Lecture n_1: Real, complex and quaternionic representations

(See also Serre, Section 13.2)

Let (ρ, V) be a representation of a finite group G. We say ρ is **real** if there is a choice of basis such that ρ can be represented with real matrices. Equivalently, $V \simeq W \otimes_{\mathbb{R}} \mathbb{C}$ where W is a representation of G over $\mathbb{R}$.

If ρ is real, then ρ is self-conjugate, i.e., $\rho \simeq \overline{\rho}$. The converse is not true.

Example. *Let ρ be the irreducible 2-dimensional representation of Q_8. Then ρ is self-conjugate but not real.*

To see this, we may realize $\rho(i) = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, $\rho(j) = w := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. *Then* $w\rho(g)w^{-1} = \overline{\rho}(g)$ *for all* g.

On the other hand, suppose there exist $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ *such that* $\rho'(g) := A\rho(g)A^{-1}$ *is real for* $g \in Q_8$. *After scaling, we may assume* $A \in \mathrm{SL}_2(\mathbb{C})$ *so* $ad - bc = 1$. *We compute*

$$\rho'(i) = A \begin{pmatrix} i & \\ & -i \end{pmatrix} A^{-1} = i \begin{pmatrix} ad+bc & -2ab \\ 2cd & -(ad+bc) \end{pmatrix},$$

$$\rho'(j) = A \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} A^{-1} = \begin{pmatrix} ad+bc & -a^2-b^2 \\ c^2+d^2 & -(ad+bc) \end{pmatrix}.$$

Consequently $ad + bc = 0$, *which combined with* $ad - bc = 1$ *means* $ad = \frac{1}{2} = -bc$. *Note that* $\rho'(i)^{-1} = \rho'(-i) = -\rho'(i)$. *Now* $iji^{-1} = -j \in Q_8$ *implies*

$$\rho'(i)\rho'(j)\rho'(i)^{-1} = -\rho'(j),$$

i.e.,

$$\begin{pmatrix} & (2ab)^2(c^2+d^2) \\ (2cd)^2(a^2+b^2) & \end{pmatrix} = \begin{pmatrix} & a^2+b^2 \\ -(c^2+d^2) & \end{pmatrix}.$$

But $2ab, 2cd \in i\mathbb{R}^\times$ *means* $2ab, 2cd < 0$ *Since both of these matrices must be real, comparing top right entries says* a^2+b^2 *and* c^2+d^2 *have opposite signs, whereas comparing the bottom left tells us the opposite. Contradiction.*

We will say ρ is **quaternionic** if ρ is self-conjugate but not real. We will say ρ is **(properly) complex** if ρ is not self-conjugate.

Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ denote the dual of V and ρ^* be the contragredient representation:

$$\langle \rho(g)v, \rho^*(g)v^* \rangle = \langle v, v^* \rangle,$$

where $\langle , \rangle : V \times V^* \to \mathbb{C}$ is a bilinear pairing, or equivalently a linear map $V \otimes V^* \to \mathbb{C}$. Different pairing give equivalent definitions of ρ^*.

Lemma 1. *Let (ρ, V) be a representation of G. Then $(\overline{\rho}, \overline{V}) \simeq (\rho^*, V^*)$.*

Proof. Recall that $\rho^*(g) \simeq {}^t\rho(g)^{-1}$. Hence if $\lambda_1(g), \ldots, \lambda_n(g)$ denote the eigenvalues of $\rho(g)$,

$$\chi_{\rho^*}(g) = \sum \lambda_i(g)^{-1} = \sum \overline{\lambda}_i(g) = \chi_{\overline{\rho}}.$$

(See also Exercise 2.3 in Serre.) $\square$

Exercise 1. *Suppose V, W are representations of G. Show that $\mathrm{Hom}_{\mathbb{C}}((V \otimes W)^G, \mathbb{C})$ (G-invariant linear forms on $V \otimes W$) is isomorphic (as a vector space) to $\mathrm{Hom}_G(V, W^*)$. (Hint: for $\ell \in \mathrm{Hom}_{\mathbb{C}}((V \otimes W)^G, \mathbb{C})$, consider the map $\alpha : V \to W^*$ given by $v \mapsto (w \mapsto \ell(v \otimes w))$.)*

Exercise 2. *Suppose V, W are irreducible representations of G. Using the previous exercise, deduce that the tensor representation $V \otimes W$ of G contains the trivial representation if and only if $V \simeq W^*$.*

Proposition 1. *Let (ρ, V) be an irreducible representation of G. Then $\overline{\rho} \simeq \rho$ if and only if there is a nondegenerate bilinear form $B : V \times V \to \mathbb{C}$ which is G-invariant, i.e.*

$$B(\rho(g)v, \rho(g)w) = B(v, w) \quad \forall v, w \in V,\ g \in G.$$

In this case, B is unique up to scaling.

