Math 3113-006 Examination 1

Name Key

ID# ____________________

pnts

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1. Solve the following initial value problem.

$$x\frac{dy}{dx}-2y=x^3\sin(x),\ y(\frac{\pi}{2})=1$$

$$\frac{dy}{dx}-2x^{-1}y=x^2\sin x$$

$$\rho=\exp\int(-2x^{-1})\,dx=\exp(-2\ln x)=x^{-2}$$

$$(x^{-2}y)'=\sin x$$

$$y=x^2\left(\frac{4}{\pi^2}-\cos x\right)$$

$$x^{-2}y=-\cos x+C$$

$$y=Cx^2-x^2\cos x$$

$$1=C\,\frac{\pi^2}{4}\ \rightarrow\ C=\frac{4}{\pi^2}$$

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2. Find the general solution of the following

$$\frac{dy}{dx}=\frac{x(2+\sqrt{x})}{y(3+\sqrt{y})}.$$

$$(3y+y^{3/2})\,dy=(2x^2+x^{3/2})\,dx$$

$$\frac{3}{2}y^2+\frac{2}{5}y^{5/2}=\frac{2}{3}x^3+\frac{2}{5}x^{5/2}+C$$

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3. Find the general solution of the following

$$(y+\cos(x))dx+(x-\sin(y))dy=0.$$

$M = y + \cos x$, $N = x - \sin y$

$\frac{\partial M}{\partial y} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = 1$

$\frac{\partial F}{\partial x} = y + \cos x \Rightarrow F(x,y) = xy + \sin x + C_0(y)$

$\frac{\partial F}{\partial y} = x + C_0'(y) = x - \sin y \Rightarrow C_0'(y) = -\sin y$

$C_0(y) = \cos y + c$

$F(x,y) = xy + \sin x + \cos y + C$

$xy + \sin x + \cos y = C$

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4. Find the solution to the initial value problem

$$xyy' = y^2 + x\sqrt{16x^2+y^2}, y(1)=0.$$

$y' = \frac{y}{x} + \sqrt{16\left(\frac{x}{y}\right)^2+1}$

$v = \frac{y}{x}$

$xv = y$

$y' = xv' + v$

$xv' + \cancel{v} = \cancel{v} + \sqrt{16\frac{1}{v^2}+1}$

$x\frac{dv}{dx} = \frac{\sqrt{v^2+16}}{v}$

$\int \frac{v}{\sqrt{v^2+16}} = \int \frac{dx}{x}$

$\left[\left(\frac{y}{x}\right)^2+16\right]^{\frac{1}{2}} = \ln x + C$

$C = 4$

$\left(\left(\frac{y}{x}\right)^2+16\right)^{\frac{1}{2}} = \ln x + 4$

$u = v^2 + 16$

$du = 2v\,dv$

$\frac{1}{2}du = v\,dv$

$\frac{1}{2}\int u^{-\frac{1}{2}} = \ln x + C$

$\cancel{\tfrac{1}{2}}\ \frac{u^{\frac{1}{2}}}{\cancel{\tfrac{1}{2}}} = \ln x + C \Rightarrow (v^2+16)^{\frac{1}{2}} = \ln x + C$

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5. Use the substitution $v = y^{-1}$ to find the general solution of the equation $y' + 2xy = xy^2$.

$v^{-1} = y$

$v' = -y^{-2} y'$

$= -v^2 y'$

$-v^{-2} v' = y'$

$-v^{-2} v' + 2x v^{-1} = x v^{-2}$

$v' - 2x v = -x$

$\rho = e^{\int -2x \, dx} = e^{-x^2}$

$(e^{-x^2} v)' = -x e^{-x^2}$

$u = -x^2$

$du = -2x \, dx$

$+\frac{1}{2} du = -x \, dx$

$e^{-x^2} v = -\int x e^{-x^2} dx$

$= \frac{1}{2} \int e^u du$

$e^{-x^2} v = \frac{1}{2} e^{-x^2} + C$

$v = \frac{1}{2} + C e^{x^2}$

$y^{-1} = \frac{1}{2} + C e^{x^2}$

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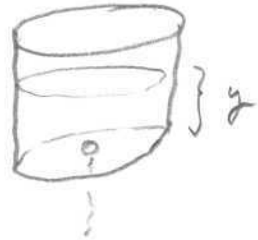
6. Suppose that a cylindrical tank initially containing 100 gallons of water drains (through a bottom hole) in 10 minutes. Use Torricelli's law ($v = \sqrt{2gy}$ where v is exit velocity and y is water depth to the hole) to show that the volume of water in the tank after $t \le 10$ minutes is $V = 100[1 - \frac{t}{10}]^2$.

$V_0 = 100$



$V = \pi R^2 y$

$\frac{dV}{dt} = \sqrt{2gy}\ a$

$\frac{V}{\pi R^2} = y$

$\frac{dV}{dt} = a\sqrt{2g}\left(\frac{V}{\pi R^2}\right)^{\frac{1}{2}}$

$\frac{dV}{dt} = \left(\frac{a}{R}\sqrt{\frac{2g}{\pi}}\right) V^{\frac{1}{2}}$

Set $k = \frac{a}{R}\sqrt{\frac{2g}{\pi}}$

$V^{-\frac{1}{2}} dV = k\, dt$

$2V^{\frac{1}{2}} = kt + c$

$V^{\frac{1}{2}} = \frac{k}{2}t + c$

$V(t) = \left[\frac{k}{2}t + c\right]^2$

$V(0) = 100 = c^2 \Rightarrow c = 10$

$V(t) = \left(\frac{k}{2}t + 10\right)^2$

$V(10) = 0 \Rightarrow \left(\frac{k(10)}{2} + 10\right)^2 = 0$

$5k + 10 = 0$

$k = -2$

$$V(t) = (10 - t)^2 = 100\left(1 - \frac{t}{10}\right)^2$$

$\frac{10}{100 \text{ pnts}}$

7. If in a culture of yeast, with initial population p_0, the rate of growth is proportional to the population $p(t)$ present at time t and the population doubles in 1 day, how much can be expected in 1 week at the same rate of growth?

$$\frac{dp}{dt} = kp$$

$$p(t) = p_0 e^{kt}$$

$$p(1) = 2p_0$$

$$2p_0 = p_0 e^{k}$$

$$\ln 2 = k$$

$$p(t) = p_0 e^{t \ln 2}$$

$$= p_0 e^{\ln 2^t} = p_0 2^t = p(t)$$