The Steinberg Tensor Product Theorem for $GL(m|n)$

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Dedicated to James E. Humphreys on the occasion of his 65th birthday.

1. Introduction

The Steinberg Tensor Product Theorem is a fundamental result in the modular representation theory of algebraic groups. The purpose of the present article is to formulate and prove the analogous theorem for the supergroup $GL(m|n)$. This result was first mentioned without proof in [2]. We emphasize that our approach closely parallels the analogous result for the supergroup $Q(n)$ proven by Brundan and Kleshchev [1], which in turn follows the approach of Cline, Parshall, and Scott [3].

The preliminaries are outlined in section 2. They are an abbreviated form of what can be found in [2] and [7]. Sections 3 and 4 contain the new results of the present article with the main theorem being the following version of the Steinberg Tensor Product Theorem.

Before stating the result, we require some notation. We direct the reader to section 2 for precise statements of definitions. Throughout, let $k$ be a fixed ground field of characteristic $p > 0$ which is algebraically closed. All objects under discussion are defined over $k$. Let $T$ be the maximal torus of $GL(m|n)$ consisting of diagonal matrices. We identify the character group $X(T) = \text{Hom}(T, G_m)$ with the free abelian group on generators $\varepsilon_1, \ldots, \varepsilon_{m+n}$, where $\varepsilon_i$ picks out the $i$th entry of a diagonal matrix. We call the set

$$X^+(T) := \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X(T) : \lambda_1 \geq \cdots \geq \lambda_m \text{ and } \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \right\},$$

the set of dominant weights. The irreducible $GL(m|n)$-supermodules are parameterized by highest weight by the set $X^+(T)$ and we write $L(\lambda)$ for the irreducible supermodule of highest weight $\lambda \in X^+(T)$. A weight is $p$-restricted if it is dominant and $\lambda_i - \lambda_{i+1} < p$ for $i = 1, \ldots, m-1$ and $i = m+1, \ldots, m+n-1$. Denote the set of $p$-restricted weights by $X^+_p(T)$.

Let $F : GL(m|n) \to GL(m) \times GL(n)$ be the Frobenius morphism given by raising entries to the $p$th power. Given a $GL(m) \times GL(n)$-supermodule $M$ we
can view it as a $GL(m|n)$-supermodule via inflation through $F$. We call this the
Frobenius twist of $M$ and denote by $F^*M$.

**Theorem 1.1 (Steinberg Tensor Product Theorem).** For $\lambda \in X^+_p(T)$ and $\mu \in X^+(T)$,
$$L(\lambda + p\mu) \cong L(\lambda) \otimes F^*L'(\mu),$$
where $L'(\mu)$ denotes the irreducible $GL(m) \times GL(n)$-supermodule of highest weight $\mu$.

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2. Definitions and Basic Results

In this section we outline the basic definitions and results we require. For an
account of the basic language of superalgebras and supergroups adopted here, we
refer the reader to [1], [2], and [7]; see also [4], [5], [9, ch.1] and [10, ch.3, §§1–2, ch.4, §1].

2.1. The supergroup $GL(m|n)$. We use the language of supergroup schemes
to define $GL(m|n)$. Our approach parallels that of [4]. Throughout, let $k$ be an
algebraically closed field of characteristic $p > 0$. All objects (superalgebras, super-
groups, . . . ) will be defined over $k$. A superspace is a $\mathbb{Z}_2$-graded $k$-vector space.
If $V$ is a superspace and $v \in V$ is a homogeneous vector, then we write $\pi \in \mathbb{Z}_2$
for the degree of $v$. A commutative superalgebra is a $\mathbb{Z}_2$-graded associative algebra
$A = A_0 \oplus A_1$ with $ab = (-1)^{ab}ba$ for all homogeneous $a, b \in A$. If $p = 2$ we also
assume that $a^2 = 0$ for all $a \in A_1$. A morphism of superalgebras is a homomor-
phism of graded algebras; that is, it is an algebra homomorphism which preserves the $\mathbb{Z}_2$-grading.

The supergroup $G = GL(m|n)$ is the functor from the category of commutative
superalgebras to the category of groups defined on a commutative superalgebra $A$
by letting $G(A)$ be the group of all invertible $(m + n) \times (m + n)$ matrices of the form

$$g = \begin{pmatrix}
W & X \\
Y & Z
\end{pmatrix}$$

where $W$ is an $m \times m$ matrix with entries in $A_0$, $X$ is an $m \times n$ matrix with entries in $A_1$, $Y$ is an $n \times m$ matrix with entries in $A_1$, and $Z$ is an $n \times n$ matrix with entries in $A_0$. If $f : A \to B$ is a superalgebra homomorphism, then $G(f) : G(A) \to G(B)$ is
the group homomorphism defined by applying $f$ to the matrix entries.

