An introduction to (principal) L-functions and L-groups

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These are some informal notes for a prep session for grad students at TORA VII (Spring 2017), meant as a prelude to Freydoon Shahidi's talk "L-functions and monoids." The specific goals were to give some overview of the work of Godement and Jacquet on principal L-functions and the notion of L-groups.

Warning: these notes have not been proofread. (So if you're going to read them, you may as well proofread them for me, and send me corrections.)

1 A brief introduction to L-functions

L-functions are certain complex functions in number theory and related fields which encode arithmetic information into an analytic function. The most famous of these is the **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad \Re(s) > 1.$$

The product is over all primes p, which is called an **Euler product**. Both the series and the product converge for $\Re(s) > 1$, and $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, and there is a functional equation relating $\zeta(s)$ to $\zeta(1-s)$.

The next best known L-functions are the Dirichlet L-functions. Let $N \in \mathbb{N}$. Suppose $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character, which we extend to $\mathbb{Z}/N\mathbb{Z}$ by $\chi(n) = 0$ if $n \in \mathbb{Z}/N\mathbb{Z}$ is not invertible, and by composition

$$\chi : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times.$$

Then we call χ a **Dirichlet character** mod N, and consider its **Dirichlet L-function**

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}, \quad \Re(s) > 1.$$

Note that if $\chi = 1$, then

$$L(s,1) = \prod_{p \nmid N} \frac{1}{1-p^{-s}} = \zeta(s) \prod_{p|N} (1-p^{-s}).$$

Hence $L(s,1)$ has meromorphic continuation to $\mathbb{C}$ with a simple pole only at $s = 1$ and a functional equation. If $\chi \neq 1$, then $L(s,\chi)$ continues to an entire function of $\mathbb{C}$ and has a functional equation relating $L(s,\chi)$ to $L(1-s,\chi)$.

The point is that the analytic behaviour of the L-functions, namely the locations of zeros and poles, encode deep arithmetic information. For instance the fact that $\zeta(s)$ has a pole at $s = 1$ implies there are infinitely many prime numbers and the location of the zeros of $\zeta(s)$ tell us about the distribution of prime numbers. The fact that Dirichlet L-functions $L(s,\chi)$ for $\chi \neq 1$ do not have a pole and are nonzero at $s = 1$ implies there are infinitely many primes in any arithmetic progression with gcd 1.

These Dirichlet L-functions are associated to algebraic objects, namely Dirichlet characters. There is another way to look at Dirichlet L-functions, which will provide impetus for the Langlands program.

By class field theory, there is a correspondence between (primitive) Dirichlet characters and *Hecke characters* of finite order. Recall the adeles $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$ are a restricted direct product

$$\mathbb{A} = \prod_v{}' \mathbb{Q}_v = \prod_p{}' \mathbb{Q}_p \times \mathbb{R}$$

and the ideles

$$\mathbb{A}^\times = \prod_v{}' \mathbb{Q}_v^\times = \prod_p{}' \mathbb{Q}_p^\times \times \mathbb{R}^\times$$

is the group of invertible adeles.[1] The restricted direct product means that for $\alpha = (\alpha_v) \in \mathbb{A}^\times$ (resp. $\mathbb{A}$) we require $\alpha_p \in \mathbb{Z}_p^\times$ (resp. $\mathbb{Z}_p$) for almost all[2] p. Note that $\mathbb{Q}^\times$ embeds as a diagonal subgroup in $\mathbb{A}^\times$. It is a discrete subgroup and approximation says $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times$.

A **Hecke (or idele class) character of finite order** $\chi = \otimes \chi_v$ is a character

$$\chi : \mathbb{A}^\times / \mathbb{Q}^\times \to \mathbb{C}^\times$$

whose kernel has finite index in the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$. It is not hard to show that χ_p must be trivial on $\mathbb{Z}_p^\times$ for almost all p. At such p, we say χ_p is **unramified**. We can define the local L-factors by

$$L(s,\chi_p) = \begin{cases} \frac{1}{1-\chi(p)p^{-s}} & \chi_p \text{ unramified} \\ 1 & \chi_p \text{ ramified.} \end{cases}$$

[1] We use p for a finite prime, i.e., a prime number in $\mathbb{N}$, and v to denote a finite prime or the "infinite prime" ∞. Here $\mathbb{Q}_\infty = \mathbb{R}$.

[2] Almost all means all but finitely many here.

Then one defines the **Hecke L-function** by

$$L(s,\chi) = \prod_p L(s,\chi_p), \quad \Re(s) > 1.$$

The correspondence between primitive Dirichlet characters λ mod N and Hecke characters χ of finite order will give a Hecke character χ which is ramified precisely the primes $p|N$. The correspondence preserves L-functions, i.e., $L(s,\lambda) = L(s,\chi)$.

Hecke considered more than just finite order characters, and thus extended Dirichlet's theory of L-functions: they still have meromorphic continuation and a functional equation. Tate, in his thesis, developed a different approach to the theory of Hecke L-functions. We will explain this at least in part below, in a more general context.

There are other L-functions out there besides Dirichlet and Hecke L-functions. If one studies Dedekind zeta functions (zeta functions for other number fields), one is naturally led to **Artin L-functions** which generalize Dirichlet L-functions. Namely, to study (say finite Galois) extensions $K/\mathbb{Q}$, we can consider irreducible representations ρ of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. If $K/\mathbb{Q}$ is abelian, then each such ρ is just a character, and can be viewed as a Dirichlet character, so we can associate to ρ a Dirichlet L-function. Artin defined L-functions $L(s,\rho)$ for more general Galois representations ρ. Using a theorem of Brauer, one gets that $L(s,\rho)$ has meromorphic continuation to $\mathbb{C}$ and has a functional equation. Artin conjectured that $L(s,\rho)$ is entire if $\rho \neq 1$. However it seems very hard to prove the analytic statement that $L(s,\rho)$ has no poles because $L(s,\rho)$ is an algebraically defined L-function.

One can ask if there is a way to generalize Hecke characters to encapsulate Artin L-functions, and Langlands conjectured there is: namely one should look at **automorphic representations** π of $\mathrm{GL}_n(\mathbb{A})$. (Here n should be the dimension of the Artin representation ρ.) These automorphic representations are analytic objects rather than algebraic objects, so it is relatively easier to study the analytic properties of their L-functions. Indeed, the analogue of Artin's conjecture is known for automorphic representations, so if one could show Artin representations correspond to automorphic representations (in the sense that their L-functions agree), one could deduce Artin's conjecture.

