Math 2423, Test I

September 29, 2004

Show all your work to receive full credit. The use of books and notes is not allowed. Good luck!

I. (20 pts) Estimate the area under the graph of \( f(x) = 1 + \frac{1}{x} \) from \( x = 1 \) to \( x = 5 \) using 4 approximating rectangles and right endpoints.

The graph of \( f(x) \) looks like the graph of \( \frac{1}{x} \) moved up one unit.

\[
R_4 = \frac{3}{2} \left( 1 \right) + \frac{4}{3} \left( 1 \right) + \frac{5}{4} \left( 1 \right) + \frac{6}{5} \left( 1 \right)
\]

\[
= \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} \quad \text{← Good Enough}
\]

\[
= \frac{9}{6} + \frac{8}{6} + \frac{15}{20} + \frac{24}{20}
\]

\[
= \frac{17}{6} + \frac{49}{20} = \frac{170}{60} + \frac{147}{60} = \frac{317}{60}
\]
II. (20 pts) Find the indefinite integrals

a) \( \int (\sqrt{y} + \frac{1}{\sqrt{y}})^2 \frac{dy}{y} = \int (y^2 + 2 + \frac{1}{y^2}) \frac{dy}{y} = \int (y^2 + 2 + y^{-2}) \frac{dy}{y} \)
   \[
   = \frac{1}{3} y^3 + 2y - y^{-1} + C \\
   = \frac{1}{3} y^3 + 2y - \frac{1}{y} + C
   \]

b) \( \int \cos x \cos(\sin x) \frac{dx}{x} \)
   
   Let \( u = \sin x \)
   
   \( du = \cos x \frac{dx}{x} \)
   
   \( = \cos(\sin x) + C \)

\[\]

c) \( \int \frac{x}{\sqrt{x^2 + 1}} \frac{dx}{u} = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2} \frac{du}{\sqrt{u}} = \frac{1}{2} \left( \frac{u^{1/2}}{1/2} \right) + C \)
   
   Let \( u = x^2 + 1 \)
   
   \( du = 2x \frac{dx}{x} \)
   
   \( \Rightarrow \frac{1}{2} \frac{du}{\sqrt{u}} = x \frac{dx}{x} \)

\[\]

d) \( \int \sqrt{y-1} \frac{dy}{u} = \int (u+1) \frac{du}{\sqrt{u}} = \int (u+1) \frac{du}{u^{1/2}} \)
   
   Let \( u = y - 1 \)
   
   \( du = dy \)

   Also note that if \( u = y - 1, \ y = u + 1 \)

   \( = \int [u^{1/2} + u^{-1/2}] \frac{du}{\sqrt{u}} = \int [u^{3/2} + u^{-1/2}] \frac{du}{\sqrt{u}} = \frac{u^{3/2}}{3/2} + \frac{u^{-1/2}}{-1/2} + C \)

   \( = \frac{2}{3} (y-1)^{3/2} + \frac{2}{3} (y-1)^{-1/2} + C \)
III. (20 pts) Find the area of the region bounded by the curves \( y^2 - y = x \) and \( y = x \).

First we need to graph \( y^2 - y = x \) and \( y = x \).

To this end, let us complete the square of \( y^2 - y = x \).

\[
\begin{align*}
\quad & y^2 - y = x \Rightarrow y^2 - y + \frac{1}{4} - \frac{1}{4} = x \\
\Rightarrow & (y - \frac{1}{2})^2 - \frac{1}{4} = x.
\end{align*}
\]

So the graph of \( x = y^2 - y \) is a parabola that has the same shape as \( x = y^2 \) but with vertex \( (-\frac{1}{4}, \frac{1}{2}) \).

Now to find the points of intersection:

\[
\begin{align*}
\quad & y^2 - y = y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y - 2) = 0 \\
\Rightarrow & y = 0 \text{ or } y = 2.
\end{align*}
\]

So,

\[
A = \int_{0}^{2} (y - (y^2 - y)) \, dy = \int_{0}^{2} (-y^2 + 2y) \, dy
\]

\[
\begin{align*}
= & \left[ -\frac{1}{3} y^3 + y^2 \right]_{0}^{2} = -\frac{1}{3}(8) + 4 - (-\frac{1}{3}(0) + 0) \\
= & -\frac{8}{3} + 4 \\
= & -\frac{8}{3} + \frac{12}{3} = \frac{4}{3}.
\end{align*}
\]
IV. (20 pts) Find the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 2x$ and $y = 0$ about the line $x = 3$.

Again, we complete the square:

$$y = -x^2 + 2x \Rightarrow y = -(x^2 - 2x)$$

$$\Rightarrow y = -(x^2 - 2x + 1) + 1$$

$$\Rightarrow y = -(x-1)^2 + 1$$

This is a parabola with the same shape as $y = -x^2$ and vertex $(1,1)$. It intersects $y = 0$ ($x$-axis) when $-x^2 + 2x = 0 \Rightarrow x(-x+2) = 0 \Rightarrow x = 0$ or $x = 2$.

The solid looks like the top half of a torus (donut). Washer method is cumbersome because we would have to integrate with respect to $y$, and everything is in terms of $x$. We will use shells. Typical shell looks like

$$V = \int_0^2 2\pi (3-x)(-x^2 + 2x)dx = \int_0^2 2\pi (-3x^2 + 6x + x^3 - 2x^2)dx$$

$$= 2\pi \int_0^2 (x^3 - 5x^2 + 6x)dx = 2\pi \left[ \frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 \right]_0^2$$

$$= 2\pi \left( \frac{1}{4}(16) - \frac{5}{3}(8) + 3(4) \right) = 2\pi \left( 4 - \frac{40}{3} + 12 \right) = 2\pi (16 - \frac{40}{3})$$

$$= 2\pi \left( \frac{48}{3} - \frac{40}{3} \right) = 2\pi \left( \frac{8}{3} \right) = \frac{16\pi}{3}.$$
V. (20 pts) An aquarium 2m long, 1m wide and 2m deep is half full of water. Find the work needed to pump all the water out of the aquarium. (Use $\rho_{\text{water}} = 1000 \text{ kg/m}^3$ and $g = 10 \text{ m/sec}^2$).

First find the work $W_i$ to lift the $i^{th}$ slice of water out of the aquarium.

$$W_i = F_i \cdot d_i = F_i \cdot (2 - y_i)$$

$$F_i = m_i \cdot acc \approx m_i \cdot (10)$$

$$m_i = V_i \cdot \rho_{\text{water}} = V_i \cdot (1000)$$

$$V_i = (1)(2) \Delta y = 2 \Delta y$$

So $m_i = 2000 \Delta y \Rightarrow F_i = 20000 \Delta y \Rightarrow W_i = 20000 (2 - y_i) \Delta y$

Thus, the total work required is

$$W = \int_0^1 20000 (2 - y) \, dy = \int_0^1 (40000 - 20000y) \, dy$$

$$= \left[40000y - 10000y^2\right]_0^1$$

$$= 40000 - 10000 = 30000.$$