Proof. A bilinear form on V is equivalent to a linear form on $V \otimes V$, so we may identify $\mathrm{Hom}_{\mathbb{C}}(V \otimes V, \mathbb{C})$ with the bilinear forms on V. Now a G-invariant form is simply an element of $\mathrm{Hom}_{\mathbb{C}}((V \otimes V)^G, \mathbb{C})$. By the first exercise, this is an element of $\mathrm{Hom}_G(V, V^*) \simeq \mathrm{Hom}_G(V, \overline{V})$, which is nonzero if and only if $\overline{V} \simeq V$. Such an element is unique up to scaling as $\dim \mathrm{Hom}_G(V, \overline{V}) \leq 1$ by Schur's Lemma. $\square$

Consider the bilinear form B from the proposition. This corresponds to an element in $\mathrm{Hom}_{\mathbb{C}}(V \otimes V, \mathbb{C}) \simeq V^* \otimes V^*$. The G invariance means that B corresponds to the trivial representation in $V^* \otimes V^*$. Recall

$$V^* \otimes V^* = \mathrm{Sym}^2(V^*) \oplus \Lambda^2(V^*).$$

Moreover, B is symmetric (resp. alternating) iff the trivial representation is contained in $\mathrm{Sym}^2(V^*)$ (resp. $\Lambda^2(V^*)$).

Theorem 1. *Let (ρ, V) be an irreducible representation of G. Then ρ is real if and only if V has a nondegenerate G-invariant* symmetric *bilinear form.*

Proof. If ρ is real, we can construct a symmetric B as follows. Choose a basis so that the matrices $\rho(g)$ are real. Let B_0 be any nondegenerate bilinear form V. Then averaging B_0 over G gives B.
Suppose V has a nondegenerate G-invariant symmetric bilinear form B. Let $(-,-)$ be a G-invariant inner product on V, conjugate linear in the second variable. Consider the map $\varphi : V \to V$ given by

$$B(v, w) = \overline{(\varphi(v), w)}.$$

This makes φ conjugate linear and bijective. Then φ^2 is $\mathbb{C}$-linear, and thus an isomorphism. Note

$$(\varphi^2(v), w) = \overline{B(\varphi(v), w)} = \overline{B(w, \varphi(v))} = (\varphi(w), \varphi(v)) = \overline{(\varphi(v), \varphi(w))}.$$

In particular,

$$(\varphi^2(v), v) = (\varphi(v), \varphi(v)) \geq 0$$

for all v. This means all diagonal matrix coefficients for φ^2 are real and positive in any basis. One also sees:

$$(\varphi^2(v), w) = \overline{(\varphi^2(w), v)}.$$

This means that φ^2 is a Hermitian positive-definite matrix.
Fact: There is a unique positive-definite Hermitian Φ such that $\Phi^2 = \varphi^2$. Moreover, $\Phi = P(\varphi^2)$ for some polynomial P. Consequently φ commutes with Φ.
Let $\sigma = \varphi\Phi^{-1}$. Then $\sigma^2 = \varphi^2\Phi^{-2} = 1$, so σ is diagonalizable. Write $V = V_+ \oplus V_-$ (over $\mathbb{R}$) where $V_\pm$ is the ± 1-eigenspace of σ. Since σ is conjugate linear, multiplication by i interchanges V_+ and V_-.
Now G-invariance implies φ and σ commute with each $\rho(g)$. Hence V_+ and V_- are G-stable $\mathbb{R}$-subspaces of V. $\square$

Theorem 2 (Frobenius–Schur)**.** *Let ρ be an irreducible representation with character χ. Then the* **Frobenius–Schur indicator**

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 1 & \text{if } \rho \text{ is real,} \\ 0 & \text{if } \rho \text{ is complex,} \\ -1 & \text{if } \rho \text{ is quaternionic.} \end{cases}$$

Proof. Let χ_σ (resp. χ_α) denote the character of the symmmetric (resp. alternating) square of ρ. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \langle 1, \chi_\sigma - \chi_\alpha\rangle = \langle 1, \chi_\sigma\rangle - \langle 1, \chi_\alpha\rangle.$$

Now ρ is real (resp. quaternionic) iff ρ possesses a non-degenerate invariant symmetric (resp. alternating) bilinear form, i.e., iff $\langle 1, \overline{\chi}_\sigma\rangle$ (resp. $\langle 1, \overline{\chi}_\alpha\rangle$ is nonzero. But note the multiplicity of the trivial representation (which is self-conjugate) in $\mathrm{Sym}^2(\rho)$ is the same as its multiplicity in $\mathrm{Sym}^2(\rho^*)$, and similarly for $\Lambda^2(\rho)$. □

Exercise 3. *Show that an irreducible representation ρ is self-conjugate if and only if χ_ρ is real.*

Exercise 4. *Let (ρ, V) be the irreducible 2-dimensional representation of Q_8. Check that there is a unique-up-to-scaling nondegenerate G-invariant bilinear form B on V. Check B is alternating (i.e., $B(v, v) = 0$ for all v).*

Exercise 5. *Let G be group of odd order, and ρ be a nontrivial irreducible representation of G. Show ρ is (properly) complex.*