Review Problems, MATH 3333, Final Exam

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1 Topics covered

1. Vector Spaces
2. Subspaces
3. Spanning Sets and Linear Independence
4. Basis and Dimension
5. Rank of a Matrix, Systems of Linear Equations
6. Inner Product Spaces
7. Orthonormal Bases (Gram - Schmidt Process)
8. Linear Transformations, their Kernel and Range
9. Matrices for Linear Transformations
10. Transition Matrices and Similarity
11. Eigenvalues and Eigenvectors
12. Diagonalization of Matrices

2 Problems

1) Determine if the following set of vectors in $M_{2\times 2}$ is a basis

\[
\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -2 & 7 \end{pmatrix}
\]

2) Consider the set $S = \{1 + x, x + x^2, 1 + 3x + 2x^2\}$ of polynomials
a) determine if the set $S$ is linearly independent;

b) find a basis of $\text{Span}(S)$;

c) find an orthonormal basis of $\text{Span}(S)$ with respect to the inner product

$$<p,q> = \int_0^1 p \cdot q \, dx.$$ 

3) Find a basis in $\mathbb{R}^3$ that includes $S = \{(1,0,2), (0,1,1)\}$.

4) Given a matrix $A = \begin{pmatrix}
1 & 0 & 7 & 0 \\
0 & 1 & 1 & 1 \\
1 & 2 & 11 & 2 \\
0 & 2 & 1 & 2
\end{pmatrix}$

a) find a basis for the row space of $A$;

b) find a basis for the column space of $A$;

c) find a basis for the solution space of a homogeneous system of linear equations $Ax = 0$;

d) find an orthonormal basis for the solution space in part c).

5) Consider a system of linear equations $Ax = b$, where

$$A = \begin{pmatrix}
2 & -4 & 5 \\
-7 & 14 & 4 \\
3 & -6 & 1
\end{pmatrix}, b = \begin{pmatrix}
8 \\
-28 \\
12
\end{pmatrix}$$

a) determine if the system is consistent;

b) if consistent, write the solution as $x = x_h + x_p$ the sum of a particular solution and a solution to the corresponding homogeneous system.

6) Let $B = \{(2,-2),(6,3)\}$, $B' = \{(1,1),(32,31)\}$ be two bases of $\mathbb{R}^2$.

a) write a transition matrix from $B$ to $B'$;

b) given $[v]_{B'} = (2,-1)$ find $[v]_B$.

7) 

a) find all the vectors in $\mathbb{R}^4$ orthogonal to $u = (1,-1,0,0)$;

b) prove that all such vectors form a subspace $u^\perp$ of $\mathbb{R}^4$;
c) find the dimension and a basis of $u^\perp$;

d) find an orthonormal basis of $u^\perp$.

8) For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (2x + 4z, x + y - z, -x - 3y + 7z)$$

a) write a standard matrix $A$ corresponding to $T$;

b) find the kernel and nullity of $T$;

c) find the range and the rank of $T$.

9) Find the kernel of a linear transformation $T$

a) $T : P_3 \to P_2$, $T(f) = f'$;

b) $T : P_2 \to \mathbb{R}$, $T(f) = \int_0^1 fdx$.

10) Let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x, y) = (2x + y, x - y)$$

a) find the standard matrix $A$ of $T$;

b) find the matrix $A'$ of $T$ relative to the basis $B' = \{(1, -1), (0, 2)\}$.

11) Let $B = \{1, x, \exp x, x \exp x\}$ be a basis of subspace $W$ of continuous functions and $D_x$ be the differential operator on $W$. Find the matrix of $D_x$ relative to $B$.

12) Let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x, y) = (x + y, 4y)$$

and $B' = \{(-4, 1), (1, -1)\}$

a) find the matrix $A'$ of $T$ relative to the basis $B'$;

b) show that $A'$ is similar to the standard matrix $A$ of $T$.

13) Let $B = \{(1, 1), (-2, 3)\}$, $B' = \{(1, -1), (0, 1)\}$ be bases for $\mathbb{R}^2$ and $A = \left(\begin{array}{cc} 3 & 2 \\ 0 & 4 \end{array}\right)$ be a matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to $B$. 

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a) find the transition matrix $P$ from $B'$ to $B$;

b) let $[v]_{B'} = (1, -3)$, find $[v]_B$;

c) find $[T(v)]_B$;

d) find $[T(v)]_{B'}$

14) Let $A = \begin{pmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{pmatrix}$

a) write the characteristic equation of $A$;

b) find the eigenvalues of $A$;

c) find the eigenvectors of $A$.

15) For the matrix

$A = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}$

find a matrix $P$, such that $P^{-1}AP$ is diagonal.

16) Find a basis $B$ in $\mathbb{R}^2$ such that the matrix of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (5x - 3y, 6x - 4y)$ is diagonal.