1. If $H$ is a finite index subgroup in a group $G$, prove that there is a subgroup $N$ of $G$ contained in $H$ and of finite index in $G$ such that $aNa^{-1} = N$ for all $a \in G$.

2. If $N$ is a normal subgroup of a group $G$, $N$ is finite, $H$ is a subgroup of $G$ of finite index and the index $[G : H]$ and the order $|N|$ are relatively prime, prove that $N \subset H$.

3. Let $\alpha = (12)(34)$ and $\beta = (24)$. Show that the group generated by $\alpha$ and $\beta$ is isomorphic to $D_4$.

4. For each bilinear symmetric function $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ below given by its matrix in the standard basis determine the dimension of the kernel and the maximal dimension of a subspace on which the function is positive definite.

- a) \[
\begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}
\]
- b) \[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]
- c) \[
\begin{pmatrix}
1 & -3 \\
-3 & 2
\end{pmatrix}
\]

5. Consider $V = \mathbb{C}$ as a vector space of dimension 2 over $\mathbb{R}$.

- a) Show that $\alpha : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by $\alpha(z, w) = Re(z\bar{w})$ is a positive definite bilinear symmetric function.
- b) Let $z \in \mathbb{C}$ and $L_z : \mathbb{C} \to \mathbb{C}$ be the map $w \to zw$. What is the matrix of $L_z$ with respect to the basis $(1, i)$ of $\mathbb{C}$ over $\mathbb{R}$?
- c) For which complex numbers $z$ do we have $\alpha(L_z(w_1), L_z(w_2)) = \alpha(w_1, w_2)$ for any $w_1, w_2 \in \mathbb{C}$?