The theory of (principal) L-functions for automorphic representations was developed by Godement–Jacquet which generalizes Tate's approach from the case of $\mathbb{A}^\times = \mathrm{GL}_1(\mathbb{A})$. We summarize the different types and approaches to L-functions in the following table.

degree	algebraic L-functions	analytic L-functions
1	Dirichlet	Hecke/Tate
n	Artin	Godement–Jacquet

Here the **degree** of an L-function is a way to measure it's complexity. For an irreducible Artin representation ρ of dimension n, $L(s,\rho)$ should have degree n. More precisely, the L-function at all unramified places should be the reciprocal of a polynomial of fixed degree in p^{-s}. This degree is the degree of the L-function.

As a final remark in this section, we have not said in general what an L-function is. The correct definition of a general L-function is a major open problem, but we list the main properties we want our L-functions $L(s)$ to possess:

- $L(s)$ should have an Euler product $\prod_p L_p(s)$ valid in some right half plane;
- $L(s)$ should have meromorphic continuation to $\mathbb{C}$, and the poles should be of finite order (in fact $L(s)$ should be entire if it does not have a "factor" of $\zeta(s)$);
- $L(s)$ should have a function equation relating $L(1-s) = W(s)L(s)$ for some simple function $W(s)$; and
- $L(s)$ should be bounded in vertical strips.[3]

2 Principal L-functions for $\mathrm{GL}(n)$

Here we give an bare-bones overview of the work of Godement and Jacquet (1972) on principal L-functions for $\mathrm{GL}(n)$.[4][5] We won't explain what the "principal" here refers to, except to say that Langlands conjectured one should be able to associate L-functions to representations π with nice properties, and there are various kinds of L-functions we could consider for a given representation π of $\mathrm{GL}(n)$, with the principal one being the most straightforward choice. (It has degree n in the global situation or the unramified local situation.) Different kinds of L-functions for π can be used to study different properties of π—e.g., one can consider exterior square L-functions attached to π, which are useful in determining if π arises from a "smaller" group (specifically $\mathrm{SO}(2n+1)$).

One can show Dirichlet L-functions satisfy the desired analytic properties using **integral representations**, which are one of the main tools in the theory of L-functions. Namely if χ is a Dirichlet character, we can write

$$L(s,\chi) = \int_0^\infty \Phi_\chi(t) t^{s/2} \, \frac{dt}{t}$$

for an suitable function Φ_χ (essentially a theta series). The above integral is known as the **Mellin transform** of Φ_χ. (The Mellin transform is also used for defining L-functions of

[3]We'll ignore this point in these notes, but it's important for things like "converse theorems" which tell you when a function is actually of L-function of something you know.

[4]I am not going to attempt to tell you enough to make you feel like you understand their approach, but just enough to give you a sense of the main results and a taste of some of the necessary ingredients. Jacquet has some other articles on this titled "Principal L-functions..." with some more details. See also the book of Gelbart–Shahidi for comparison with other methods. Other approaches to L-functions (primarily Rankin–Selberg) are also explained in various notes of Cogdell, Bump's book, and Bump's survey articles.

[5]We should also note that Godement and Jacquet in fact worked in the more general setting of inner forms of $\mathrm{GL}(n)$, i.e., groups of the form $\mathrm{GL}(m,D)$, where D is some division algebra. Godement and Jacquet also work over number fields, but I'll stick to $\mathbb{Q}$ for expository purposes.

modular forms). Then the idea is to break up the integral over $(0, 1)$ and $(1, \infty)$ and use a change of variable to express both parts as integrals over $(1, \infty)$. This rewrites the integral in such a way that it is defined for all s, and a change of variable also gives the functional equation relating $L(s, \chi)$ with $L(1 - s, \chi)$.

Tate's thesis developed an adelic approach to integral representations for GL(1). This was adapted to the case of GL(2) by Jacquet and Langlands, and to GL(n) by Godement and Jacquet. Many other cases are known, with different approaches to getting integral representations such as the Rankin–Selberg method, the Langlands–Shahidi method and the doubling method. Here we just discuss the approach of Godement and Jacquet for GL(n). One advantage of this method over others is it does not require Whittaker models. For groups besides GL(n), not all representations of interest will have Whittaker models (i.e., are not generic), e.g., automorphic representations of GSp(4) associated to holomorphic Siegel modular forms.

Let π be a **cuspidal automorphic representation** of $\mathrm{GL}_n(\mathbb{A})$. This factors as $\pi = \otimes \pi_v$, where π_v is a smooth irreducible (infinite-dimensional) representation of $\mathrm{GL}_n(\mathbb{Q}_v)$ for each v. If you don't know what all this means, the key point for us now is that π is a representation on some infinite dimensional (complex) vector space V, and the vectors in here can be viewed as L^2 functions φ on $\mathrm{GL}_n(\mathbb{A})$.[6] If $\varphi \in \pi$, the action of π is by right translations, so $(\pi(g)\varphi)(x) := \varphi(xg)$. Moreover V is (topologically) generated by factorizable functions $\varphi = \otimes \varphi_v$, where each φ_v is a function on $\mathrm{GL}_n(\mathbb{Q}_v)$.[7]

Hope: We can define $L(s, \pi)$ as a Mellin transform, i.e., a certain integral $I(\varphi, s)$, of some nice vector $\varphi = \otimes \varphi_v \in V$. Furthermore, we would like a factorization $I(\varphi, s) = \prod_v I(\varphi_v, s)$ which gives an Euler product $L(s, \pi) = \prod_v L(s, \pi_v)$, i.e., $I(\varphi_v, s) = L(s, \pi_v)$ for each v.

Of course you might first want to know, what should the L-function be? How can we define it? This is also part of what Godement and Jacquet do, generalizing what Tate did for GL(1). What is important is that this L-function is the right function generalizing Dirichlet L-functions and L-functions of modular forms. While for Dirichlet characters or modular (new)forms, it is easy to define a natural L-function in terms of a Dirichlet series $\sum \frac{a_n}{n^s}$, in general it is much easier to define the desired L-function by the Euler product.

Let me just describe what the factors should be at most places. For almost all p, π_p is an **unramified principal series**. This means there are characters $\chi_1, \ldots, \chi_n$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ such that π_p is induced from the character Ξ of the standard **Borel** (upper triangular)

[6] The automorphic part means that $\varphi(\gamma x) = \varphi(x)$ for $\gamma \in \mathrm{GL}_n(\mathbb{Q})$, and when $n = 2$ this corresponds to the transformation law for modular forms.

