

## ROOT NUMBER BIAS FOR NEWFORMS

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ABSTRACT. Previously we observed that newforms obey a strict bias towards root number  $+1$  in squarefree levels: at least half of the newforms in  $S_k(\Gamma_0(N))$  with root number  $+1$  for  $N$  squarefree, and it is strictly more than half outside of a few special cases. Subsequently, other authors treated levels which are cubes of squarefree numbers. Here we treat arbitrary levels, and find that if the level is not the square of a squarefree number, this strict bias still holds for any weight. In fact the number of such exceptional levels is finite for fixed weight, and 0 if  $k < 12$ . We also investigate some variants of this question to better understand the exceptional levels.

### 1. INTRODUCTION

Throughout,  $N$  and  $k$  are positive integers with  $k$  even. Let  $S_k(N) = S_k(\Gamma_0(N))$  be the space of weight  $k$ , level  $N$  cusp forms, and  $S_k^{\text{new}}(N)$  be the new subspace. The root number of a newform  $f \in S_k^{\text{new}}(N)$  is the sign in the functional equation for  $L(s, f)$ , and equals  $(-1)^{k/2}$  times the eigenvalue of the Fricke involution  $W_N = \prod_{p|N} W_p$ . It is known that, on average, 50% of newforms have root number  $+1$  and 50% of newforms have root number  $-1$ . We examine the distribution of root numbers more precisely.

Let  $S_k^{\text{new}}(N)^\pm$  denote the subspace spanned by newforms with root number  $\pm 1$ . Set

$$\Delta(N, k) = \dim S_k^{\text{new}}(N)^+ - \dim S_k^{\text{new}}(N)^-.$$

In [Mar18a], we observed that the trace formula for  $W_N$  implies that newforms obey a strict bias towards root number  $+1$  for squarefree levels  $N$ , in the sense that  $\Delta(N, k) \geq 0$  for all  $k$  and squarefree  $N$ . Further,  $\Delta(N, k)$  is an elementary expression in the class number of  $\mathbb{Q}(\sqrt{-N})$ , and is strictly positive unless (i)  $N = 2, 3$  and  $k$  obeys a certain congruence, or (ii)  $k = 2$  and  $\dim S_2^{\text{new}}(N) = 0$  or  $N = 37, 58$ .

Subsequently, these results were extended to the case of cubes of squarefree levels by [PQ21] ( $k > 2$ ) and [LPW23] ( $k \geq 2$ , which also considers Hilbert modular forms). The methods in those papers are much more involved, and use Petersson and Jacquet–Zagier trace formulas, respectively. Here we explicitly compute the trace of  $W_N$  on  $S_k^{\text{new}}(N)$  for arbitrary  $(N, k)$ , which yields the following.

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**Theorem 1.1.**

- (1) *If  $N$  is not the square of a squarefree number, or if  $k \leq 10$  or  $k = 14$ , then there is a strict bias of newforms toward root number  $+1$ , i.e.,  $\Delta(N, k) \geq 0$ .*
- (2) *Suppose  $N$  is the square of a squarefree number, i.e., a cubefree square. For sufficiently large  $k$ ,  $\Delta(N, k) < 0$  if and only if  $(-1)^{k/2} = -\mu(\sqrt{N}) = -\prod_{p|N}(-1)$ .*
- (3) *For a fixed  $k$ , there are only finitely many  $N$  such that  $\Delta(N, k) < 0$ .*

Moreover, in (1),  $\Delta(N, k)$  is typically strictly positive. Precise conditions are given in Corollary 3.3. We will discuss possible reasons for the curious difference in behaviour for cubefree square levels in Section 1.2.

It is not too hard to make statements (2) and (3) effective. We do not explicate this, but refer the interested reader to the proof of Corollary 3.3(2).

As in the squarefree case, we in fact get an elementary expression for the exact size of the bias, and it is essentially the class number of  $\mathbb{Q}(\sqrt{-N})$ . The precise formula for  $\Delta(N, k)$  breaks up into several cases according to whether  $N$  is 1, 2, 3 or 4 times the square of a squarefree number, or none of these (the generic case). Here we just describe the formula in the generic case, and refer to Proposition 3.2 for all cases.

