The Steinberg Tensor Product Theorem for $GL(m|n)$

Jonathan Kujawa

Dedicated to James E. Humphreys on the occasion of his 65th birthday.

1. Introduction

The Steinberg Tensor Product Theorem is a fundamental result in the modular representation theory of algebraic groups. The purpose of the present article is to formulate and prove the analogous theorem for the supergroup $GL(m|n)$. This result was first mentioned without proof in [**2**]. We emphasize that our approach closely parallels the analogous result for the supergroup $Q(n)$ proven by Brundan and Kleshchev [**1**], which in turn follows the approach of Cline, Parshall, and Scott [**3**].

The preliminaries are outlined in section 2. They are an abbreviated form of what can be found in [**2**] and [**7**]. Sections 3 and 4 contain the new results of the present article with the main theorem being the following version of the Steinberg Tensor Product Theorem.

Before stating the result, we require some notation. We direct the reader to section 2 for precise statements of definitions. Throughout, let k be a fixed ground field of characteristic $p > 0$ which is algebraically closed. All objects under discussion are defined over k. Let T be the maximal torus of $GL(m|n)$ consisting of diagonal matrices. We identify the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ with the free abelian group on generators $\varepsilon_1, \dots, \varepsilon_{m+n}$, where ε_i picks out the ith entry of a diagonal matrix. We call the set

$$X^+(T) := \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X(T) \;:\; \lambda_1 \geq \cdots \geq \lambda_m \text{ and } \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \right\},$$

the set of *dominant weights*. The irreducible $GL(m|n)$-supermodules are parameterized by highest weight by the set $X^+(T)$ and we write $L(\lambda)$ for the irreducible supermodule of highest weight $\lambda \in X^+(T)$. A weight is *p-restricted* if it is dominant and $\lambda_i - \lambda_{i+1} < p$ for $i = 1, \dots, m-1$ and $i = m+1, \dots, m+n-1$. Denote the set of p-restricted weights by $X_p^+(T)$.

Let $F : GL(m|n) \to GL(m) \times GL(n)$ be the *Frobenius morphism* given by raising entries to the pth power. Given a $GL(m) \times GL(n)$-supermodule M we

---

2000 *Mathematics Subject Classification.* 20G05.

Research supported in part by NSF grant DMS-0402916.

can view it as a $GL(m|n)$-supermodule via inflation through F. We call this the *Frobenius twist* of M and denote by F^*M.

THEOREM 1.1 (Steinberg Tensor Product Theorem). *For* $\lambda \in X_p^+(T)$ *and* $\mu \in X^+(T)$,

$$L(\lambda + p\mu) \cong L(\lambda) \otimes F^*L'(\mu),$$

where $L'(\mu)$ *denotes the irreducible* $GL(m) \times GL(n)$*-supermodule of highest weight* μ.

Acknowledgements. This work was done as part of the author's Ph.D. thesis at the University of Oregon [**8**]. The author is grateful for the guidence and patience of his advisor, Jonathan Brundan.

2. Definitions and Basic Results

In this section we outline the basic definitions and results we require. For an account of the basic language of superalgebras and supergroups adopted here, we refer the reader to [**1**], [**2**], and [**7**]; see also [**4**], [**5**], [**9**, ch.I] and [**10**, ch.3, §§1–2, ch.4, §1].

2.1. The supergroup $GL(m|n)$. We use the language of supergroup schemes to define $GL(m|n)$. Our approach parallels that of [**4**]. Throughout, let k be an algebraically closed field of characteristic $p > 0$. All objects (superalgebras, supergroups, ...) will be defined over k. A *superspace* is a $\mathbb{Z}_2$-graded k-vector space. If V is a superspace and $v \in V$ is a homogeneous vector, then we write $\overline{v} \in \mathbb{Z}_2$ for the degree of v. A *commutative superalgebra* is a $\mathbb{Z}_2$-graded associative algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with $ab = (-1)^{\bar{a}\bar{b}}ba$ for all homogeneous $a, b \in A$. If $p = 2$ we also assume that $a^2 = 0$ for all $a \in A_{\bar{1}}$. A morphism of superalgebras is a homomorphism of graded algebras; that is, it is an algebra homomorphism which preserves the $\mathbb{Z}_2$-grading.

The supergroup $G = GL(m|n)$ is the functor from the category of commutative superalgebras to the category of groups defined on a commutative superalgebra A by letting $G(A)$ be the group of all invertible $(m+n) \times (m+n)$ matrices of the form

$$g = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right) \tag{2.1}$$

where W is an $m \times m$ matrix with entries in $A_{\bar{0}}$, X is an $m \times n$ matrix with entries in $A_{\bar{1}}$, Y is an $n \times m$ with entries in $A_{\bar{1}}$, and Z is an $n \times n$ matrix with entries in $A_{\bar{0}}$. If $f : A \to B$ is a superalgebra homomorphism, then $G(f) : G(A) \to G(B)$ is the group homomorphism defined by applying f to the matrix entries.