[7] Often one works with smooth automorphic representations rather than L^2 ones, and in this framework φ_∞ is not a function of $\mathrm{GL}_n(\mathbb{R})$. Instead, one works with $(\mathfrak{g}, K)$-modules at ∞. However, for cuspidal representations, it doesn't really matter whether one works in the smooth or the L^2 context.

subgroup B_p given by

$$\Xi \begin{pmatrix} a_1 & * & \cdots & * \\ & a_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & a_n \end{pmatrix} = \chi_1(a_1)\chi_2(a_2)\cdots\chi_n(a_n).$$

A general property one wants for L-functions is that they are "inductive," which means in this case we should have

$$L(s,\pi_p) = \prod_{i=1}^{n} L(s,\chi_i) = \prod_{i=1}^{n} \frac{1}{1-\chi_i(p)p^{-s}}. \tag{1}$$

In particular, $L(s,\pi_p)$ should be the reciprocal of a polynomial of degree n in p^{-s} when π_p is unramified. In general, at any prime p, $L(s,\pi_p)$ should be the reciprocal of a polynomial of degree $\leq n$ in p^{-s}, so we will say $L(s,\pi)$ is an L-function of degree n. (At the infinite place, $L(s,\pi_\infty)$ should be a product of Gamma functions.)

Next I'll briefly describe the global theory including the main result, and then come back and talk about the local theory that gives (1).

2.1 Global theory

Let Φ be a **Schwartz–Bruhat function** on $M_n(\mathbb{A})$. This means that Φ is smooth and rapidly decreasing. We can assume $\Phi = \otimes\Phi_v$, and for a finite prime p the Schwartz–Bruhat condition means each Φ_p is locally constant of compact support. In fact we can assume Φ_p is the characteristic function of the standard maximal compact subgroup $K_p = \mathrm{GL}_n(\mathbb{Z}_p)$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ for almost all p. Then we define the **zeta integral**, for $\varphi \in \pi$, by

$$Z(\Phi,s,\varphi) = \int_{\mathrm{GL}_n(\mathbb{A})} \Phi(x)\varphi(x)|\det x|^s\,dx,$$

where dx is a Haar measure on $\mathrm{GL}_n(\mathbb{A})$. One of the main issues in this theory is convergence, and the Schwartz–Bruhat function makes this integral converge for $\Re(s) > n$. We remark that if $\varphi = \otimes\varphi_v$, then the above global zeta integral factors into local zeta integrals

$$Z(\Phi,s,\varphi) = \prod_v Z(\Phi_v,s,\varphi_v) = \prod_v \int_{\mathrm{GL}_n(F_v)} \Phi_v(x)\varphi_v(x)|\det x|^s\,dx.$$

In this right half-plane of convergence, one can rewrite $Z(\Phi,s,\varphi)$ as a sum of two integrals, one involving Φ and φ, and one involving $\hat{\Phi}$ and $\check{\varphi}$, where $\hat{\Phi}$ is a **Fourier transform**[8] of Φ and $\check{\varphi}(g) = \varphi(g^{-1})$. We note that $\check{\varphi}$ is does not lie in V in general, but rather the space $\check{V}$ for the contragredient representation $\check{\pi}$.

[8] The Fourier transform on M_n is defined by an integral over $n \times n$ matrices of the form $\hat{\Phi}(x) = \int \Phi(y)\psi(\mathrm{tr}xy)\,dy$, where ψ is an additive character and dy is a suitable measure.

Theorem 1. *$Z(\Phi, s, \varphi)$ has analytic continuation to all $s \in \mathbb{C}$, and satisfies the functional equation*

$$Z(\Phi, s, \varphi) = Z(\hat{\Phi}, n - s, \check{\varphi}).$$

Ideally, we would like to be able to choose Φ, φ so that $Z(\Phi, s, \varphi)$ is our desired L-function. However, this is not always possible. Nevertheless, it is possible to write the desired L-function $L(s, \pi) = \sum Z(\Phi_i, s, \varphi_i)$ for some finite collection of "test functions" (Φ_i, φ_i). This is sufficient to get the desired theorem

Theorem 2. *There is a* **principal L-function** *$L(s, \pi)$ associated to π which is entire, satisfies the functional equation*

$$L(s, \pi) = \varepsilon(s, \pi) L(1 - s, \check{\pi})$$

for a suitable function $\varepsilon(s, \pi)$, and has an Euler product

$$L(s, \pi) = \prod_v L(s, \pi_v), \quad \Re(s) > \frac{n-1}{2}.$$

Here each $L(s, \pi_v)$ is an entire holomorphic function, with $L(s, \pi_p)$ given by (1) *for π_p unramified.*

We note that Godement and Jacquet do not explicitly compute the local factors $L(s, \pi_p)$ in all cases when π_p is ramified, as a complete classification of the local components π_p was not known at the time. They note that their description of $L(s, \pi)$ is therefore incomplete. However, the explicit description of $L(s, \pi_p)$ is not too hard (e.g., $L(s, \pi_p) = 1$ for π_p supercuspidal), and knowing $L(s, \pi_p)$ for almost all p determines it for all p by Strong Multiplicity One for $\mathrm{GL}(n)$. The correct way to say what the local L-factors $L(s, \pi_p)$ should be at all places comes via the local Langlands correspondence, which was proven many years after Godement–Jacquet.[9]

2.2 Local theory

Here we briefly describe some of the local input that goes into the above global results.

We just describe the nonarchimedean case. Let ψ be a nontrivial additive character of $\mathbb{Q}_p$. Let (π, V) be a smooth irreducible (infinite-dimensional) representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then there is an invariant bilinear form

$$\langle \cdot, \cdot \rangle : V \times \check{V} \to \mathbb{C}.$$

[9] The local Langlands correspondence for GL(2) was proven in 1980 by Kutzko (or Sally–Shalika for $p \neq 2$), for GL(prime) in 1985 by Kutzko–Moy, and for general $\mathrm{GL}(n)$ independently by Harris–Taylor (2001) and Henniart (2000). (Despite the dates, the Harris–Taylor proof came slightly before Henniart's.)

A **matrix coefficient** of π is a function $\varphi : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ of the form

$$\varphi(g) = \langle \pi(g)v, \check{v} \rangle,$$

for some $v \in V$ and $\check{v} \in \check{V}$. (If π were a finite-dimensional representation, then we could write

$$\pi(g) = \begin{pmatrix} \varphi_{11}(g) & \cdots & \varphi_{1n}(g) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(g) & \cdots & \varphi_{nn}(g) \end{pmatrix},$$

and each of the φ_{ij} are matrix coefficients under this definition, hence the terminology.)