Write  $N = N_1 N_2^2$ , where  $N_1$  is squarefree. Define  $\beta(N) \in \{1, 2, 3, 4\}$  by

$$(1.1) \quad \beta(N) = \begin{cases} 1 & \text{if } N_1 \equiv 1, 2 \pmod{4}, \text{ or } 2 \parallel N_2; \\ 2 & \text{if } N_1 \equiv 3 \pmod{4} \text{ and } 4 \mid N_2; \\ 3 - \left(\frac{-N_1}{2}\right) & \text{if } N_1 \equiv 3 \pmod{4} \text{ and } 2 \nmid N_2. \end{cases}$$

For a fundamental discriminant  $-D < 0$ , let  $h'(-D)$  be the “unit group weighted” class number, i.e., one half of the number of integral units in  $\mathbb{Q}(\sqrt{-D})$  times the usual class number  $h(-D)$ . Precisely, set  $h'(-4) = \frac{1}{2}$  and  $h'(-3) = \frac{1}{3}$ , and take  $h'(-D) = h_{\mathbb{Q}(\sqrt{-D})}$  to be usual class number for  $D > 4$ .

**Theorem 1.2.** *Suppose  $N$  is not 1, 2, 3 or 4 times the square of a squarefree number. Let  $-D \in \{-N_1, -4N_1\}$  be the discriminant of  $\mathbb{Q}(\sqrt{-N})$ . Then*

$$\Delta(N, k) = \frac{1}{2}\beta(N) \prod_{p|N_2} \left( \phi(p^{v_p(N_2)}) - \phi(p^{v_p(N_2)-1}) \left(\frac{-D}{p}\right) \right) h'(-D) - \delta,$$

where  $\phi$  is the Euler phi function,  $\delta = 1$  if  $(N, k) = (1, 2)$  and  $\delta = 0$  otherwise.

In particular, we always have  $\Delta(N, k) \geq 0$ . Moreover,  $\Delta(N, k) = 0$  if and only if (i)  $k = 2$  and either  $\dim S_k^{\text{new}}(N) = 0$  or  $N \in \{37, 58\}$ ; or (ii)  $2 \parallel N_2$  and  $N_1 \equiv 7 \pmod{8}$ .

Note that if  $v_p(N)$  is odd for all  $p \mid N$ , then  $\left(\frac{-D}{p}\right) = 0$  for all  $p \mid N_2$ , and thus the main term in the above formula is just  $\frac{1}{2}\beta(N)\phi(N_2)h'(-D)$ . In particular, this recovers the formulas in [Mar18a] and [PQ21].

**1.1. Remarks on proof.** The proof has in essence two steps. The quantity  $\Delta(N, k)$  equals the trace of  $(-1)^{k/2}W_N$  on  $S_k^{\text{new}}(N)$ . First, we use a trace formula due to Yamauchi [Yam73] and Skoruppa–Zagier [SZ88] which expresses the trace of  $W_N$  on  $S_k(N)$  as an alternating sum of class numbers. Using class number relations and elementary but careful analysis, we rewrite this trace in terms of a single weighted class number  $h'(-D)$  (see Proposition 3.1, which can also be derived from the recent work [ZZ22]). Second, one can express the trace of  $W_N$  on

the new subspace as an alternating sum over  $M^2 \mid N$  of traces on  $S_k(N/M^2)$ . Once again, we use class number relations and elementary analysis to obtain our formula for  $\Delta(N, k)$  (see Proposition 3.2).

We remark that our approach is much more straightforward and simple than the approaches taken in [PQ21] and [LPW23]. In [Mar18a], we restricted to squarefree levels because our main focus in that paper was dimension formulas with prescribed local signs at multiple ramified places, and that problem seems considerably more complicated for non-squarefree levels. The result therein about root number bias for squarefree levels was merely a curious observation we made along the way, which was already immediate from the Yamauchi/Skoruppa–Zagier trace formulas (though, to our knowledge, it had not been noticed before). For squarefree levels, both steps in our present proof are trivial because (i) the trace formula for  $W_N$  on  $S_k(N)$  is very simple for squarefree  $N$  and (ii) there is no oldform contribution to this trace that we need to subtract off.