Let Mat be the affine superscheme with $Mat(A)$ consisting of *all* (not necessarily invertible) $(m+n)\times(m+n)$ matrices of the above form. For $1 \leq i, j \leq m+n$, let $T_{i,j}$ be the morphism defined by having $T_{i,j} : Mat(A) \to A$ map a matrix to its ij-entry. Then the coordinate ring $k[Mat]$ is the free commutative superalgebra on the generators $T_{i,j}$ $(1 \leq i, j \leq m+n)$ with $T_{i,j}$ having parity $\overline{i} + \overline{j}$, where we write $\overline{i} = \bar{0}$ for $i = 1, \dots, m$ and $\overline{i} = \bar{1}$ for $i = m+1, \dots, m+n$. By [**9**, I.7.2], a matrix $g \in Mat(A)$ of the form (2.1) is invertible if and only if $\det W \det Z \in A^{\times}$, where here det denotes the usual matrix determinant. Hence, G is the principal open subset of Mat defined by the function $\det : g \mapsto \det W \det Z$. In particular, the coordinate ring $k[G]$ is the localization of $k[Mat]$ at det.

Just as for group schemes [**4**, I.2.3], the coordinate ring $k[G]$ has the naturally induced structure of a Hopf superalgebra. Explicitly, the comultiplication and counit are the unique superalgebra maps satisfying

$$\Delta(T_{i,j}) = \sum_{h=1}^{m+n} T_{i,h} \otimes T_{h,j}, \tag{2.2}$$

$$\varepsilon(T_{i,j}) = \delta_{i,j} \tag{2.3}$$

for all $1 \leq i, j \leq m+n$.

By definition a *representation* of G means a natural transformation $\rho : G \to GL(M)$ for some vector superspace M, where $GL(M)$ is the supergroup with $GL(M)(A)$ being equal to the group of all even automorphisms of the A-supermodule $M \otimes A$, for each commutative superalgebra A. Equivalently, as with group schemes [**4**, I.2.8], M is a right $k[G]$-cosupermodule. That is, there is a $\mathbb{Z}_2$-grading preserving structure map $\eta : M \to M \otimes k[G]$ satisfying the usual comodule axioms. We will usually refer to such an M as a *G-supermodule*.

If $\rho : G \to GL(M)$ and $\rho' : G \to GL(M')$ are two representations of G, then a morphism of representations is a linear map $f : M \to M'$ such that for any commutative superalgebra A we have $\rho'(g)(f(m)) = f(\rho(g)(m))$ for all $g \in G(A)$ and all $m \in M \otimes A$. In the language of $k[G]$-cosupermodules, if $\eta : M \to M \otimes k[G]$ and $\eta' : M' \to M' \otimes k[G]$ are the cosupermodule structure maps, then $f : M \to M'$ is a morphism if $f \otimes 1 \circ \eta = \eta' \circ f$.

We denote by G-$\mathfrak{mod}$ the category of all G-supermodules. We emphasize that we allow *all* morphisms and not just graded (i.e. *even*) morphisms. However, note that for superspaces M and M' the space $\mathrm{Hom}_k(M, M')$ is naturally $\mathbb{Z}_2$-graded by declaring $f \in \mathrm{Hom}_k(M, M')_r$ if $f(M_s) \subseteq M'_{s+r}$ for all $s \in \mathbb{Z}_2$. This gives a $\mathbb{Z}_2$-grading on $\mathrm{Hom}_G(M, M') \subseteq \mathrm{Hom}_k(M, M')$. We remark that G-$\mathfrak{mod}$ is not an abelian category. However, the *underlying even category* of G-$\mathfrak{mod}$, consisting of the same objects as G-$\mathfrak{mod}$ but only the even morphisms, is an abelian category. This, along with the parity change functor Π, which, roughly speaking, interchanges the $\mathbb{Z}_2$-grading of a supermodule, allows one to make use of the tools of homological algebra.

The underlying purely even group G_{ev} of G is by definition the functor from superalgebras to groups given by $G_{\mathrm{ev}}(A) = G(A_{\bar{0}})$. Thus, $G_{\mathrm{ev}}(A)$ consists of all invertible matrices of the form (2.1) with $X = Y = 0$, so $G_{\mathrm{ev}} \cong GL(m) \times GL(n)$. Let T be the usual maximal torus of G_{ev} consisting of diagonal matrices. The character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ as defined in [**4**, I.2.4] can then be identified with the free abelian group on generators $\varepsilon_1, \dots, \varepsilon_{m+n}$, where ε_i is the function which picks out the ith diagonal entry of a diagonal matrix. Let B denote the subgroup of G given by letting $B(A)$ equal the set of all of all upper triangular invertible matrices of the form (2.1). We call this the *standard Borel subgroup.* Note that the underlying purely even subgroup, B_{ev}, is given by the upper triangular matrices in G_{ev}.

The root system of G is the set $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq m+n, i \neq j\}$. There are even and odd roots, the parity of the root $\varepsilon_i - \varepsilon_j$ being $\bar{i} + \bar{j}$. Our choice of Borel subgroup, B, defines a set,

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq m+n\}, \tag{2.4}$$

of positive roots. The simple roots then are $\varepsilon_i - \varepsilon_{i+1}$ where $i = 1, \ldots, m+n-1$. The corresponding dominance order on $X(T)$ is denoted $\leq$, defined by $\lambda \leq \mu$ if $\mu - \lambda$ can be written as the sum of positive roots.