Let $Mat$ be the affine superscheme with $Mat(A)$ consisting of all (not neces-
sarily invertible) $(m+n) \times (m+n)$ matrices of the above form. For $1 \leq i, j \leq m + n$, let $T_{i,j}$ be the morphism defined by having $T_{i,j} : Mat(A) \to A$ map a matrix to its $ij$-entry. Then the coordinate ring $k[Mat]$ is the free commutative superalgebra on the generators $T_{i,j}$ ($1 \leq i, j \leq m + n$) with $T_{i,j}$ having parity $i + j$, where we write $\bar{i} = 0$ for $i = 1, \ldots, m$ and $\bar{i} = 1$ for $i = m + 1, \ldots, m + n$. By [9, I.7.2], a
matrix $g \in Mat(A)$ of the form (2.1) is invertible if and only if $\det W \det Z \in A^\times$, where here $\det$ denotes the usual matrix determinant. Hence, $G$ is the principal
open subset of $Mat$ defined by the function $\det : g \mapsto \det W \det Z$. In particular,
the coordinate ring $k[G]$ is the localization of $k[Mat]$ at $\det$. 


Just as for group schemes \([4,1.2.3]\), the coordinate ring \(k[G]\) has the naturally induced structure of a Hopf superalgebra. Explicitly, the comultiplication and counit are the unique superalgebra maps satisfying

\[
\Delta(T_{i,j}) = \sum_{h=1}^{m+n} T_{i,h} \otimes T_{h,j},
\]

\[
\varepsilon(T_{i,j}) = \delta_{i,j}
\]

for all \(1 \leq i, j \leq m+n\).

By definition a representation of \(G\) means a natural transformation \(\rho : G \to GL(M)\) for some vector superspace \(M\), where \(GL(M)\) is the supergroup with \(GL(M)(A)\) being equal to the group of all even automorphisms of the \(A\)-superspace \(M \otimes A\), for each commutative superalgebra \(A\). Equivalently, as with group schemes \([4,1.2.8]\), \(M\) is a right \(k[G]\)-cosuperspace. That is, there is a \(\mathbb{Z}_2\)-grading preserving structure map \(\eta : M \to M \otimes k[G]\) satisfying the usual comodule axioms. We will usually refer to such an \(M\) as a \(G\)-superspace.

If \(\rho : G \to GL(M)\) and \(\rho' : G \to GL(M')\) are two representations of \(G\), then a morphism of representations is a linear map \(f : M \to M'\) such that for any commutative superalgebra \(A\) we have \(\rho'(g)(f(m)) = f(\rho(g)(m))\) for all \(g \in G(A)\) and all \(m \in M \otimes A\). In the language of \(k[G]\)-cosuperspaces, if \(\eta : M \to M \otimes k[G]\) and \(\eta' : M' \to M' \otimes k[G]\) are the cosuperspace structure maps, then \(f : M \to M'\) is a morphism if \(f \otimes 1 \circ \eta = \eta' \circ f\).

We denote by \(G\)-mod the category of all \(G\)-superspaces. We emphasize that we allow all morphisms and not just graded (i.e. even) morphisms. However, note that for superspaces \(M\) and \(M'\) the space \(\text{Hom}_k(M,M')\) is naturally \(\mathbb{Z}_2\)-graded by declaring \(f \in \text{Hom}_k(M,M')\) if \(f(M_s) \subseteq M'_{s+r}\) for all \(s \in \mathbb{Z}_2\). This gives a \(\mathbb{Z}_2\)-grading on \(\text{Hom}_k(M,M') \subseteq \text{Hom}_k(M,M')\). We remark that \(G\)-mod is not an abelian category. However, the underlying even category of \(G\)-mod, consisting of the same objects as \(G\)-mod but only the even morphisms, is an abelian category. This, along with the parity change functor \(\Pi\), which, roughly speaking, interchanges the \(\mathbb{Z}_2\)-grading of a superspace, allows one to make use of the tools of homological algebra.

The underlying purely even group \(G_{ev}\) of \(G\) is by definition the functor from superalgebras to groups given by \(G_{ev}(A) = G(A_\bar{0})\). Thus, \(G_{ev}(A)\) consists of all invertible matrices of the form \((2.1)\) with \(X = Y = 0\), so \(G_{ev} \cong GL(m) \times GL(n)\). Let \(T\) be the usual maximal torus of \(G_{ev}\) consisting of diagonal matrices. The character group \(X(T) = \text{Hom}(T, G_m)\) as defined in \([4,1.2.4]\) can then be identified with the free abelian group on generators \(\varepsilon_1, \ldots, \varepsilon_{m+n}\), where \(\varepsilon_i\) is the function which picks out the \(i\)th diagonal entry of a diagonal matrix. Let \(B\) denote the subgroup of \(G\) given by letting \(B(A)\) equal the set of all of all upper triangular invertible matrices of the form \((2.1)\). We call this the standard Borel subgroup. Note that the underlying purely even subgroup, \(B_{ev}\), is given by the upper triangular matrices in \(G_{ev}\).