Proposition 1. *For a matrix coefficient φ of π and a locally constant $\Phi : M_n(\mathbb{Q}_p) \to \mathbb{C}$ of compact support, the zeta integral $Z(\Phi, s, \varphi)$ converges in a right-half plane. Moreover:*

- *$Z(\Phi, s + \frac{n-1}{2}, \varphi) \in \mathbb{C}(p^{-s})$; and*
- *there is a rational function $\gamma(s, \pi, \psi)$, such that for all φ and Φ,*

$$Z(\hat{\Phi}, n - s, \check{\varphi}) = \gamma(s, \pi, \psi) Z(\Phi, s, \varphi).$$

One then deduces that as Φ and φ vary, the integrals $Z(\Phi, s + \frac{n-1}{2}, \varphi)$ generate a fractional ideal in $\mathbb{C}[p^s, p^{-s}]$. In fact this ideal has a generator $P(p^{-s})^{-1}$, with $P(x)$ a polynomial, which we normalize to be monic. This generator can be obtained by taking a zeta integral $Z(\Phi, s + \frac{n-1}{2}, \varphi)$ where Φ has sufficiently small support around the identity.

Definition. *We define the* **local L-factor** *attached to π to be*

$$L(s, \pi) = \frac{1}{P(p^{-s})},$$

where P is the generator of the zeta integral ideal in $\mathbb{C}[p^s, p^{-s}]$ described above.

Thus we often say that the local L-factors $L(s, \pi)$ are defined to be as gcd's of zeta integrals.

Defining the **local epsilon factor** by

$$\varepsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) \frac{L(s, \pi)}{L(1 - s, \check{\pi})},$$

one can deduce from the function equation of the zeta function that $\varepsilon(s, \pi, \psi)$ is a monomial in p^{-s}. Multiplying the definition for $\varepsilon(s, \pi, \psi)$ by $L(1 - s, \check{\pi})$ gives a local functional equation. Putting together the local functional equations will give the desired global functional equation. (The γ factors disappear globally, as well as the local dependence on ψ.)

This is, very roughly, what is needed locally to get the analytic properties of our global L-function. To check that the local factors are what we want, we need to calculate the local factors at unramified places.

For simplicity, we will take $n = 2$, and suppose π is an unramified principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is induced from the pair of characters (χ_1, χ_2) on the Borel B. Let $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Then there is a unique (up to scaling) nonzero vector $v \in V$, called the spherical vector, which is K-invariant, i.e., $\pi(k)v = v$ for all $k \in K$. Similarly, let $\check{v}$ be the spherical vector in $\check{v}$. Let φ_π be the matrix coefficient associate to the pair of spherical vectors $(v, \check{v})$.

The **unramified calculation** goes as follows. The Iwasawa decomposition for GL(2) says that $\mathrm{GL}_2(\mathbb{Q}_p) = BK$. We can describe φ_π explicitly as

$$\varphi_\pi(\begin{pmatrix} a & x \\ & b \end{pmatrix} k) = \left|\frac{a}{b}\right|^{1/2} \chi_1(a)\chi_2(b).$$

Let Φ be the characteristic function of K. Then K-invariance of Φ and φ imply that the integral over $\mathrm{GL}_2(\mathbb{Q}_p)$ reduces to the following easily-computable integral over B:

$$\begin{aligned} Z(\Phi, s, \varphi_\pi) &= \int_B \Phi(\begin{pmatrix} a & x \\ & b \end{pmatrix})\varphi_\pi(\begin{pmatrix} a & x \\ & b \end{pmatrix})\chi_1(a)\chi_2(b)|ab|^s \, dx \, d^\times a \, d^\times b \\ &= \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \chi_1(a)\chi_2(b)|ab|^s d^\times a \, d^\times b \\ &= \frac{1}{(1-\chi_1(p)p^{-s})(1-\chi_2(p)p^{-s})}. \end{aligned}$$

In this case, this particular zeta integral is $L(s, \pi)$, which agrees with what we wanted from (1).

3 L-groups

We have explained (roughly) how one can attach L-functions to automorphic representations of $\mathrm{GL}(n)$, and that one can show they are nice (have a functional equation and Euler product, with local factors admitting a simple description at least at unramified places). Automorphic representations for GL(1) correspond to Hecke characters. For GL(2) they correspond to things like elliptic curves, modular forms and Maass forms. However, we are also interested in automorphic representations of other groups G. For instance, Siegel modular forms correspond to automorphic representations on symplectic groups. Generalizing the modularity theorem for elliptic curves, higher dimensional abelian varieties conjecturally correspond to automorphic representations on orthogonal (or spin) groups.

Given an arbitrary (connected) reductive linear algebraic group G defined over $\mathbb{Q}$, one can define automorphic forms on and automorphic representations of $G(\mathbb{A})$. So one would like to associate L-functions to (say cuspidal) automorphic representations π of $G(\mathbb{A})$. In the case where $G = \mathrm{GL}(n)$, the basic idea is to define $L(s, \pi)$ so it equals $L(s, \rho)$ when ρ is a Galois representation into $\mathrm{GL}(n)$ that "corresponds" to π. Globally, not all π will correspond to Galois representations (i.e., representations of $\mathrm{Gal}(\overline{F}/F)$ into GL_n over $\mathbb{C}$ or maybe $\overline{\mathbb{Q}}_p$), but locally this is essentially true. Namely, Langlands conjectured a correspondence between local representations (up to equivalence)

$$\{\text{smooth irred. rep.s } \pi_p \text{ of } \mathrm{GL}_n(\mathbb{Q}_p)\} / \sim \quad \leftrightarrow \quad \{\text{rep.s } \varphi : WD(\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{C})\} / \sim,$$

where $WD(\mathbb{Q}_p)$ is the Weil–Deligne group of $\mathbb{Q}_p$. (We won't define this precisely, but it is a certain subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ augmented with a factor of $\mathrm{SL}_2(\mathbb{C})$.) This is known as the **local Langlands correspondence** for $\mathrm{GL}(n)$ and is now proven. This local Langlands correspondence for $\mathrm{GL}(n)$ is one-to-one, and sends the principal series π_p induced from $\chi_1, \dots, \chi_n$ to a reducible representation corresponding to the direct sum of the χ_i's.[10]

This classifies or parameterizes local components of automorphic representations (or rather the somewhat larger class of smooth local representations), and the point for us is that one knows how to associate local L-functions to the objects on the right. So this tells us what the local L-factors of a global automorphic representation should be in terms local parameters $\varphi = \varphi(\pi_p)$.

At least conjecturally, there is an analogous local Langlands correspondence for more general groups G. Namely, we should have a correspondence

$$\{\text{smooth irred. rep.s } \pi_p \text{ of } G(\mathbb{Q}_p)\} / \sim \quad \leftrightarrow \quad \left\{\text{rep.s } \varphi : WD(\mathbb{Q}_p) \to {}^L G\right\} / \sim,$$

where ${}^L G$ is the ***L*-group** of G. (Technically, one should restrict to "admissible" φ.) For $G = \mathrm{GL}(n)$, this is just $\mathrm{GL}_n(\mathbb{C})$. For general groups, this correspondence is no longer one-to-one, but one should get a finite-to-one surjective map from the left to the right. The fibers of these maps are called ***L*-packets**. Within a packet, all π_p's should have the same local L- and ε-factors, and these should match with the L- and ε-factor for the corresponding parameter $\varphi = \varphi(\pi_p)$. The local Langlands correspondence is now known for many groups besides $\mathrm{GL}(n)$.