2.2. The Superalgebra of Distributions. Just as for algebraic groups [**4**, I.7.7] one can abstractly define the superalgebra of distributions $\mathrm{Dist}(G)$ of G. We sketch how this is done. Let $\mathcal{I}$ be the kernel of the counit $\varepsilon : k[G] \to k$, a superideal of $k[G]$. For $r \geq 0$, let

$$\begin{aligned} \mathrm{Dist}_r(G) &= \{x \in k[G]^* \,:\, x(\mathcal{I}^{r+1}) = 0\} \cong (k[G]/\mathcal{I}^{r+1})^*, \\ \mathrm{Dist}(G) &= \bigcup_{r \geq 0} \mathrm{Dist}_r(G). \end{aligned}$$

There is a multiplication on $k[G]^*$ dual to the comultiplication on $k[G]$, defined by $(xy)(f) = (x\bar{\otimes}y)(\Delta(f))$ for $x, y \in k[G]^*$ and $f \in k[G]$. Note here (and elsewhere) we are implicitly using the superalgebra rule of signs: $(x\bar{\otimes}y)(f \otimes g) = (-1)^{\bar{y}\bar{f}} x(f)y(g)$ where y and f are assumed to be homogeneous. The general case is obtained via linearity. In fact, $\mathrm{Dist}(G)$ is a subsuperalgebra of $k[G]^*$ (see [**2**]).

In the case when $G = GL(m|n)$, however, we can describe $\mathrm{Dist}(G)$ explicitly as the reduction modulo p of the universal enveloping superalgebra of the Lie superalgebra $\mathfrak{gl}(m|n, \mathbb{C})$. We now describe how this can be done.

Recall that $\mathfrak{gl}(m|n, \mathbb{C})$ is the Lie superalgebra given by letting $\mathfrak{gl}(m|n, \mathbb{C})$ be the set of all $(m+n) \times (m+n)$ matricies over $\mathbb{C}$. If for $1 \leq i, j \leq m+n$ we write $e_{i,j}$ for the ij matrix unit, then the $e_{i,j}$ provide a homogeous basis with the degree of $e_{i,j}$ defined to be $\bar{i} + \bar{j}$. The bracket is given by

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \delta_{i,l} e_{k,j} \tag{2.5}$$

By the PBW theorem for Lie superalgebras (see [**5**]) we have that the universal enveloping superalgebra of $\mathfrak{gl}(m|n, \mathbb{C})$, $U_{\mathbb{C}}$, has basis consisting of all monomials

$$\prod_{\substack{1 \leq i,j \leq m+n \\ \bar{i}+\bar{j}=\bar{0}}} e_{i,j}^{a_{i,j}} \prod_{\substack{1 \leq i,j \leq m+n \\ \bar{i}+\bar{j}=\bar{1}}} e_{i,j}^{d_{i,j}}$$

where $a_{i,j} \in \mathbb{Z}_{\geq 0}$, $d_{i,j} \in \{0, 1\}$, and the product is taken in any fixed order. We shall write $h_i = e_{i,i}$ for short.

Define the *Kostant $\mathbb{Z}$-form* $U_{\mathbb{Z}}$ to be the $\mathbb{Z}$-subalgebra of $U_{\mathbb{C}}$ generated by elements $e_{i,j}$ $(1 \leq i, j \leq m+n, \bar{i}+\bar{j} = \bar{1})$, $e_{i,j}^{(r)}$ $(1 \leq i, j \leq m+n, i \neq j, \bar{i}+\bar{j} = \bar{0}, r \geq 1)$, and $\binom{h_i}{r}$ $(1 \leq i \leq m+n, r \geq 1)$. Here, $e_{i,j}^{(r)} := e_{i,j}^r/(r!)$ and $\binom{h_i}{r} := h_i(h_i - 1)\cdots(h_i - r + 1)/(r!)$. Following the proof of [**11**, Th.2], one verifies the following lemma.

Lemma 2.1. *The superalgebra $U_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with basis given by the set of all monomials of the form*

$$\prod_{\substack{1 \leq i,j \leq m+n \\ i \neq j, \bar{i}+\bar{j}=\bar{0}}} e_{i,j}^{(a_{i,j})} \prod_{1 \leq i \leq m+n} \binom{h_i}{r_i} \prod_{\substack{1 \leq i,j \leq m+n \\ \bar{i}+\bar{j}=\bar{1}}} e_{i,j}^{d_{i,j}}$$

for all $a_{i,j}, r_i \in \mathbb{Z}_{\geq 0}$ and $d_{i,j} \in \{0, 1\}$, where the product is taken in any fixed order.

The enveloping superalgebra $U_{\mathbb{C}}$ is a Hopf superalgebra in a canonical way, hence $U_{\mathbb{Z}}$ is a Hopf superalgebra over $\mathbb{Z}$. Consequently, $k \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$ is naturally a Hopf superalgebra over k. It is known, for example by [**2**, Thm. 3.2], that

$$\mathrm{Dist}(G) \cong k \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$$

as Hopf superalgebras. We *identify* these Hopf superalgebras and will abuse notation by using the same symbols $e_{i,j}^{(r)}, \binom{h_i}{r}$, etc. for the canonical images of these elements of $U_{\mathbb{Z}}$ in $\mathrm{Dist}(G)$. Note that the monomials given in Lemma 2.1 form a homogeneous basis of $\mathrm{Dist}(G)$.