The root system of \(G\) is the set \(\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq m+n, i \neq j\}\). There are even and odd roots, the parity of the root \(\varepsilon_i - \varepsilon_j\) being \(i + j\). Our choice of Borel subgroup, \(B\), defines a set,

\[
\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq m+n\},
\]
of positive roots. The simple roots then are \( \varepsilon_i - \varepsilon_{i+1} \) where \( i = 1, \ldots, m + n - 1 \). The corresponding dominance order on \( X(T) \) is denoted \( \leq \), defined by \( \lambda \leq \mu \) if \( \mu - \lambda \) can be written as the sum of positive roots.

2.2. The Superalgebra of Distributions. Just as for algebraic groups [4, I.7.7] one can abstractly define the superalgebra of distributions \( \text{Dist}(G) \) of \( G \). We sketch how this is done. Let \( I \) be the kernel of the counit \( \varepsilon : k[G] \to k \), a superideal of \( k[G] \). For \( r \geq 0 \), let

\[
\text{Dist}_r(G) = \{ x \in k[G]^* : x(I^{r+1}) = 0 \} \cong (k[G]/I^{r+1})^*,
\]

\[
\text{Dist}(G) = \bigcup_{r \geq 0} \text{Dist}_r(G).
\]

There is a multiplication on \( k[G]^* \) dual to the comultiplication on \( k[G] \), defined by \( (xy)(f) = (x \otimes y)(\Delta(f)) \) for \( x, y \in k[G]^* \) and \( f \in k[G] \). Note here (and elsewhere) we are implicitly using the superalgebra rule of signs: \( (x \otimes y)(f \otimes g) = (-1)^{\delta(x,y)} f(y)g \) where \( y \) and \( f \) are assumed to be homogeneous. The general case is obtained via linearity. In fact, \( \text{Dist}(G) \) is a subsuperalgebra of \( k[G]^* \) (see [2]).

In the case when \( G = GL(m|n) \), however, we can describe \( \text{Dist}(G) \) explicitly as the reduction modulo \( p \) of the universal enveloping superalgebra of the Lie superalgebra \( \mathfrak{gl}(m|n, \mathbb{C}) \). We now describe how this can be done.

Recall that \( \mathfrak{gl}(m|n, \mathbb{C}) \) is the Lie superalgebra given by letting \( \mathfrak{gl}(m|n, \mathbb{C}) \) be the set of all \( (m + n) \times (m + n) \) matrices over \( \mathbb{C} \). If for \( 1 \leq i, j \leq m + n \) we write \( e_{i,j} \) for the \( ij \) matrix unit, then the \( e_{i,j} \) provide a homogeneous basis with the degree of \( e_{i,j} \) defined to be \( \bar{\bar{i}} + \bar{\bar{j}} \). The bracket is given by

\[
[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - (-1)^{(\bar{\bar{i}}+\bar{\bar{j}})(\bar{\bar{k}}+\bar{\bar{l}})}\delta_{i,l}e_{k,j}
\]

By the PBW theorem for Lie superalgebras (see [5]) we have that the universal enveloping superalgebra of \( \mathfrak{gl}(m|n, \mathbb{C}) \), \( U_C \), has basis consisting of all monomials

\[
\prod_{1 \leq i,j \leq m+n} e_{i,j}^{\alpha_{i,j}} \prod_{1 \leq i,j \leq m+n} e_{i,j}^{d_{i,j}}
\]

where \( \alpha_{i,j} \in \mathbb{Z}_{\geq 0}, d_{i,j} \in \{0,1\} \), and the product is taken in any fixed order. We shall write \( h_i = e_{i,i} \) for short.

Define the Kostant \( \mathbb{Z} \)-form \( U_Z \) to be the \( \mathbb{Z} \)-subalgebra of \( U_C \) generated by elements \( e_{i,j} (1 \leq i, j \leq m + n, \bar{i} + \bar{j} = 1), e_{i,j}^{(r)} (1 \leq i, j \leq m + n, \bar{i} + \bar{j} = 0, r \geq 1) \), and \( h_i \) \((1 \leq i \leq m + n, r \geq 1) \). Here, \( e_{i,j}^{(r)} := e_{i,j}^{r}/(r!) \) and \( h_i := h_i(h_i - 1) \cdots (h_i - r + 1)/(r!) \). Following the proof of [11, Th.2], one verifies the following lemma.