The rest of these notes will focus on explaining what this L-group is. This will require understanding some structure and families of algebraic groups, which we look at first.

3.1 Algebraic groups

The theory of algebraic groups is technical, and we'll just try to give a practical overview of the relevant terms and concepts, rather than defining everything precisely or in the best possible way. The focus to know what these things mean in bunch of examples. Three

[10] Langlands also conjectured the existence of a group $L_{\mathbb{Q}}$, now called the *Langlands group*, whose n-dimensional representations should correspond to all automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. This would be a global Langlands correspondence, but even what $L_{\mathbb{Q}}$ should be is unclear.

standard texts are by Borel, Humphries and Springer; see also Milne's online notes. For surveys, see articles by Borel and Springer the Boulder and Corvallis proceedings.[11]

Let F be a field of characteristic 0, and $\overline{F}$ be the algebraic closure. A **(linear) algebraic (matrix) group** G is a subgroup of the **general linear group** $\mathrm{GL}_n(F)$ cut out by polynomial equations. Consequently, for any field extension K/F, we can consider the group $G(K) \subset \mathrm{GL}_n(K)$ of K-points obtained by just using the same polynomial equations over K.[12] For instance the **special linear group** $\mathrm{SL}(n)$ is the algebraic subgroup of $\mathrm{GL}(n)$ consisting of $g \in G$ satisfying the polynomial equation $\det g = 1$. (By polynomial, we mean polynomial in the entries of g.)

Any matrix $B \in \mathrm{GL}(n)$ defines a nondegenerate bilinear form $B : F^n \times F^n \to F$ given by $(u, v) \mapsto {}^t u B v$. This defines a linear algebraic group

$$O_B = \left\{ g \in \mathrm{GL}_n(F) : {}^t g B g = B \right\}.$$

If B is a symmetric matrix, the corresponding form is symmetric, and we call O_B an **orthogonal group**. If O_B is an orthogonal group, the **special orthogonal group** SO_B is the subgroup of matrices of determinant 1, i.e., the intersection with $\mathrm{SL}(n)$. If B is anti-symmetric, i.e., ${}^t B = -B$, then the form is skew symmetric and we call O_B a **symplectic group**. Necessarily symplectic groups are contained in some $\mathrm{SL}(n)$. The linear, orthogonal and symplectic groups are collectively known as the **classical groups**.

Besides looking at automorphic forms on classical groups, there are some other well-known families of algebraic groups we often look at automorphic forms on. First, there are the **projective linear groups**, $\mathrm{PGL}(n) = \mathrm{GL}(n)/Z$ and $\mathrm{PSL}(n) = \mathrm{SL}(n)/Z$, where Z denotes the center of the larger groups. In the case of $\mathrm{GL}(n)$ the center is just the set of scalar matrices (so isomorphic to $\mathrm{GL}(1) \simeq F^\times$) and in $\mathrm{SL}(n)$ the center is the finite set of scalar matrices corresponding to elements in $F^\times$ which are n-th roots of 1. (It is not obvious that the projective linear groups are in fact linear algebraic groups, but it's a theorem they are, i.e., that they are embeddable in $\mathrm{GL}(N)$ for some N as the zero locus of some finite set of polynomials.) There are also (orthogonal and symplectic) **similitude groups**

$$GO_B = \left\{ g \in \mathrm{GL}_n(F) : {}^g B g = \lambda B \text{ for some } \lambda \in F^\times \right\}$$

[11] The Boulder and Corvallis proceedings, PSPM (*Proceedings of Symposia in Pure Mathematics*, by the AMS) IX and PSPM 33, were two of the main places to learn about automorphic forms traditionally, with Corvallis being the closest thing to a textbook for many years. The more recent Edinburgh proceedings (PSPM 61) also has good stuff, but less about the structure of algebraic groups. E.g., two of Jacquet's articles on Godement–Jacquet are in the Corvallis and Edinburgh proceedings. Now there are various other (often easier) places to learn many of these things, but at least Corvallis is still the primary reference for many generalities.

[12] A more standard way to think of an algebraic group G is as a functor from field extensions K/F to groups associating each field K to the subgroup $G(K) \subset \mathrm{GL}_n(K)$. Hence in standard treatments there is a difference between G and $G(F)$, the latter of which is called the F-rational points of G. We'll suppress this distinction, though when I write things like $\mathrm{GL}(n)$ rather than $\mathrm{GL}_n(F)$, I'm thinking of it as a functor—that is, as the formal concept of a general linear group rather than a specific $\mathrm{GL}_n(F)$.

associated to bilinear forms B. One can also consider projectivizations of orthogonal, symplectic and similitude groups by quotienting out by the center (e.g., PGSp(4)).

The last major family of algebraic groups in automorphic forms are **unitary groups** U_H associated to some quadratic extension E/F and a Hermitian (sequilinear) form H. (Again, one can consider similitudes and projectivizations.) I won't define them, but it's similar to the definition of O_B.

To explain some structure theory, we will assume our algebraic group G is **connected** (in the Zariski topology). All of the examples of algebraic groups above except the orthogonal groups O_B are connected. (An element g of an orthogonal group can have determinant ± 1, and there are two connected components of O_B corresponding to the two discrete possibilities for $\det g$, so the special orthogonal groups are connected.)

We say G is **reductive** if it has no nontrivial connected normal unipotent subgroups (unipotent subgroups are one consisting of only unipotent elements). The general Langlands framework considers automorphic forms/representations on reductive groups (however some nonreductive groups are of interest, such as the *metaplectic groups*). This contains the class of **semisimple** groups (no nontrivial connected normal subgroups), and all of the examples we have given above are reductive. Since the center Z of G is a normal subgroup, G being semisimple means connected component of the identity in the center is trivial, which means Z is finite. Hence groups like $\mathrm{GL}(n)$ and similitude groups are reductive but not semisimple. The special linear, special orthogonal and symplectic groups are all semisimple.

To study the above families of reductive groups, one needs to work with various subgroups of our given algebraic group of interest G.

A **torus** T is an algebraic group which is connected and consists only of semisimple (diagonalizable) elements. Necessarily, T is commutative and $T(\overline{F}) \simeq (\overline{F}^\times)^d$ for some r, i.e., T is diagonalizable over the algebraic closure of F (in fact over a finite extension). We call d the **dimension** of T. Note T is not necessarily diagonalizable over F. E.g., if $F = \mathbb{R}$, then we can take the torus

$$T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\} \simeq \mathbb{C}^\times,$$

which is not diagonalizable over $\mathbb{R}$. If T is diagonalizable over F, i.e., $T(F) \simeq (F^\times)^d$ for some d, we say T is **split**.

