

# Probability (MATH 4733 - 01) Fall 2011

## Exam 2 - Practice Problems: Selected answers and hints

Due: never

Here are some sample problems for you. I recommend *studying first*, using previous homeworks, exams and exam practice problems. Then treat this like a mock exam (i.e., without any aids, but don't worry about a time limit). Afterward, go through with aids and figure out the ones you couldn't do on your own. Finally, check your answers and make sure you can do each problem on your own.

$X$  and  $Y$  denote random variables.  $A$  and  $B$  denote events in a sample space  $S$ .

1. T F I read the above recommendations.
2. T F I followed the above recommendations.
3. T F If  $P$  is a probability function on  $S$ , then  $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ .

Answer: T. Draw a Venn diagram.

4. T F If  $X = Y^2$ , then  $E(X) = E(Y)^2$ .

Answer: F.  $E(XY) = E(X)E(Y)$  if  $X$  and  $Y$  are independent, but not in general. For a counterexample to this statement, think  $X = 0, 1$  according to a coin flip being heads or tails.

5. State the axioms of a probability function.
6. State the definition of a random variable  $X$ . State what it means for  $X$  to be discrete or continuous.
7. State the Central Limit Theorem.
8. Derive the pdf for a binomial distribution with parameters  $(n, p)$ .
9. Derive the pdf for a geometric distribution with parameters  $p$ .
10. Roll one die 3 times. What is the probability of a sum of 12 given that the first two rolls were the same?

Answer:  $1/12$

11. Assume there are 20 people in the class, and I assign grades at random in one of the following ways?
  - (a) If I give each individual an A, B, C, D or F, each with probability 0.2, what is the expected number of A's? What is the probability I give at least 5 A's?
  - (b) If decide to give 3 A's, 4 B's, 6 C's, 4 D's and 3 F's, what is the probability you get at least a B?
  - (c) Which grading scheme do you like better?

Answer: (a) is really binomial as the only relevant possibilities are A or not an A. So expected number of A's is 4, and probability of at least 5 A's is most easily computed as  $1 - P(4A's) - P(3A's) - \dots P(0A's)$ .

(b)  $7/20$

(c) In scheme (a) you have a better chance of getting an A, but a better chance of passing for scheme (b), so I vote (b).

12. There are 5 men and 5 women invited to a banquet, who will be seated at a long table with 5 seats on the left side and 5 on the right side. If all men are seated on the left side, how many ways are there to seat the guests?

Answer:  $5! \cdot 5!$

13. Suppose an urn contains 5 white balls and 3 red balls. Draw 3 without replacement. What is the probability at least 2 are red?

Answer: Use hypergeometric distribution

14. Determine the mean and variance of  $X$  in the following cases:

(a)  $X$  is the number of 1's in  $n$  die rolls.

(b)  $X$  has pdf  $f_X(x) = 4e^{-4x}$ ,  $x > 0$ .

(c)  $X$  has pdf  $f_X(x) = \frac{1}{2\sqrt{\pi}}e^{-(x-1)^2/4}$ ,  $x \in \mathbb{R}$ .

Answer: (a)  $E(X) = n/6$ ,  $Var(X) = 5n/36$  (write as sum of  $n$  independent variables)

(b)  $E(X) = 1/4$ ,  $Var(X) = 1/16$  (cf. Example 3.5.6 and Exercise 3.6.11)

(c)  $E(X) = 1$  and  $Var(X) = 2$  (normal with  $\mu = 1$ ,  $\sigma = \sqrt{2}$ )

15. Let  $X_1$ ,  $X_2$  and  $X_3$  be independent random variables with uniform distribution on  $[0, 1]$ .

(a) What is the expected value of the sum?

(b) What is the expected value of the largest value of  $X_1$ ,  $X_2$  and  $X_3$ ?

Answer: (a)  $3/2$  as expected values are additive

(b) Use order statistics

16. Define a continuous random variable whose expected value is infinite?

Answer: Hint: you want a function  $f(x)$  whose integral on  $(-\infty, \infty)$  is finite but  $\int_{-\infty}^{\infty} xf(x)dx = \infty$ .

17. Suppose  $X$  and  $Y$  are random variables with  $f_{X,Y}(x, y) = \frac{4}{3}(x + xy)$ , where  $0 \leq x, y \leq 1$ .

(a) Find  $f_X(x)$ .

(b) Find  $P(X \leq \frac{1}{2}, Y \leq \frac{1}{3})$ .

(c) Find  $P(X < \frac{1}{2} | Y = 1)$ .

(d) Are  $X$  and  $Y$  independent?

Answer: Review joint and marginal pdfs.

18. Let  $X$  and  $Y$  be independent random variables with  $f_X(x) = f_Y(x) = \frac{4}{x^5}$  for  $x > 1$ . Compute the pdf for  $X + Y$ .

Answer: See Theorem 3.8.1

19. Suppose an urn contains 10 white balls and 5 red balls. You play a game where you first draw one ball at random. If a white ball is drawn, you roll 1 die. If a red ball is drawn, you roll 2 dice. You win  $x$  dollars where  $x$  is the sum of the faces rolled. If you win 3 dollars, what is the probability you drew a red ball?

Answer: Use Bayes' Theorem

20. You are playing 5 card stud with Coolio. You are each dealt 5 cards, one at time. The first face down, and the last 4 face up. After the first four cards are dealt, you have: 5 of spades (face down), queen of hearts, jack of spades and a 5 of diamonds. Coolio has a 3 of spades, 6 of spades and a king of spades showing. What is the probability he will get a flush (5 spades)?

Answer: Since you don't know his face down card, you can treat the problem as him choosing 2 more cards in addition to his 3 that are face up. There are 8 spades and 45 total cards remaining, so the answer is  $(8/2) / (45/2)$ .

21. Okay, I'm afraid I'm not making any more problems, but go over your homework from Chapter 4.