**Lemma 2.1.** The superalgebra \( U_Z \) is a free \( \mathbb{Z} \)-module with basis given by the set of all monomials of the form

\[
\prod_{1 \leq i,j \leq m+n} e_{i,j}^{(\alpha_{i,j})} \prod_{1 \leq i \leq m+n} h_i \prod_{1 \leq i,j \leq m+n} e_{i,j}^{d_{i,j}}
\]

for all \( \alpha_{i,j}, h_i \in \mathbb{Z}_{\geq 0} \) and \( d_{i,j} \in \{0,1\} \), where the product is taken in any fixed order.
The enveloping superalgebra \( U_\mathbb{Z} \) is a Hopf superalgebra in a canonical way, hence \( U_\mathbb{Z} \) is a Hopf superalgebra over \( \mathbb{Z} \). Consequently, \( k \otimes_\mathbb{Z} U_\mathbb{Z} \) is naturally a Hopf superalgebra over \( k \). It is known, for example by [2, Thm. 3.2], that

\[
\text{Dist}(G) \cong k \otimes_\mathbb{Z} U_\mathbb{Z}
\]
as Hopf superalgebras. We identify these Hopf superalgebras and will abuse notation by using the same symbols \( e_{i,j}^{(r)} \), \( h_{i} \), etc. for the canonical images of these elements of \( U_\mathbb{Z} \) in \( \text{Dist}(G) \). Note that the monomials given in Lemma 2.1 form a homogeneous basis of \( \text{Dist}(G) \).

It is also easy to describe the superalgebras of distributions of our various natural subgroups of \( G \) as subalgebras of \( \text{Dist}(G) \). For example, \( \text{Dist}(T) \) is the subalgebra generated by all \( \{ h_{i} \} \, 1 \leq i \leq m+n, r \geq 1 \), \( \text{Dist}(B_{ev}) \) is the subalgebra generated by \( \text{Dist}(T) \) and all \( e_{i,j}^{(r)} \, 1 \leq i, j \leq m+n, i < j, i+j = 0, r \geq 1 \), and \( \text{Dist}(B) \) is the subalgebra generated by \( \text{Dist}(B_{ev}) \) and all \( e_{i,j} \, 1 \leq i, j \leq m+n, i+j = 1, i < j \).

Let us describe the category of \( \text{Dist}(G) \)-supermodules. The objects are all left \( \text{Dist}(G) \)-modules which are \( \mathbb{Z}_2 \)-graded: that is, \( k \)-superspaces, \( M \), satisfying \( \text{Dist}(G)_{r}, M_s \subseteq M_{r+s} \) for \( r, s \in \mathbb{Z}_2 \). A morphism of \( \text{Dist}(G) \)-supermodules is a linear map \( f : M \to M' \) satisfying \( f(xm) = (-1)^{T} x f(m) \) for all \( m \in M \) and all \( x \in \text{Dist}(G) \). Note that this definition makes sense as stated only for homogeneous elements; it should be interpreted via linearity in the general case. We emphasize that morphisms are not necessarily even. However, the \( \text{Hom} \)-spaces are naturally \( \mathbb{Z}_2 \)-graded and our remarks about the category \( \mathcal{O} \) made in the previous subsection apply here as well.

For \( \lambda = \sum_{i=1}^{m+n} \lambda_i e_i \in X(T) \) and a \( \text{Dist}(G) \)-supermodule \( M \), define the \( \lambda \)-\textit{weight space} of \( M \) to be

\[
(2.6) \quad M_{\lambda} = \left\{ m \in M : \left( \frac{h_i}{r} \right) m = \left( \frac{\lambda_i}{r} \right) m \text{ for all } 1 \leq i \leq m+n, r \geq 1 \right\}.
\]

We call a \( \text{Dist}(G) \)-supermodule \( M \) \textit{integrable} if it is locally finite over \( \text{Dist}(G) \) and satisfies \( M = \sum_{\lambda \in X(T)} M_{\lambda} \).

If \( M \) is a \( G \)-supermodule then we can view \( M \) as a \( \text{Dist}(G) \)-supermodule as follows. Given a \( G \)-supermodule \( M \) with structure map \( \eta : M \to M \otimes k[G] \), we can view \( M \) as a \( \text{Dist}(G) \)-supermodule by \( xm = (1 \otimes x \circ \eta)(m) \). In fact, in this way we obtain a functor from \( \mathcal{G} \mathcal{M} \mathcal{O} \) to the category of \( \text{Dist}(G) \)-supermodules. Moreover, the notion of weight space defined above for \( \text{Dist}(G) \)-supermodules coincides with the usual notion of weight space of \( M \) with respect to the torus \( T \). It is then straightforward to verify that the \( G \)-supermodule \( M \) is integrable when viewed as a \( \text{Dist}(G) \)-supermodule. We prove the following theorem in [2, Corollary 3.5].

\textbf{Theorem 2.2.} The category \( \mathcal{G} \mathcal{M} \mathcal{O} \) is isomorphic to the full subcategory of integrable \( \text{Dist}(G) \)-supermodules via the aforementioned functor.

In view of this result, we will not distinguish between \( G \)-supermodules and integrable \( \text{Dist}(G) \)-supermodules in what follows.

