Review Problems, MATH 3333, Midterm II

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1 Topics covered

1. Vector Spaces
2. Subspaces
3. Spanning Sets and Linear Independence
4. Basis and Dimension
5. Rank of a Matrix, Systems of Linear Equations
6. Inner Product Spaces
7. Orthonormal Bases (Gram - Schmidt Process)

2 Problems

1) Write \( v = (-1,7,2) \) as a linear combination of \( u_1 = (1,3,5), u_2 = (2,-1,3), u_3 = (-3,2,-4) \).

2) Determine whether the given set with indicated operations is a vector space
   
a) The set of all fifth-degree polynomials with the standard operations;
   
b) The set of all polynomials with the standard operations;
   
c) The set of \( 2 \times 2 \) diagonal matrices with the standard operations.

3) Determine whether the set \( W \) is a subspace of \( \mathbb{R}^3 \) with the standard operations
   
a) \( W = \{(s,t,s-t)|s,t \in \mathbb{R}\} \);
b) \( W = \{(s,t, st)|s, t \in \mathbb{R}\} \).

4) Determine whether the set \( S \) is linearly independent

a) \( S = \{(6,7,6),(3,2,-4),(1,-3,2)\} \) in \( \mathbb{R}^3 \);

b) \( S = \{2 - x, 2x - x^2, 6 - 5x + x^2\} \) in \( P_2 \).

5) Determine whether the set \( S \) is a basis

a) \( S = \{(1,5,3),(0,1,2),(0,0,6)\} \) in \( \mathbb{R}^3 \);

b) \( S = \{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\} \) in \( P_3 \).

6) Find a basis and the dimension of the subspace \( W = \{(0,6t,t - s,2t)|s, t \in \mathbb{R}\} \) of \( \mathbb{R}^4 \).

7) Prove that if \( S = \{v_1, v_2, \ldots, v_n\} \) is a basis for a vector space \( V \) and \( c \) is a non-zero scalar, then \( S_1 = \{cv_1, cv_2, \ldots, cv_n\} \) is also a basis of \( V \).

8) Find a basis and the dimension of the solution space of \( Ax = 0 \), where
\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}
\]

9) Let \( u = (1,1,2), v = (-1,3,0) \) be vectors in \( \mathbb{R}^3 \). Find

a) \( ||u||^2 \);

b) a vector \( w \) in the direction of \( u \) of length 2;

c) distance between \( u \) and \( v \);

d) \( (u \cdot v)v \);

e) the angle between \( u \) and \( v \).

10) Prove that if \( u \) and \( v \) are vectors in \( \mathbb{R}^n \) then
\[
u \cdot v = 1/4 ||u + v|| - 1/4 ||u - v||.
\]

11) Consider the set \( C[-1,1] \) of continuous functions on the interval \([ -1,1 ] \) with the inner product given by \( \langle f, g \rangle = \int_{-1}^1 f(x)g(x) \, dx \). If \( f(x) = x \) and \( g(x) = e^x \), find
a) \|f\|;

b) \(d(f, g)\);

c) \(<f, g>\);

d) \(proj_{fg}\).

12) Is \(<u, v> = u_1 v_1 - u_2 v_2\) an inner product on \(\mathbb{R}^2\)?

13) Find the coordinates of \(x = (2, -1, 4, 3)\) with respect to the orthonormal basis \(B = \{(5/13, 0, 12/13, 0), (0, 1, 0, 0), (-12/13, 0, 5/13, 0), (0, 0, 0, 1)\}\).

14) Use Gram-Schmidt orthonormalization process to transform the given basis \(B = \{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}\) of \(\mathbb{R}^3\) into an orthonormal basis.

15) Use the inner product \(<u, v> = 2u_1 v_1 + u_2 v_2\) in \(\mathbb{R}^2\) and the Gram-Schmidt orthonormalization process to transform \{(2, -1), (-2, 10)\} into an orthonormal basis.