It is also easy to describe the superalgebras of distributions of our various natural subgroups of G as subalgebras of $\mathrm{Dist}(G)$. For example, $\mathrm{Dist}(T)$ is the subalgebra generated by all $\binom{h_i}{r}$ $(1 \le i \le m+n, r \ge 1)$, $\mathrm{Dist}(B_{\mathrm{ev}})$ is the subalgebra generated by $\mathrm{Dist}(T)$ and all $e_{i,j}^{(r)}$ $(1 \le i,j \le m+n, i<j, \bar{\imath}+\bar{\jmath} = \bar{0}, r \ge 1)$, and $\mathrm{Dist}(B)$ is the subalgebra generated by $\mathrm{Dist}(B_{\mathrm{ev}})$ and all $e_{i,j}$ $(1 \le i,j \le m+n, \bar{\imath}+\bar{\jmath} = \bar{1}, i<j)$.

Let us describe the category of $\mathrm{Dist}(G)$-supermodules. The objects are all left $\mathrm{Dist}(G)$-modules which are $\mathbb{Z}_2$-graded: that is, k-superspaces, M, satisfying $\mathrm{Dist}(G)_r M_s \subseteq M_{r+s}$ for $r,s \in \mathbb{Z}_2$. A morphism of $\mathrm{Dist}(G)$-supermodules is a linear map $f : M \to M'$ satisfying $f(xm) = (-1)^{\bar{f}\,\bar{x}} x f(m)$ for all $m \in M$ and all $x \in \mathrm{Dist}(G)$. Note that this definition makes sense as stated only for homogeneous elements; it should be interpreted via linearity in the general case. We emphasize that morphisms are not necessarily even. However, the Hom-spaces are naturally $\mathbb{Z}_2$-graded and our remarks about the category G-$\mathfrak{mod}$ made in the previous subsection apply here as well.

For $\lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X(T)$ and a $\mathrm{Dist}(G)$-supermodule M, define the λ-*weight space of* M to be

$$M_\lambda = \left\{ m \in M \;:\; \binom{h_i}{r} m = \binom{\lambda_i}{r} m \text{ for all } 1 \le i \le m+n, r \ge 1 \right\}. \tag{2.6}$$

We call a $\mathrm{Dist}(G)$-supermodule M *integrable* if it is locally finite over $\mathrm{Dist}(G)$ and satisfies $M = \sum_{\lambda \in X(T)} M_\lambda$.

If M is a G-supermodule then we can view M as a $\mathrm{Dist}(G)$-supermodule as follows. Given a G-supermodule M with structure map $\eta : M \to M \otimes k[G]$, we can view M as a $\mathrm{Dist}(G)$-supermodule by $xm = (1 \bar{\otimes} x \circ \eta)(m)$. In fact, in this way we obtain a functor from G-$\mathfrak{mod}$ to the category of $\mathrm{Dist}(G)$-supermodules. Moreover, the notion of weight space defined above for $\mathrm{Dist}(G)$-supermodules coincides with the usual notion of weight space of M with respect to the torus T. It is then straightforward to verify that the G-supermodule M is integrable when viewed as a $\mathrm{Dist}(G)$-supermodule. We prove the following theorem in [**2**, Corollary 3.5].

THEOREM 2.2. *The category G-$\mathfrak{mod}$ is isomorphic to the full subcategory of integrable* $\mathrm{Dist}(G)$*-supermodules via the aforementioned functor.*

In view of this result, we will not distinguish between G-supermodules and integrable $\mathrm{Dist}(G)$-supermodules in what follows.

2.3. Classification of irreducible $GL(m|n)$-supermodules. Now we describe the classification of the irreducible representations of G by their highest weights. It seems to be more convenient to work first in the category $\mathcal{O}_p$: the full subcategory of all $\mathrm{Dist}(G)$-supermodules M such that $M = \bigoplus_{\lambda \in X(T)} M_\lambda$ and

M is locally finite over $\mathrm{Dist}(B)$. This is an analogue of Bernstein, Gelfand and Gelfand's category $\mathcal{O}$ in classical Lie theory. We remark that Theorem 2.2 implies that G-$\mathfrak{mod}$ can be viewed as a full subcategory of $\mathcal{O}_p$. From now on we will assume all $\mathrm{Dist}(G)$-supermodules under discussion are objects in $\mathcal{O}_p$.

For $\lambda \in X(T)$, we have the *Verma supermodule*

$$M(\lambda) := \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} k_\lambda,$$

where k_λ denotes k viewed as a $\mathrm{Dist}(B)$-supermodule of weight λ concentrated in degree $\bar{0}$. Note that by Lemma 2.1 it follows that $M(\lambda)$ is an object in $\mathcal{O}_p$. We say that a homogeneous vector v in a $\mathrm{Dist}(G)$-supermodule M is a *primitive vector of weight* λ if $\mathrm{Dist}(B)v \cong k_\lambda$ as a $\mathrm{Dist}(B)$-supermodule. Familiar arguments show that $M(\lambda)$ is universal among all supermodules of $\mathcal{O}_p$ which are generated by a primitive vector of weight λ and $M(\lambda)$ has a unique maximal subsupermodule, hence an irreducible quotient which we denote by $L(\lambda)$. Taken together these imply that $\{L(\lambda) : \lambda \in X(T)\}$ gives a complete set of pairwise non-isomorphic irreducibles in $\mathcal{O}_p$. In this way, we get a parametrization of the irreducible objects in $\mathcal{O}_p$ by their highest weights with respect to the ordering $\leq$.