A **maximal split torus** T in a reductive group G is a split torus in G of maximum possible dimension. All maximal split tori in G have the same dimension r, and are conjugate in G. We call this dimension r the **(F-)rank** of G. Note that the rank of G may be less than the rank of $\overline{G}$, i.e., a maximal split torus may have smaller dimension than a maximal non-split torus. E.g., if $G = T \simeq \mathbb{C}^\times$ is the torus in GL(2) over $F = \mathbb{R}$ given above, then there is a unique maximal split torus, the center $\mathbb{R}^\times$, but G itself is a torus. So in this example the rank of G is 1, but the rank of $\overline{G} = G(\mathbb{C})$ is 2 ($G(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$). If this phenomena doesn't happen, i.e., if there exist a maximal torus in G which is split, we say G is **split**.

The linear groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{PGL}(n)$, and $\mathrm{PSL}(n)$ are split groups. For $\mathrm{GL}(n)$, a split maximal torus just the subgroup of diagonal matrices, so $\mathrm{GL}(n)$ has rank n. We can also take conjugates of the diagonal subgroup to get other split maximal tori. We remark there are also lots of nonsplit maximal tori: if E/F is a field extension of of degree d and $n = dm$, then we can embed $(E^\times)^m$ as a subgroup T of $\mathrm{GL}(n)$. This will be a nonsplit maximal torus if $d > 1$, but it will become split over $\mathrm{GL}_n(E)$. For $\mathrm{SL}(n)$, the diagonal subgroup consists of elements of the form $\mathrm{diag}(a_1, a_2, \ldots, a_{n-1}, (a_1 \cdots a_{n-1})^{-1})$, which is a maximal split torus. Hence $\mathrm{SL}(n)$ has rank $n-1$. Similarly, $\mathrm{PGL}(n)$ and $\mathrm{PSL}(n)$ also have rank $n-1$.

A **Borel subgroup** of $\overline{G}$ is an algebraic subgroup $\overline{B}$ of $\overline{G}$ (over $\overline{F}$) which is a maximal connected solvable subgroup of $\overline{G}$. All Borel subgroups of $\overline{G}$ are conjugate in $\overline{G}$. A Borel $\overline{B}$ will be defined as a subgroup of some $\mathrm{GL}_n(\overline{F})$ satisfying some polynomial equations with coefficients in $\overline{F}$. It may or may not be possible (even after conjugation) to define $\overline{B}$ using polynomials with coefficients in F. If it is, we say $\overline{B}$ is **defined over** F, and let B be the corresponding algebraic group over F. Then we say B is a **Borel subgroup** of G. If G has a Borel subgroup defined over F, we say G is **quasi-split**. In particular, $G = \mathrm{GL}(n)$ is quasi-split, and a Borel subgroup B is conjugate to the standard Borel B consisting of upper triangular matrices.

Let G be a quasi-split group. A **parabolic subgroup** of G is a closed subgroup P containing some Borel B. In particular $P = G$ and $P = B$ are parabolic subgroups for any Borel B. For $G = \mathrm{GL}(n)$, a standard parabolic is a subgroup of block upper triangular matrices. Any parabolic is conjugate to a standard one, and two standard parabolics are conjugate if and only if their collection of block sizes are the same. For instance, for $G = \mathrm{GL}(3)$, the conjugacy classes of standard parabolics are represented by the following three standard parabolics:

$$P_0 = B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \quad P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad P_2 = G = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

Proper parabolic subgroups are examples of non-reductive groups, but we can decompose them in to a reductive part M, called the **Levi subgroup** of P, and the **unipotent radical** N of P, so that $P = MN$. The unipotent radical of P is defined to be the maximal connected unipotent normal subgroup of P. For a standard parabolic P of $\mathrm{GL}(n)$, the Levi component is the block diagonal matrices (thus of the form $\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_m)$), e.g., for the $P = P_1$ in $\mathrm{GL}(3)$ above, the decomposition is

$$P_1 = M_1 N_1, \quad M_1 = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ & & 1 \end{pmatrix}.$$

Parabolics can also be described as subgroups stabilizing flags, which are nested sequences of subspaces. In our GL(3) example, the minimal parabolic P_0 stabilizes a flag

$$0 \subset \langle \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \rangle \subset \langle \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \rangle \subset F^3$$

of maximal length and the maximal proper parabolic P_1 stabilizes the flag

$$0 \subset \langle \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \rangle \subset F^3.$$

We remark that for GL(2) and GL(3) there is a unique conjugacy class of maximal proper ($\neq G$) parabolics, but for higher GL(n) and higher this is not the case. For instance, for GL(4) there two non-conjugate maximal parabolics: one with two 2×2 blocks in the Levi, and one with a 3×3 block in the Levi.

Since Levi component of any Borel subgroup will be a maximal split torus, any split group is quasi-split. Orthogonal groups and symplectic groups can be quasi-split or not, and in special cases can be quasi-split but not split.

Let $I_n \in \mathrm{GL}(n)$ be the identity matrix. We define the **split odd special orthogonal groups** to be

$$\mathrm{SO}(2n+1) = SO_{B_n}, \quad B_n = \begin{pmatrix} 1 & & \\ & & I_n \\ & I_n & \end{pmatrix} \in \mathrm{GL}(2n+1),$$

the **split even special orthogonal groups** to be

$$\mathrm{SO}(2n) = SO_{D_n}, \quad D_n = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix},$$

and the **split symplectic groups** to be

$$\mathrm{Sp}(2n) = O_{C_n} = SO_{C_n}, \quad C_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}.$$

These groups are all split, and of rank n.[13] All split special orthogonal and symplectic groups are isomorphic to the ones given above. The above choice of the forms means there is a maximal split torus consist of elements of the form $\mathrm{diag}(1, a_1, \ldots, a_n, a_1^{-1}, \ldots a_n^{-1})$ for $\mathrm{SO}(2n+1)$ or $\mathrm{diag}(1, a_1, \ldots, a_n, a_1^{-1}, \ldots a_n^{-1})$ for $\mathrm{SO}(2n)$ or $\mathrm{Sp}(2n)$.

[13] Different authors take different forms for these groups, and sometimes this notation is used for non-split groups as well. E.g., in real Lie groups, this notation is usually used for compact (in particular non-split) forms. Some people also denote Sp($2n$) by Sp(n). You have been warned!

