$$y(0) = A = 0$$

$$y(\pi) = B\sin\left(\frac{\sqrt{\lambda}}{3}\pi\right) = 0$$

$$\frac{\sqrt{\lambda}}{3}\pi = n\pi$$

$$\sqrt{\lambda} = 3n$$

$$\lambda_n = 9n^2$$

$$y_n = \sin(nx)$$