Now we pass from $\mathcal{O}_p$ to G-$\mathfrak{mod}$. Recall that

$$X^+(T) = \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X(T) \;:\; \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \right\}$$

denotes the set of all *dominant integral weights*. The proof of the following lemma is due to Kac [**6**] (see also [**2**]).

Lemma 2.3. *Given any $\lambda \in X(T)$, $L(\lambda)$ is finite dimensional if and only if $\lambda \in X^+(T)$. In particular, the supermodules $\{L(\lambda)\}_{\lambda \in X^+(T)}$ form a complete set of pairwise non-isomorphic irreducible supermodules in G-$\mathfrak{mod}$.*

3. Frobenius Kernels

For $r \geq 1$, we define the *Frobenius morphism* $F^r : G \to G_{\mathrm{ev}}$ by having $F^r : G(A) \to G_{\mathrm{ev}}(A)$ raise each matrix entry to the p^rth power for any commutative superalgebra A. Note that for $a \in A_{\bar{1}}$, $a^{p^r} = 0$ so the morphism makes sense. Let G_r denote the kernel of F^r, the r^{th} *Frobenius kernel*, a normal subgroup of G. Similarly, let $G_{\mathrm{ev},r}$ denote the kernel of $F^r|_{G_{\mathrm{ev}}}$, B_r denote the kernel of $F^r|_B$, etc.

Lemma 3.1. *$F^r : G \to G_{\mathrm{ev}}$ is a quotient of G by G_r in the category of superschemes. That is, for any morphism $f : G \to S$ of superschemes which is constant on $G_r(A)$-cosets of $G(A)$ for all commutative superalgebras A, there is a unique morphism $\tilde{f} : G_{\mathrm{ev}} \to S$ such that $f = \tilde{f} \circ F^r$.*

Proof. Let $\pi : G \to G_{\mathrm{ev}}$ be the superscheme morphism defined by projection. That is, if $g \in G(A)$ is as in (2.1), then π acts as the identity on the entries of W and Z, and sends the entries of X and Y to zero. Let $f : G \to S$ be a morphism of superschemes which is constant on $G_r(A)$-cosets. For any element $g \in G(A)$ written as in (2.1) we have

$$\begin{pmatrix} I_m & XZ^{-1} \\ YW^{-1} & I_n \end{pmatrix}^{-1} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. That is, $hg = \pi(g)$ for some $h \in G_r(A)$. Thus $f = f|_{G_{\mathrm{ev}}} \circ \pi$. However from the purely even theory (see [**4**, I.9.5]), $F^r|_{G_{\mathrm{ev}}}$

is a quotient of $G_{\rm ev}$ by $G_{{\rm ev},r}$. Consequently, since $f|_{G_{\rm ev}}$ is constant on $G_{{\rm ev},r}$-cosets of $G_{\rm ev}$, there is a unique morphism $\tilde{f} : G_{\rm ev} \to S$ such that $f|_{G_{\rm ev}} = \tilde{f} \circ F^r|_{G_{\rm ev}}$. Therefore $f = f|_{G_{\rm ev}} \circ \pi = \tilde{f} \circ F^r|_{G_{\rm ev}} \circ \pi = \tilde{f} \circ F^r$. □

Observe that $k[G_r] \cong k[G]/I_r$ where I_r is the ideal generated by $\{T_{i,j}^{p^r}, T_{k,k}^{p^r} - 1 : 1 \le i,j,k \le m+n, i \ne j\}$. Consequently, a basis for $k[G_r]$ is given by the monomials in $T_{i,j}^{a_{i,j}}$ for $1 \le i,j \le m+n$, where $a_{i,j} \in \{0,1,\ldots,p^r-1\}$ if $\bar{i}+\bar{j} = \bar{0}$ and $a_{i,j} \in \{0,1\}$ if $\bar{i}+\bar{j} = \bar{1}$, with the product taken in any fixed order. In particular, the dimension of $k[G_r]$ is finite so by definition G_r is a *finite* algebraic supergroup. Moreover the p^r-th power of any element of $\mathcal{I} := \mathrm{Ker}(\varepsilon : k[G_r] \to k)$ lies in I_r so $\mathcal{I}$ is nilpotent. That is, G_r is *infinitesimal* and, consequently, $\mathrm{Dist}(G_r)$ can be identified with the Hopf superalgebra dual of $k[G_r]$. It follows as in [**4**, I.8.1-6] that the category of G_r-supermodules is isomorphic to the category of $\mathrm{Dist}(G_r)$-supermodules. Also, under this identification we can take as our basis for $\mathrm{Dist}(G_r) \subset \mathrm{Dist}(G)$ the ordered PBW monomials

$$\prod_{\substack{1\le i,j\le m+n\\ i<j}} e_{j,i}^{(a_{j,i})} \prod_{1\le k\le m+n} \binom{h_k}{d_k} \prod_{\substack{1\le i,j\le m+n\\ i<j}} e_{i,j}^{(a_{i,j})}, \tag{3.1}$$

where $a_{i,j}, d_k \in \{0,\ldots,p^r-1\}$ for $1 \le i,j,k \le m+n$ when $\bar{i}+\bar{j} = \bar{0}$, and $a_{i,j} \in \{0,1\}$ when $\bar{i}+\bar{j} = \bar{1}$. Similarly we can describe bases for $\mathrm{Dist}(B_r)$, etc. From this we observe the following lemma.