\textbf{2.3. Classification of irreducible} \( GL(m|n) \)-\textit{supermodules.} Now we describe the classification of the irreducible representations of \( G \) by their highest weights. It seems to be more convenient to work first in the category \( \mathcal{O}_p \) : the full subcategory of all \( \text{Dist}(G) \)-supermodules \( M \) such that \( M = \bigoplus_{\lambda \in X(T)} M_{\lambda} \) and
$M$ is locally finite over $\text{Dist}(B)$. This is an analogue of Bernstein, Gelfand and Gelfand’s category $\mathcal{O}$ in classical Lie theory. We remark that Theorem 2.2 implies that $G$-$\mathbf{mod}$ can be viewed as a full subcategory of $O_p$. From now on we will assume all $\text{Dist}(G)$-supermodules under discussion are objects in $O_p$.

For $\lambda \in X(T)$, we have the Verma supermodule

$$M(\lambda) := \text{Dist}(G) \otimes_{\text{Dist}(B)} k_\lambda,$$

where $k_\lambda$ denotes $k$ viewed as a $\text{Dist}(B)$-supermodule of weight $\lambda$ concentrated in degree 0. Note that by Lemma 2.1 it follows that $M(\lambda)$ is an object in $O_p$. We say that a homogeneous vector $v$ in a $\text{Dist}(G)$-supermodule $M$ is a primitive vector of weight $\lambda$ if $\text{Dist}(B)v \cong k_\lambda$ as a $\text{Dist}(B)$-supermodule. Familiar arguments show that $M(\lambda)$ is universal among all supermodules of $O_p$ which are generated by a primitive vector of weight $\lambda$ and $M(\lambda)$ has a unique maximal subsupermodule, hence an irreducible quotient which we denote by $L(\lambda)$. Taken together these imply that $\{L(\lambda) : \lambda \in X(T)\}$ gives a complete set of pairwise non-isomorphic irreducibles in $O_p$. In this way, we get a parametrization of the irreducible objects in $O_p$ by their highest weights with respect to the ordering $\leq$.

Now we pass from $O_p$ to $G$-$\mathbf{mod}$. Recall that

$$X^+(T) = \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X(T) : \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \right\}$$

denotes the set of all dominant integral weights. The proof of the following lemma is due to Kac [6] (see also [2]).

**Lemma 2.3.** Given any $\lambda \in X(T)$, $L(\lambda)$ is finite dimensional if and only if $\lambda \in X^+(T)$. In particular, the supermodules $\{L(\lambda)\}_{\lambda \in X^+(T)}$ form a complete set of pairwise non-isomorphic irreducible supermodules in $G$-$\mathbf{mod}$.

### 3. Frobenius Kernels

For $r \geq 1$, we define the Frobenius morphism $F^r : G \to G_{ev}$ by having $F^r : G(A) \to G_{ev}(A)$ raise each matrix entry to the $p^r$th power for any commutative superalgebra $A$. Note that for $a \in A_{T_n}$, $a^{p^r} = 0$ so the morphism makes sense. Let $G_r$ denote the kernel of $F^r$, the $r$th Frobenius kernel, a normal subgroup of $G$. Similarly, let $G_{ev,r}$ denote the kernel of $F^r|_{G_{ev}}$, $B_r$ denote the kernel of $F^r|_B$, etc.

**Lemma 3.1.** $F^r : G \to G_{ev}$ is a quotient of $G$ by $G_r$ in the category of superschemes. That is, for any morphism $f : G \to S$ of superschemes which is constant on $G_r(A)$-cosets of $G(A)$ for all commutative superalgebras $A$, there is a unique morphism $\tilde{f} : G_{ev} \to S$ such that $f = \tilde{f} \circ F^r$.

**Proof.** Let $\pi : G \to G_{ev}$ be the superscheme morphism defined by projection. That is, if $g \in G(A)$ is as in (2.1), then $\pi$ acts as the identity on the entries of $W$ and $Z$, and sends the entries of $X$ and $Y$ to zero. Let $f : G \to S$ be a morphism of superschemes which is constant on $G_r(A)$-cosets. For any element $g \in G(A)$ written as in (2.1) we have

$$\begin{pmatrix} I_m & XZ^{-1} \\ YW^{-1} & I_n \end{pmatrix}^{-1} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix},$$

where $I_k$ denotes the $k \times k$ identity matrix. That is, $h g = \pi(g)$ for some $h \in G_r(A)$. Thus $f = f|_{G_{ev}} \circ \pi$. However from the purely even theory (see [4, 1.9.5]), $F^r|_{G_{ev}}$
is a quotient of $G_{ev}$ by $G_{ev,r}$. Consequently, since $f|_{G_{ev}}$ is constant on $G_{ev,r}$-cosets of $G_{ev}$, there is a unique morphism $\tilde{f} : G_{ev} \to S$ such that $f|_{G_{ev}} = \tilde{f} \circ F^r|_{G_{ev}}$. Therefore $f = f|_{G_{ev}} \circ \pi = \tilde{f} \circ F^r|_{G_{ev}} \circ \pi = \tilde{f} \circ F^r$. □