If $F = \overline{F}$, e.g, $F = \mathbb{C}$, then there are no non-split special orthogonal or symplectic groups over F, but for general F there are, and the classification depends on F. In particular, for $F = \mathbb{Q}$, $F = \mathbb{R}$ and $F = \mathbb{Q}_p$, there are always non-split special orthogonal and symplectic groups over F, by taking an appropriate bilinear form B in SO_B. If I_n is the identity in $\mathrm{GL}(n)$, then SO_B for I_n is non-split over $\mathbb{R}$ (or $\mathbb{Q}$) for $n > 1$. (This is the compact form, and corresponds to what is referred to in real Lie groups as $\mathrm{SO}(n)$, i.e., the orientation-preserving isometries of the sphere in $\mathbb{R}^n$.) The even special orthogonal group

$$\mathrm{SO}^*(2n) = SO_B, \quad B = \begin{pmatrix} & & I_{n-1} \\ & I_2 & \\ I_{n-1} & & \end{pmatrix}$$

over $\mathbb{R}$ (or $\mathbb{Q}$) is quasi-split but not split. On the other hand, any odd special orthogonal or any symplectic group which is quasi-split must also be split.

3.2 Root data and dual groups

The L-group is of a reductive group G is defined in terms of the *root datum* of G. We first explain what this root datum is, for simplicity in the case of split group. This consists of characters, cocharacters, roots and coroots.

Assume G is a split group over F, and T a maximal split torus. Let $X = X^*(T)$ be the group of **(rational) characters** of T, i.e., the group of homomorphisms from T to $F^\times$. If we identify $T = (F^\times)^r$, then the characters are of the form $(a_1, \ldots, a_r) \mapsto \prod a_i^{m_i}$ for some $m_i \in \mathbb{Z}$, so $X \simeq \mathbb{Z}^r$. Let $X^\vee = X_*(T)$ be the group of **cocharacters** of T, i.e., the group of (rational) homomorphisms $F^\times \to T$. So, with the above identification of $T = (F^\times)^r$ a cocharacter is of form $a \mapsto (a^{m_1}, \ldots, a^{m_r})$ for $m_i \in \mathbb{Z}$, and we have $X^\vee \simeq \mathbb{Z}^r$. There is a natural pairing

$$\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$$
$$a^{\langle \alpha, \alpha^\vee \rangle} = \alpha(\alpha^\vee(a)), \quad a \in F^\times.$$

Now we define roots and coroots.

Just like for Lie groups, we can associate a **Lie algebra** $\mathfrak{g}$ to an algebraic group G. This will be an associative F-algebra with a Lie bracket $[x, y] = xy - yx$. It can be defined similar to the case of Lie groups, e.g., in terms of derivations. We summarize what $\mathfrak{g}$ is as a subset of $M_n(F)$ for basic cases of $G \subset \mathrm{GL}_n(F)$.

G	$\mathfrak{g}$
$\mathrm{GL}(n)$	$\mathfrak{gl}(n) = M_n$
$\mathrm{SL}(n)$	$\mathfrak{sl}(n) = \{x \in M_n : \mathrm{tr} x = 0\}$
$\mathrm{SO}(2n+1)$	$\mathfrak{so}(2n+1) = \left\{x \in M_{2n+1} : {}^t x B_n + B_n x = 0,\ \mathrm{tr} x = 0\right\}$
$\mathrm{SO}(2n)$	$\mathfrak{so}(2n) = \left\{x \in M_{2n} : {}^t x D_n + D_n x = 0,\ \mathrm{tr} x = 0\right\}$
$\mathrm{Sp}(2n)$	$\mathfrak{sp}(2n) = \left\{x \in M_{2n} : {}^t x C_n + C_n x = 0,\ \mathrm{tr} x = 0\right\}$

Since $G \subset \mathrm{GL}_n(F)$, we can let G act on $V = \mathfrak{g} \subset M_n(F)$ by conjugation, and this is the **adjoint representation**:

$$\begin{array}{rcl} \mathrm{Ad} : G & \to & \mathrm{GL}(\mathfrak{g}) \\ \mathrm{Ad}(g)x & = & gxg^{-1}, \quad x \in \mathfrak{g}. \end{array}$$

Now we restrict the adjoint representation to T acting on $V = \mathfrak{g}$ and can decompose

$$\mathfrak{g} = \mathfrak{g}_\chi \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0$ is the space of T-invariant vectors in $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the space of vectors in $\mathfrak{g}$ upon which T acts by a nontrivial character $\alpha \in X = X^*(T)$. The characters α which arise this way are called the **roots** of G relative to T, which we denote by $\Phi = \Phi(G, T)$.

For instance, for $G = \mathrm{GL}(2)$ and T the diagonal torus, it is easy to see $\mathfrak{g}_0 = \mathfrak{t}$, the Lie algebra of T, i.e., the diagonal subalgebra of $M_2(F)$. Also note

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & b^{-1} \end{pmatrix} = ab^{-1} \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 0 & \\ y & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & b^{-1} \end{pmatrix} = a^{-1} \begin{pmatrix} 0 & \\ y & 0 \end{pmatrix}.$$

Hence the roots of G relative to T are given by

$$\alpha \begin{pmatrix} a & \\ & b \end{pmatrix} := \frac{a}{b}, \quad \beta \begin{pmatrix} a & \\ & b \end{pmatrix} := \frac{b}{a},$$

and the above decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta = \begin{pmatrix} * & \\ & * \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \\ * & 0 \end{pmatrix}.$$

We use *additive notation* for $X^*(T)$, so $\beta = -\alpha$ in the above example.

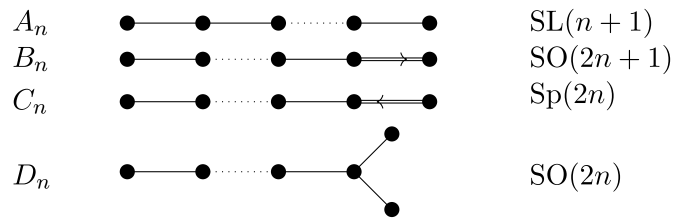
Similarly, for $G = \mathrm{SL}(2)$, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta = \begin{pmatrix} * & \\ & * \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \\ * & 0 \end{pmatrix}.$$

where

$$\alpha \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} := a^2, \quad \beta \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} := a^{-2}.$$

In general, we can choose a collection of **simple roots** Δ, which is a minimal collection of roots so that any root is either a positive or negative integral linear combination of simple roots. In GL(2) and SL(2) examples above, we can just take $\Delta = \{\alpha\}$ (though we could have also chosen $\Delta = \{\beta\}$). For semisimple groups, $|\Delta| = r$, where $r = \dim T$ is the rank of G. There is a standard way to associate to a root system a **Dynkin diagram**, which is a certain graph on Δ. For the classical groups in SL(n), there are 4 types, called A_n, B_n, C_n and D_n. Here is the picture along with the associated classical group:



For $\alpha \in \Phi$, we associate a **coroot** $\alpha^\vee$, which is an element of $X^\vee$ such that:

- $\langle \alpha, \alpha^\vee \rangle = 2$; and
- the homomorphism $s_\alpha : X \to X$ given by $s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$ maps Φ to Φ;

This is not the formal definition of the coroot $\alpha^\vee$—one needs to do more to specify $\alpha^\vee$ uniquely, but I will not explain exactly what.