Lemma 3.2. $\mathrm{Dist}(G_r)$ *is a free right* $\mathrm{Dist}(B_r)$*-supermodule with basis given by the ordered monomials*

$$\prod_{\substack{1\le i,j\le m+n\\ i<j}} e_{j,i}^{(a_{j,i})},$$

where $a_{i,j} \in \{0,\ldots,p^r-1\}$ *when* $\bar{i}+\bar{j} = \bar{0}$, *and* $a_{i,j} \in \{0,1\}$ *when* $\bar{i}+\bar{j} = \bar{1}$.

Having identified the representations of G_r and B_r with $\mathrm{Dist}(G_r)$-supermodules and $\mathrm{Dist}(B_r)$-supermodules, respectively, we have the induction functor given by

$$\mathrm{ind}_{B_r}^{G_r} M = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} M.$$

From Lemma 3.2 we see this is an exact functor which is left adjoint to restriction. Given $\lambda \in X(T)$, let k_λ denote k viewed as a T_r-supermodule of weight λ concentrated in degree $\bar{0}$. The classical theory [**4**, II.3.7] gives the following lemma.

Lemma 3.3. *The set* $\{k_\lambda : \lambda \in X(T)\}$ *is a complete family of irreducible* T_r*-supermodules. Moreover,* $k_\lambda \cong k_\mu$ *if and only if* $\lambda - \mu \in p^r X(T)$.

Furthermore, $\{k_\lambda : \lambda \in X(T)\}$ provides a complete set of irreducible B_r-supermodules via inflation. For $\lambda \in X(T)$, define

$$Z_r(\lambda) = \mathrm{ind}_{B_r}^{G_r} k_\lambda.$$

Let $L_r(\lambda)$ denote the G_r-head of $Z_r(\lambda)$.

Proposition 3.4. $\{L_r(\lambda) : \lambda \in X(T)\}$ *is a complete set of irreducible* G_r*-supermodules. Furthermore,* $L_r(\lambda) \cong L_r(\mu)$ *if and only if* $\lambda - \mu \in p^r X(T)$.

Proof. Let U^- denote the unipotent radical of the lower Borel; that is, $U^-(A)$ is the subgroup of $G(A)$ given by lower triangular matricies with ones along the diagonal. Then by definition U_r^- is the kernel of F^r restricted to U^-.

Observe that by Lemma 3.2 we have that $Z_r(\lambda) \cong \operatorname{Dist}(U_r^-)$ as U_r^--supermodules. Consequently we have

$$\dim_k \operatorname{Hom}_{U_r^-}(Z_r(\lambda), k) = \dim_k \operatorname{Hom}_{U_r^-}(\operatorname{Dist}(U_r^-), k) = 1.$$

Thus $Z_r(\lambda)$ has an irreducible U_r^--head and it then follows that $Z_r(\lambda)$ has an irreducible G_r-head. That is, $L_r(\lambda)$ is irreducible.

Now if L is an irreducible G_r-supermodule then we can choose $\lambda \in X(T)$ so that $\operatorname{Hom}_{B_r}(k_\lambda, L) \neq 0$. By Frobenius reciprocity L is isomorphic to a quotient of $Z_r(\lambda)$, hence $L \cong L_r(\lambda)$. Finally, from the classification of the irreducible supermodules of B_r we see that $L_r(\lambda) \cong L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$. □

4. The Steinberg Tensor Product Theorem

We are now able to prove the Steinberg Tensor Product Theorem for $GL(m|n)$.

Lemma 4.1. *Let L be an irreducible G-supermodule. Then L is completely reducible as a G_1-supermodule.*

Proof. Let L_1 be an irreducible supermodule in the G_1-socle of L. Since G_1 is a normal subgroup of G each translate, gL_1, by an element $g \in G(k)$ is an irreducible G_1-subsupermodule of L. Thus

$$M := \sum_{g \in G(k)} gL_1$$

is a completely reducible G_1-subsupermodule of L. It suffices, then, to prove $M = L$. Since L is irreducible it suffices to to show M is $\operatorname{Dist}(G)$-stable. Clearly M is $G(k)$-stable. Since $G(k) = G_{\mathrm{ev}}(k)$ is dense in G_{ev}, M is necessarily a G_{ev}-supermodule by [**4**, I.6.16, I.2.12(5)]. That is, M is $\operatorname{Dist}(G_1)$ and $\operatorname{Dist}(G_{\mathrm{ev}})$-stable. However $\operatorname{Dist}(G)$ is generated by $\operatorname{Dist}(G_1)$ and $\operatorname{Dist}(G_{\mathrm{ev}})$, so M is $\operatorname{Dist}(G)$-stable. □

Lemma 4.2. *Let $\lambda \in X^+(T)$. Then $\operatorname{Dist}(G_1)L(\lambda)_\lambda$ is a G_1-subsupermodule of $L(\lambda)$ isomorphic to $L_1(\lambda)$.*

Proof. As a B-supermodule $L(\lambda)_\lambda \cong k_\lambda$, so they are isomorphic as B_1-supermodules as well. Thus there is a B_1-supermodule homomorphism $k_\lambda \to L(\lambda)$ with image $L(\lambda)_\lambda$. By Frobenius reciprocity we have a nonzero G_1-supermodule homomorphism $Z_1(\lambda) \to L(\lambda)$ with image $\operatorname{Dist}(G_1)L(\lambda)_\lambda$. By Lemma 4.1 $\operatorname{Dist}(G_1)L(\lambda)_\lambda$ is completely reducible as a G_1-supermodule while $Z_1(\lambda)$ has irreducible $G_1(\lambda)$-head, $L_1(\lambda)$. Consequently $\operatorname{Dist}(G_1)L(\lambda)_\lambda$ is an irreducible G_1-supermodule isomorphic to $L_1(\lambda)$. □

Recall that we say a weight $\lambda \in X^+(T)$ is *p-restricted* if it is dominant and $\lambda_i - \lambda_{i+1} < p$ for $i = 1, \dots, m-1$ and $i = m+1, \dots, m+n-1$ and that we denote the set of p-restricted weights by $X_p^+(T)$.