Observe that $k[G_r] \cong k[G]/I_r$ where $I_r$ is the ideal generated by $\{T_{k,k}^{p^r}, T_{k,k}^{p^r} - 1 : 1 \leq i, j, k \leq m+n, i \neq j\}$. Consequently, a basis for $k[G_r]$ is given by the monomials in $T_{i,j}^{p^r}$ for $1 \leq i, j \leq m+n$, where $a_{i,j} \in \{0, 1, \ldots, p^r-1\}$ if $i + j = 0$ and $a_{i,j} \in \{0, 1\}$ if $i + j = 1$, with the product taken in any fixed order. In particular, the dimension of $k[G_r]$ is finite so by definition $G_r$ is a finite algebraic supergroup. Moreover the $p^r$-th power of any element of $\mathcal{I} := \text{Ker}(\epsilon : k[G_r] \to k)$ lies in $I_r$ so $\mathcal{I}$ is nilpotent. That is, $G_r$ is infinitesimal and, consequently, $\text{Dist}(G_r)$ can be identified with the Hopf superalgebra dual of $k[G_r]$. It follows as in [4, I.8.1-6] that the category of $G_r$-supermodules is isomorphic to the category of $\text{Dist}(G_r)$-supermodules. Also, under this identification we can take as our basis for $\text{Dist}(G_r) \subset \text{Dist}(G)$ the ordered PBW monomials

$$
\prod_{1 \leq i,j \leq m+n} e_{i,j}^{a_{i,j}} \prod_{1 \leq k \leq m+n} (h_k) \prod_{1 \leq i,j \leq m+n} e_{i,j}^{a_{i,j}},
$$

where $a_{i,j} \in \{0, \ldots, p^r-1\}$ for $1 \leq i, j, k \leq m+n$ when $i + j = 0$, and $a_{i,j} \in \{0, 1\}$ when $i + j = 1$. Similarly we can describe bases for $\text{Dist}(B_r)$, etc. From this we observe the following lemma.

**Lemma 3.2.** $\text{Dist}(G_r)$ is a free right $\text{Dist}(B_r)$-supermodule with basis given by the ordered monomials

$$
\prod_{1 \leq i,j \leq m+n} e_{i,j}^{a_{i,j}},
$$

where $a_{i,j} \in \{0, \ldots, p^r-1\}$ when $i + j = 0$, and $a_{i,j} \in \{0, 1\}$ when $i + j = 1$.

Having identified the representations of $G_r$ and $B_r$ with $\text{Dist}(G_r)$-supermodules and $\text{Dist}(B_r)$-supermodules, respectively, we have the induction functor given by

$$
\text{ind}_{B_r}^{G_r} M = \text{Dist}(G_r) \otimes_{\text{Dist}(B_r)} M.
$$

From Lemma 3.2 we see this is an exact functor which is left adjoint to restriction. Given $\lambda \in X(T)$, let $k_\lambda$ denote $k$ viewed as a $T_r$-supermodule of weight $\lambda$ concentrated in degree 0. The classical theory [4, II.3.7] gives the following lemma.

**Lemma 3.3.** The set $\{k_\lambda : \lambda \in X(T)\}$ is a complete family of irreducible $T_r$-supermodules. Moreover, $k_\lambda \cong k_\mu$ if and only if $\lambda - \mu \in p^r X(T)$.

Furthermore, $\{k_\lambda : \lambda \in X(T)\}$ provides a complete set of irreducible $B_r$-supermodules via inflation. For $\lambda \in X(T)$, define

$$
Z_r(\lambda) = \text{ind}_{B_r}^{G_r} k_\lambda.
$$

Let $L_r(\lambda)$ denote the $G_r$-head of $Z_r(\lambda)$.

**Proposition 3.4.** $\{L_r(\lambda) : \lambda \in X(T)\}$ is a complete set of irreducible $G_r$-supermodules. Furthermore, $L_r(\lambda) \cong L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$.
To prove the Steinberg Tensor Product Theorem for $GL(m|n)$. 

**Lemma 4.1.** Let $L$ be an irreducible $G$-supermodule. Then $L$ is completely reducible as a $G_1$-supermodule.

**Proof.** Let $L_1$ be an irreducible supermodule in the $G_1$-socle of $L$. Since $G_1$ is a normal subgroup of $G$ each translate, $gL_1$, by an element $g \in G(k)$ is an irreducible $G_1$-subsupermodule of $L$. Thus

$$M := \sum_{g \in G(k)} gL_1$$

is a completely reducible $G_1$-subsupermodule of $L$. It suffices, then, to prove $M = L$. Since $L$ is irreducible it suffices to show $M$ is Dist($G$)-stable. Clearly $M$ is $G(k)$-stable. Since $G(k) = G_{ev}(k)$ is dense in $G_{ev}$, $M$ is necessarily a $G_{ev}$-supermodule by [4, I.6.16, I.2.12(5)]. That is, $M$ is Dist($G_1$) and Dist($G_{ev}$)-stable. However Dist($G$) is generated by Dist($G_1$) and Dist($G_{ev}$), so $M$ is Dist($G$)-stable. 