Denote the set of coroots by $\Phi^\vee$. Then the **root datum** for G (relative to T) is the quadruple $(X, \Phi, X^\vee, \Phi^\vee)$. (Suppressed here is the map from $\Phi \to \Phi^\vee$ sending α to $\alpha^\vee$.)

The first condition on coroots means that $\alpha(\alpha^\vee(a)) = a^2$ for $a \in F^\times$. So for our both our SL(2) example above, we are forced to take

$$\alpha^\vee(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad \beta^\vee(a) = \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}.$$

These are also the coroots for the GL(2) example, but one cannot determine this just from the two properties of coroots above. Note that even though the roots for SL(2) and GL(2) look similar, the root systems are different (the character and cocharacter groups have different ranks as free $\mathbb{Z}$-modules).

To see a different simple example, for $G = \mathrm{PGL}(2)$, we can identify a split torus with $T = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right\}$. The roots are given by $\Phi \{\alpha, -\alpha\}$ where $\alpha \begin{pmatrix} a & \\ & 1 \end{pmatrix} = a$ and the coroot of α is $\alpha^\vee(a) = \begin{pmatrix} a^2 & \\ & 1 \end{pmatrix}$. This example is dual to SL(2) in the following sense. For SL(2), the root α does not generate X (it involves squaring), but the coroot $\alpha^\vee$ does generate $X^\vee$. However for PGL(2), the root α generates X but the coroot $\alpha^\vee$ does not (it involves squaring).

For F a local or global field and G reductive over F, e.g., $F = \mathbb{Q}_p$, $F = \mathbb{R}$ or $F = \mathbb{Q}$. Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum for $\overline{G}$. The **(Langlands) dual group** $\hat{G}$ is defined to be the *complex* (base field $\mathbb{C}$) reductive group with *dual root datum* $(X^\vee, \Phi^\vee, X, \Phi)$. This is well defined because a reductive group over an algebraically closed field is determined up to isomorphism by its root datum, and all possible root data come from reductive groups. Since $\hat{G}$ is complex, unless $F = \mathbb{C}$, the dual group of the dual group of G is not G, but it will be $G(\mathbb{C})$. In particular the dual group of SL(2) is $\mathrm{PGL}_2(\mathbb{C})$ and the dual group of PGL(2) is $\mathrm{SL}_2(\mathbb{C})$.

We summarize dual groups for some important families/examples of groups

G	$\hat{G}$	type of Φ	type of $\hat{\Phi}$
GL(n)	$\mathrm{GL}_n(\mathbb{C})$		
SL(n)	$\mathrm{PGL}_n(\mathbb{C})$	A_{n-1}	A_{n-1}
SO($2n+1$)	$\mathrm{Sp}_{2n}(\mathbb{C})$	B_n	C_n
Sp($2n$)	$\mathrm{SO}_{2n+1}(\mathbb{C})$	C_n	B_n
SO($2n$)	$\mathrm{SO}_{2n}(\mathbb{C})$	D_n	D_n
GSp(4)	$\mathrm{GSp}_4(\mathbb{C})$		

Thus dualizing "switches" root systems B_n and C_n but fixes root systems A_n and D_n, which amounts to switching the direction of the arrows in the 4 classical Dynkin diagrams. We remark that if G is simply connected then $\hat{G}$ is adjoint, and vice versa. This is why the dual group for SL(n) is $\mathrm{PGL}_n(\mathbb{C})$, rather than $\mathrm{SL}_n(\mathbb{C})$. Similarly, the dual group of PGL(n) is $\mathrm{SL}_n(\mathbb{C})$.

The dual group does not encode any rational structure of G over F, but the classification of representations of $G(F)$ or $G(\mathbb{A})$ depends on this. The L-group is a way to remedy this. There are different ways to define the L-group (not all of which are equivalent). One simple way is the following. It is a fact that G splits over a finite Galois extension E/F. If one refines the notion of root datum to *based root datum* (which basically means choose a set of simple roots Δ and simple coroots $\Delta^\vee$), one can define an action of $\mathrm{Gal}(E/F)$ on based root data, which leads to an action of $\mathrm{Gal}(E/F)$ on $\hat{G}$.

Thus we can define the ***L*-group**

$$^L G = \hat{G} \rtimes \mathrm{Gal}(E/F).$$

In particular, if G is split, we simply have $^L G = \hat{G}$. So the **local Langlands conjectures** (or correspondence) for split groups G over $\mathbb{Q}_p$ say that (up to equivalence) the smooth irreducible representations (or rather finite packets of representations when $G \neq \mathrm{GL}(n)$) should be parametrized by homomorphisms $\varphi : WD_{\mathbb{Q}_p} \to \hat{G}$, which are called ***L*-parameters**. The analogue over $\mathbb{R}$ was proven by Langlands. The local Langlands conjectures are now essentially known for the classical groups due to recent work of Arthur, Waldspurger, Moeglin, ...[14]

[14]For GL(n) it was known before as mentioned above; Arthur's book treats the quasi-split symplectic and special orthogonal groups, though he did not quite specify the local Langlands correspondence (which should have several prescribed properties I am not describing here) for even orthogonal groups.

If G is not split, then there should be fewer representations than in the split case, but they should be parametrized by *admissible* homomorphisms $\varphi : WD_{\mathbb{Q}_p} \to {}^L G$, which are homomorphisms that behave a certain way under the Galois action.

Besides their use in classifying local components of automorphic representations and prescribing local L- and ε-factors, L-groups are important in the statement of Langlands' **functoriality** conjectures. These conjectures are central to the Langlands program and are about (locally and globally) how we can transfer automorphic (or in the local case smooth irreducible) representations of some group G to another group H. Functoriality is a vast generalization of classical lifts in number theory, such as associating modular forms to Dirichlet characters, or Siegel modular forms to elliptic modular forms.

Some places to read more about L-groups and Langlands' conjectures are Cogdell's survey *Dual groups and Langlands functoriality* (from *An introduction to the Langlands program*, ed.s Bernstein, Gelbart), the Arthur–Gelbart article in the Durham proceedings (*L-functions and arithmetic*, ed.s Coates and Taylor), the Gelbart–Shahidi book, the Blasius–Rogawski article in the *Motives* volumes (PSMP 55), and the Corvallis volumes (particularly Borel's article).