Math 4333, Final

Due May 7, 2003, 1:30pm

Show all your work to receive full credit. Good luck!

1. a) Prove that the set of all polynomials all of whose coefficients are even is a prime ideal in \(\mathbb{Z}[x]\).
   
   b) Let \(I = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}, a - b \text{ is even}\}\). Show that \(I\) is a maximal ideal in \(\mathbb{Z}[\sqrt{-5}]\).

2. a) Describe all elements in \(\mathbb{Q}(\pi)\).
   b) Let \(F = \mathbb{Q}(\pi^3)\). Find a basis for \(F(\pi)\) over \(F\).
   c) Show that \(\mathbb{Q}(\sqrt{2})\) is not field isomorphic to \(\mathbb{Q}(\pi)\).

3. a) Show that \(\sqrt{2} + \sqrt[3]{5}\) is algebraic over \(\mathbb{Q}\).
   b) What is the degree \([\mathbb{Q}(\sqrt{2} + \sqrt[3]{5}) : \mathbb{Q}]\)?

4. Suppose that \([E : \mathbb{Q}] = 2\). Show that there is an integer \(d\) such that \(E = \mathbb{Q}(\sqrt{d})\) and \(d\) is not divisible by the square of any prime.

5. Let \(a\) be a complex number that is algebraic over \(\mathbb{Q}\) and let \(p(x)\) denote the minimal polynomial for \(a\) over \(\mathbb{Q}\). Show that \(\sqrt{a}\) is algebraic over \(\mathbb{Q}\) and determine the minimal polynomial for \(\sqrt{a}\) over \(\mathbb{Q}\).

6. a) Suppose that \(F\) is a field of order 1024 and \(F^* = \langle \alpha \rangle\). List the elements of each subfield of \(F\).
   b) Suppose \(L, K\) are subfields of \(\mathbb{F}_{p^n}\). If \(L\) has \(p^s\) elements and \(K\) has \(p^t\) elements, how many elements does \(L \cap K\) have?

7. Use the fact that
   
   \[4\cos^2(2\pi/5) + 2\cos(2\pi/5) - 1 = 0\]

   to prove that a regular pentagon is constructible.
8. a) Let \( f(x) \in F[x] \) and let the zeros of \( f(x) \) be \( a_1, a_2, ... a_n \). If \( K = F(a_1, a_2, ... a_n) \), show that \( \text{Gal}(K/F) \) is isomorphic to a subgroup of the group of permutations of the \( a_i \)'s.

b) Show that the Galois group of a polynomial of degree \( n \) has order dividing \( n! \).

9. Let \( E \) be the splitting field of \( x^4 + 1 \) over \( \mathbb{Q} \).

a) Find \( \text{Gal}(E/F) \).

b) Find all subfields of \( E \).

c) Find automorphisms of \( E \) that have fixed fields \( \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(i) \).

d) Is there automorphism of \( E \) whose fixed field is \( \mathbb{Q} \)?

10. a) Find the splitting field \( E_1 \) of \( p_1(x) = x^3 - 1 \) over \( \mathbb{Q} \) and \( E_2 \) of \( p_2(x) = x^3 - 2 \) over \( \mathbb{Q} \).

b) Find \( \text{Gal}(E_1/\mathbb{Q}) \) and \( \text{Gal}(E_2/\mathbb{Q}) \).