Lemma 4.3. *For $\lambda \in X_p^+(T)$, the irreducible G-supermodule $L(\lambda)$ is irreducible as a G_1-supermodule and $L(\lambda) \cong L_1(\lambda)$ as G_1-supermodules.*

PROOF. Throughout the proof we write e_α for $e_{i,j}$ where $\varepsilon_i - \varepsilon_j = \alpha$ is a root. Given a monomial of the PBW basis $e_{\alpha_1}^{(s_1)} \cdots e_{\alpha_k}^{(s_k)}$ for roots $\alpha_1, \dots, \alpha_k$, we define the *total degree* of the monomial to be the nonnegative integer $s_1 + \cdots + s_k$.

Let $M = \mathrm{Dist}(G_1)L(\lambda)_\lambda$. By Lemma 4.2 M is isomorphic to $L_1(\lambda)$. Consequently it suffices to show $M = L(\lambda)$. We do this by showing M is $\mathrm{Dist}(G)$ invariant—hence equal to $L(\lambda)$ by irreducibility.

First we make several reductions. Note that $\mathrm{Dist}(G)$ is generated by $\mathrm{Dist}(G_1)$ and $\mathrm{Dist}(G_{\mathrm{ev}})$ and M is clearly $\mathrm{Dist}(G_1)$-stable, so it suffices to check that it is $\mathrm{Dist}(G_{\mathrm{ev}})$-stable. Since $\mathrm{Dist}(G_{\mathrm{ev}})$ is generated by $\mathrm{Dist}(B_{\mathrm{ev}})$ and

$$\mathcal{A} := \left\{ e_{-\alpha}^{(r)} \ : \ r \in \mathbb{Z}_{>0}, \ \alpha \text{ an even simple root} \right\},$$

it suffices to show M is invariant under the action of $\mathrm{Dist}(B_{\mathrm{ev}})$ and the elements of $\mathcal{A}$. However, B_{ev} normalizes G_1 and $L(\lambda)_\lambda$ is a B_{ev}-subsupermodule of $L(\lambda)$ so M is $\mathrm{Dist}(B_{\mathrm{ev}})$-stable. Therefore we have reduced the problem to proving that M is invariant under the action of the elements of $\mathcal{A}$.

Fix $0 \neq v_\lambda \in L(\lambda)_\lambda$. By Lemma 3.2 M is spanned by vectors of the form Xv_λ where X is a monomial in the $e_{-\beta}^{(s)}$'s for β a positive root and $s \in \{0, 1, \dots, p-1\}$ if β is even and $s \in \{0, 1\}$ if β is odd. Consequently, it suffices to prove $e_{-\alpha}^{(r)} X v_\lambda \in M$ for $e_{-\alpha}^{(r)} \in \mathcal{A}$ and such monomials X. We prove this by inducting on the total degree of $e_{-\alpha}^{(r)} X$. The base case when the total degree is zero is immediate.

Now assume the total degree of $e_{-\alpha}^{(r)} X$ is greater than zero. If the total degree of X is zero, then we have $e_{-\alpha}^{(r)} v_\lambda$. If $r < p$ then $e_{-\alpha}^{(r)} \in \mathrm{Dist}(G_1)$ by (3.1) and the result is immediate. Now say $\alpha = \varepsilon_i - \varepsilon_{i+1}$ and say $r > \lambda_i - \lambda_{i+1}$, then $e_{-\alpha}^{(r)} v_\lambda = 0$ by $SL(2)$ theory. Since $\lambda \in X_p^+(T)$, our two cases cover all possibilities. Thus the result always holds.

Now assume the total degree of X is greater than zero. We can then write $X = e_{-\beta}^{(s)} Y$ where β is a positive root and $s \in \{1, \dots, p-1\}$ if β is even and $s = 1$ if β is odd, and Y is a monomial of total degree strictly less than the total degree of X. If $\alpha + \beta$ is not a root, then $e_{-\alpha}^{(r)} e_{-\beta}^{(s)} = e_{-\beta}^{(s)} e_{-\alpha}^{(r)}$ and the result holds by induction. If $\alpha + \beta$ is a root, then using (2.5) we have

$$e_{-\alpha}^{(r)} e_{-\beta}^{(s)} = \sum a_{b,c,d} e_{-\beta}^{(b)} e_{-\alpha}^{(c)} e_{-(\alpha+\beta)}^{(d)}$$

where the sum is over all $b, c, d \in \mathbb{Z}_{\geq 0}$ with $r\alpha + s\beta = b\beta + c\alpha + d(\alpha + \beta)$ for some integral coefficients $a_{b,c,d}$ (c.f. [**11**, Lemma 8]). Observe that $b + d = s$ so $s \geq b, d$ which implies $e_{-\beta}^{(b)}, e_{-(\alpha+\beta)}^{(d)} \in \mathrm{Dist}(G_1)$. Also observe that $c + d = r < r + s$ so by the inductive assumption $e_{-\alpha}^{(c)} e_{-(\alpha+\beta)}^{(d)} Y v_\lambda \in M$. Therefore all terms of the sum lie in M, proving the desired result.