**Lemma 4.2.** Let $\lambda \in X^+(T)$. Then Dist($G_1$) $L(\lambda)_\lambda$ is a $G_1$-subsupermodule of $L(\lambda)$ isomorphic to $L_1(\lambda)$.

**Proof.** As a $B$-supermodule $L(\lambda)_\lambda \cong k_\lambda$, so they are isomorphic as $B_1$-supermodules as well. Thus there is a $B_1$-supermodule homomorphism $k_\lambda \rightarrow L(\lambda)$ with image $L(\lambda)_\lambda$. By Frobenius reciprocity we have a nonzero $G_1$-supermodule homomorphism $Z_1(\lambda) \rightarrow L(\lambda)$ with image Dist($G_1$) $L(\lambda)_\lambda$. By Lemma 4.1 Dist($G_1$) $L(\lambda)_\lambda$ is completely reducible as a $G_1$-supermodule while $Z_1(\lambda)$ has irreducible $G_1$($\lambda$)-head, $L_1(\lambda)$. Consequently Dist($G_1$) $L(\lambda)_\lambda$ is an irreducible $G_1$-supermodule isomorphic to $L_1(\lambda)$. 

Recall that we say a weight $\lambda \in X^+(T)$ is $p$-restricted if it is dominant and $\lambda_i - \lambda_{i+1} < p$ for $i = 1, \ldots, m-1$ and $i = m+1, \ldots, m+n-1$ and that we denote the set of $p$-restricted weights by $X^+_p(T)$.

**Lemma 4.3.** For $\lambda \in X^+_p(T)$, the irreducible $G$-supermodule $L(\lambda)$ is irreducible as a $G_1$-supermodule and $L(\lambda) \cong L_1(\lambda)$ as $G_1$-supermodules.
Throughout the proof we write $e_\alpha$ for $e_{i,j}$ where $\varepsilon_i - \varepsilon_j = \alpha$ is a root. Given a monomial of the PBW basis $e^{(s_1)}_{\alpha_1} \cdots e^{(s_k)}_{\alpha_k}$ for roots $\alpha_1, \ldots, \alpha_k$, we define the total degree of the monomial to be the nonnegative integer $s_1 + \cdots + s_k$. Let $M = \text{Dist}(G_1) L(\lambda)_\lambda$. By Lemma 4.2 $M$ is isomorphic to $L_1(\lambda)$. Consequently it suffices to show $M = L(\lambda)$. We do this by showing $M$ is Dist($G$) invariant—hence equal to $L(\lambda)$ by irreducibility.

First we make several reductions. Note that Dist($G$) is generated by Dist($G_1$) and Dist($G_{ev}$) and $M$ is clearly Dist($G_1$)-stable, so it suffices to check that it is Dist($G_{ev}$)-stable. Since Dist($G_{ev}$) is generated by Dist($B_{ev}$) and

$$A := \left\{ e^{(r)}_{-\alpha} : r \in \mathbb{Z}_{\geq 0}, \alpha \text{ an even simple root} \right\},$$

it suffices to show $M$ is invariant under the action of Dist($B_{ev}$) and the elements of $A$. However, $B_{ev}$ normalizes $G_1$ and $L(\lambda)_\lambda$ is a $B_{ev}$-subsupersubmodule of $L(\lambda)$ so $M$ is Dist($B_{ev}$)-stable. Therefore we have reduced the problem to proving that $M$ is invariant under the action of the elements of $A$.

Fix $0 \neq v_\lambda \in L(\lambda)_\lambda$. By Lemma 3.2 $M$ is spanned by vectors of the form $Xv_\lambda$ where $X$ is a monomial in the $e^{(s)}_\beta$’s for $\beta$ a positive root and $s \in \{0, 1, \ldots, p-1\}$ if $\beta$ is even and $s \in \{0, 1\}$ if $\beta$ is odd. Consequently, it suffices to prove $e^{(r)}_{-\alpha} Xv_\lambda \in M$ for $e^{(r)}_{-\alpha} \in A$ and such monomials $X$. We prove this by inducting on the total degree of $e^{(r)}_{-\alpha} X$. The base case when the total degree is zero is immediate.

Now assume the total degree of $e^{(r)}_{-\alpha} X$ is greater than zero. If the total degree of $X$ is zero, then we have $e^{(r)}_{-\alpha} Xv_\lambda$. If $r < p$ then $e^{(r)}_{-\alpha} \in \text{Dist}(G_1)$ by (3.1) and the result is immediate. Now say $\alpha = \varepsilon_i - \varepsilon_{i+1}$ and say $r > \lambda_i - \lambda_{i+1}$, then $e^{(r)}_{-\alpha} v_\lambda = 0$ by $SL(2)$ theory. Since $\lambda \in X_{\lambda}^p(T)$, our two cases cover all possibilities. Thus the result always holds.

