Probability (MATH 4733 - 01) Fall 2011
Exam 2 - Practice Problems: Selected answers

Don't read until you have sincerely attempted these problems.

It's possible there are some typos/errors here, so if you think you find one, let me know!

In the wild

X and Y denote random variables, discrete or continuous.

1. T F $E(X+Y) = E(X) + E(Y)$

   Answer: T

2. T F If X and Y are independent, then $E(XY) = E(X)E(Y)$.

   Answer: T

3. T F $Var(X+Y) = Var(X) + Var(Y)$

   Answer: F. It is true if X and Y are independent.

4. T F If X and Y are independent, then $F_{X+Y}(t) = F_X(t) + F_Y(t)$.

   Answer: F. Think about what $\lim_{t\to\infty} F_{X+Y}(t)$ needs to be.

5. Suppose X is continuous. Define $E(X)$.

6. Define $Var(X)$.

7. Suppose X is continuous. Define the pdf of X. (**Bonus:** What to the letters "pdf" stand for?)

8. Let X be a binomial random variable with parameters (n, p), i.e., the number of successes on n independent random trials each with success probability p. Determine

   (a) the pdf of X;

   (b) the cdf of X;

   (c) $E(X)$;

   (d) $Var(X)$.

   Show your work for (b)(c)(d).

   Answer: (a) $p_X(k) = \binom{n}{p} p^k (1-p)^{n-k}$ for $k = 0, 1, \ldots, n$. (Theorem 3.2.1)

   (b) $F_X(k) = \sum_{j=0}^{k} \binom{n}{p} p^j (1-p)^{n-j}$ for $k = 0, 1, \ldots, n$. (No simple expression)

   (c) See Example 3.9.3

   (d) See Example 3.9.8

9. Repeat the above problem when X is a hypergeometric random variable with parameters (n, r, w), i.e., the number of red balls in n draws (without replacement) from an urn with r red balls and w white balls.

   Answer: (a) See Theorem 3.2.2.

   (b) Write as sum as in previous problem.

   (c) See Exercise 3.9.7

   (d) $nrw(r+w-n)/((r+w)^2(r+w-1))$ (This is too involved for an exam problem, though a special case might be reasonable.)

10. Repeat the above problem when X is the exponential distribution with pdf $\lambda e^{-\lambda x}$ $(\lambda > 0, x > 0)$.

    Answer: (b) $1 - e^{-\lambda x}$

    (c) $1/\lambda$. See Example 3.5.6 where $\lambda = \frac{1}{\mu}$.

    (d) See Exercise 3.6.11

11. Repeat the above problem when $X = Y^2$ where Y is a continuous random variable with uniform distribution on $[0, 1]$.

    Answer: It's easier to do (b) first:

    (b) $F_Y(y) = y$ for $0 < y < 1$. Thus

    $$F_X(x) = P(X < x) = P(Y^2 < x) = P(Y < \sqrt{x}) = F_Y(\sqrt{x}) = \sqrt{x}$$

    for $0 < x < 1$.

    (a) $f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{2\sqrt{x}}$ for $0 < x < 1$.

    (c) $E(X) = \int_0^1 x f_X(x) dx = \frac{1}{2} \int_0^1 \sqrt{x} = \frac{1}{3} x^{3/2}|_0^1 = \frac{1}{3}$.

    (d) $Var(X) = E(X^2) - E(X)^2 = \int_0^1 x^2 f_X(x) dx - \frac{1}{9} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$.

12. Suppose X_i is the face value of a die on the i roll. Find the pdf of $X_1 + X_2$.

    Answer: See Example 3.7.2. Also note Theorem 3.8.1.

13. Let X be a continuous random variable with the uniform distribution on $[0, 1]$. Let X_1, X_2, X_3 be a random sample of size 3. Find the mean and median for the 1st order statistic $X_{min} = X_1'$.

    Answer: We know $f_X(x) = 1$ and $F_X(x) = x$ for $0 < x < 1$. Then $f_{X_{min}}(x) = n(1-x)^{n-1}$ and $F_{X_{min}}(x) = 1 - (1-x)^n$ for $0 < x < 1$. The mean

    $$E(X_{min}) = \int_0^1 nx(1-x)^{n-1} dx = -\int_1^0 n(1-u)u^{n-1} du = n(1/n - 1/(n+1)) = \frac{1}{n+1}.$$

    The median is x_0 such that $F_{X_{min}}(x_0) = 1 - (1-x_0)^n = 0.5$, i.e., $x_0 = 1 - \sqrt[n]{0.5}$.

By the book

Section 3.7: 13, 15, 21, 28, 43

Answer: 28 (a) $u^2/4$. (b) $v \ln u - v \ln v + v$. (c) When $v \leq 1-u$, $3u^2 v$; when $v > 1-u$ $3u^2 - 2u^3 - (1-v)^3$.

Section 3.8: 2

Answer: 2. $f_{X+Y}(w) = w^2 e^{-w}/2$ for $w \geq 0$.

Section 3.9: 1, 3, 6, 11, 14, 20

Answer: 6. $(n/8)(p-q)$

Answer: 20. $E(T) = n(n+1)p/2$ and $Var(T) = n(n+1)(2n+1)(p-p^2)/6$. (The second part requires $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$, which I don't expect you to remember; though I might expect you to know $\sum_{k=1}^n n = \frac{n(n+1)}{2}$, used for $E(T)$.)

Section 3.10: 1, 3