□

Given a G_{ev}-supermodule, M, we can inflate M to a G-supermodule through the Frobenius morphism $F = F^1 : G \to G_{\mathrm{ev}}$. We denote the resulting G-supermodule by F^*M and call it the *Frobenius twist* of M. This defines a functor from the category of G_{ev}-supermodules to the category of G-supermodules. For example, if we let $L_{\mathrm{ev}}(\mu)$ be the irreducible G_{ev}-supermodule of highest weight μ, which is simply the irreducible G_{ev}-module viewed as a supermodule concentrated in degree $\bar{0}$, we have the G-supermodule $F^* L_{\mathrm{ev}}(\mu)$. Conversely, if N is a G-supermodule, then there is an induced $G_{\rm ev}$ structure on the fixed point space N^{G_1}. Namely, the representation $G \to GL(N^{G_1})$ is constant on G_1-cosets so factors through to give a representation $G_{\rm ev} \to GL(N^{G_1})$ by Lemma 3.1. Therefore by taking G_1-fixed points we have a functor from G-supermodules to $G_{\rm ev}$-supermodules which is right adjoint to F^*. We are now prepared to prove the main result.

THEOREM 4.4. *For $\lambda \in X_p^+(T)$ and $\mu \in X^+(T)$,*

$$L(\lambda + p\mu) \cong L(\lambda) \otimes F^* L_{\rm ev}(\mu),$$

where $L_{\rm ev}(\mu)$ denotes the irreducible $G_{\rm ev}$-supermodule of highest weight μ.

PROOF. For $\lambda \in X_p^+(T)$, $L(\lambda)$ is irreducible as a G_1-supermodule by Lemma 4.3. By Lemma 4.2 and Proposition 3.4 we know

$$H := \operatorname{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu))_{\bar{0}} \neq 0.$$

We view H as a G-supermodule by conjugation: the action of $u \in \operatorname{Dist}(G)$ is given by $(uf)(x) = \sum_i u_i f(\sigma(v_i)x)$ for $f \in H$ and $x \in L(\lambda)$, where $\Delta(u) = \sum_i u_i \otimes v_i$ and Δ and σ are the comultiplication and antipode of $\operatorname{Dist}(G)$, respectively. Checking directly one can verify that the map $H \otimes L(\lambda) \to L(\lambda + p\mu)$ given by $f \otimes x \mapsto f(x)$ is an even G-supermodule homomorphism. Since H is nonzero, the map must be nonzero hence, by the irreducibility of $L(\lambda + p\mu)$, surjective. On the other hand by the complete reducibility of $L(\lambda + p\mu)$ by Lemma 4.1 and the super version of Schur's Lemma,

$$\begin{aligned}\dim_k (H \otimes L(\lambda)) &= \dim_k (\operatorname{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu))_{\bar{0}} \otimes L(\lambda)) \\ &\leq (\dim_k L(\lambda + p\mu)/ \dim_k L(\lambda)) \cdot \dim_k L(\lambda) \\ &= \dim_k L(\lambda + p\mu),\end{aligned}$$

so our map must be an isomorphism. Finally, since the action G_1 on H is trivial, we have $H \cong F^* M$ for some $G_{\rm ev}$-supermodule M. Since $L(\lambda + p\mu)$ is irreducible, M must be irreducible. Since H has highest weight $p\mu$, $M \cong L_{\rm ev}(\mu)$. □

References

[1] J. Brundan and A. Kleshchev, Modular representations of the supergroup $Q(n)$, I, *J. Algebra* **260** (2003), 64-98.

[2] J. Brundan and J. Kujawa, A new proof of the Mullineux conjecture, *J. Alg. Combinatorics* **18** (2003), 13-39.

[3] E. Cline, B. Parshall, and L. Scott, On the tensor product theorem for algebraic groups, *J. Algebra,* **63** (1980), 264-257.

[4] J. C. Jantzen, *Representations of algebraic groups,* second edition, Math. Surveys **107**, American Math. Soc., 2003.

[5] V. Kac, Lie superalgebras, *Advances Math.* **26** (1977), 8–96.

[6] V. Kac, Representations of classical Lie superalgebras, in: Lecture Notes in Mathematics, vol. 676, 597–626, Springer, 1978.

[7] J. Kujawa, Crystal structures arising from representations of $GL(m|n)$, submitted.

[8] J. Kujawa, *The representation theory of the supergroup $GL(m|n)$*, PhD thesis, University of Oregon, 2003.

[9] D.A. Leites, Introduction to the theory of supermanifolds, *Russian Math. Surveys* **35** (1980).

[10] Yu I. Manin, *Gauge field theory and complex geometry*, Grundlehren der mathematischen Wissenschaften 289, second edition, Springer, 1997.

[11] R. Steinberg, *Lectures on Chevalley groups,* Yale University, 1967.

DEPT. OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602

E-mail address: kujawa@math.uga.edu