Now assume the total degree of $X$ is greater than zero. We can then write $X = e^{(s)}_{-\beta} Y$ where $\beta$ is a positive root and $s \in \{1, \ldots, p-1\}$ if $\beta$ is even and $s = 1$ if $\beta$ is odd, and $Y$ is a monomial of total degree strictly less than the total degree of $X$. If $\alpha + \beta$ is not a root, then $e^{(r)}_{-\alpha} e^{(s)}_{-\beta} = e^{(s)}_{-\beta} e^{(r)}_{-\alpha}$ and the result holds by induction. If $\alpha + \beta$ is a root, then using (2.5) we have

$$e^{(r)}_{-\alpha} e^{(s)}_{-\beta} = \sum a_{b,c,d} e^{(b)}_{-\beta} e^{(c)}_{-\alpha} e^{(d)}_{-(\alpha+\beta)}$$

where the sum is over all $b, c, d \in \mathbb{Z}_{\geq 0}$ with $r \alpha + s \beta = b \beta + c \alpha + d (\alpha + \beta)$ for some integral coefficients $a_{b,c,d}$ (c.f. [11, Lemma 8]). Observe that $b + d = s$ so $s \geq b, d$ which implies $e^{(b)}_{-\beta}, e^{(d)}_{-(\alpha+\beta)} \in \text{Dist}(G_1)$. Also observe that $c + d = r < r + s$ so by the inductive assumption $e^{(r)}_{-\alpha} e^{(d)}_{-(\alpha+\beta)} Y v_\lambda \in M$. Therefore all terms of the sum lie in $M$, proving the desired result.

Given a $G_{ev}$-supersupersubmodule, $M$, we can inflate $M$ to a $G$-supersubmodule through the Frobenius morphism $F = F^1 : G \rightarrow G_{ev}$. We denote the resulting $G$-supersubmodule by $F^* M$ and call it the Frobenius twist of $M$. This defines a functor from the category of $G_{ev}$-supersupersubmodules to the category of $G$-supersubmodules. For example, if we let $L_{ev}(\mu)$ be the irreducible $G_{ev}$-supersubmodule of highest weight $\mu$, which is simply the irreducible $G_{ev}$-module viewed as a supersubmodule concentrated in degree 0, we have the $G$-supersubmodule $F^* L_{ev}(\mu)$. Conversely, if $N$ is a $G$-supersubmodule, then there is
an induced $G_{ev}$ structure on the fixed point space $N^{G_1}$. Namely, the representation $G \to GL(N^{G_1})$ is constant on $G_1$-cosets so factors through to give a representation $G_{ev} \to GL(N^{G_1})$ by Lemma 3.1. Therefore by taking $G_1$-fixed points we have a functor from $G$-supermodules to $G_{ev}$-supermodules which is right adjoint to $F^*$. We are now prepared to prove the main result.

**Theorem 4.4.** For $\lambda \in X^+_p(T)$ and $\mu \in X^+(T)$,

$$L(\lambda + p\mu) \cong L(\lambda) \otimes F^* L_{ev}(\mu),$$

where $L_{ev}(\mu)$ denotes the irreducible $G_{ev}$-supermodule of highest weight $\mu$.

**Proof.** For $\lambda \in X^+_p(T)$, $L(\lambda)$ is irreducible as a $G_1$-supermodule by Lemma 4.3. By Lemma 4.2 and Proposition 3.4 we know

$$H := \text{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu)) \neq 0.$$

We view $H$ as a $G$-supermodule by conjugation: the action of $u \in \text{Dist}(G)$ is given by $(uf)(x) = \sum_i u_i f(\sigma(v_i)x)$ for $f \in H$ and $x \in L(\lambda)$, where $\Delta(u) = \sum_i u_i \otimes v_i$ and $\Delta$ and $\sigma$ are the comultiplication and antipode of $\text{Dist}(G)$, respectively. Checking directly one can verify that the map $H \otimes L(\lambda) \to L(\lambda + p\mu)$ given by $f \otimes x \mapsto f(x)$ is an even $G$-supermodule homomorphism. Since $H$ is nonzero, the map must be nonzero hence, by the irreducibility of $L(\lambda + p\mu)$, surjective. On the other hand by the complete reducibility of $L(\lambda + p\mu)$ by Lemma 4.1 and the super version of Schur’s Lemma,

$$\dim_k (H \otimes L(\lambda)) = \dim_k (\text{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu))) \otimes L(\lambda)) \leq (\dim_k L(\lambda + p\mu)/ \dim_k L(\lambda)) \cdot \dim_k L(\lambda),$$

so our map must be an isomorphism. Finally, since the action $G_1$ on $H$ is trivial, we have $H \cong F^* M$ for some $G_{ev}$-supermodule $M$. Since $L(\lambda + p\mu)$ is irreducible, $M$ must be irreducible. Since $H$ has highest weight $p\mu$, $M \cong L_{ev}(\mu)$. \qed

**References**


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