**1.2. Further questions.** Theorem 1.1 naturally prompts 2 questions: (i) what is the reason for this bias, and (ii) what is the reason for these exceptions.

We briefly speculated on (i) in [Mar18a], and mentioned two ideas that at least support the existence of a bias towards root number +1. First, the “ $L$ -functions from nothing” perspective suggests that  $L$ -functions in small weight and level tend to have root number +1 (e.g., see [Mar18a, p. 10]). Second, when  $k = 2$ , at least for prime levels,  $\Delta(p, 2) \geq 0$  is forced upon us by the Jacquet–Langlands correspondence together with the fact that the type number is at least  $\frac{1}{2}$  times the class number for maximal orders in quaternion algebras of prime discriminant.

Anna Medvedovsky has since suggested to me that one can also reinterpret the  $k = 2$  case for squarefree  $N$  by comparing the genus of  $X_0(N)$  with its quotient by the Fricke involution. One can note that the formulas for  $\Delta(N, k)$  are essentially independent of  $k$ , and thus are controlled by the  $k = 2$  case. One heuristic reason for this latter fact is that the main term in dimension formulas of spaces of modular forms is a product of local factors, including one at infinity for the weight  $k$ .

Thus there are at least some hints for the above root number bias that do not only rely on the trace formula, but I do not have a compelling existential explanation for this bias. On the other hand, if one just counts elliptic curves up to isogeny by conductor, i.e., weight 2 rational newforms, examining Cremona’s database (see [LMFDB]) shows there is at least an initial bias towards root number  $-1$ .

Moving on to (ii), let me first explain how the exceptions to the strict bias arise in the proof. First, the trace of  $(-1)^{k/2}W_N$  on  $S_k(N)$  is non-negative (see Proposition 3.1). The only question now is whether this still holds when we subtract off the oldform contribution, i.e., if  $(-1)^{k/2} \sum_{M^2 \mid N} \mu(M) \operatorname{tr} W_{N/M^2} \geq 0$ . For a fixed  $N$ , the the terms for each  $M^2 \mid N$  are all similar looking expressions in  $h'(-D)$  that can be collected together, except in the special case that  $N$  is a perfect square. In that case, the term with  $M = \sqrt{N}$  is  $\mu(M)$  times the trace of  $W_{N/M^2} = W_1 = \operatorname{Id}$  on  $S_k(1)$ , i.e.,  $\mu(M) \dim S_k(1)$ . Since  $\dim S_k(1) \simeq \frac{k}{12}$ , we find that  $\Delta(N, k) \approx \frac{1}{4}\phi(M) + (-1)^{k/2}\mu(M)\frac{k}{12}$ . This yields parts (2) and (3) of Theorem 1.1.

One thing that is special about levels which are perfect squares is that these are precisely the levels where the newspaces are not generated by theta series attached to definite quaternion algebras (see [HPS89] and [Mar20]). In particular if the level

$N = M^2$ , where  $M$  is squarefree, then  $S_k^{\text{new}}(N)$  is generated by theta series together with twists of forms of level 1 and level  $M$  (now including forms with nebentypus).

The discussion of the proof above, combined with this consideration about theta series, suggests that the reason the strict bias towards root number  $+1$  does not persist in (certain) square levels may be due to newforms which are twists from full level.

In Section 4, we describe what happens if one removes the contribution from twists of level 1 forms. Specifically, denote by  $S_k^{\text{new}}(N)'$  be the subspace of  $S_k^{\text{new}}(N)$  generated by newforms which are not twists from level 1. Let  $\Delta(N, k)'$  be the number of newforms in  $S_k^{\text{new}}(N)'$  with root number  $+1$  minus the number of newforms with root number  $-1$ .

**Proposition 1.3.** *Let  $M > 1$  be squarefree. The spaces  $S_k^{\text{new}}(M^2)'$  have a strict bias towards root number  $+1$ , i.e.,  $\Delta(M^2, k)' \geq 0$ , for all  $k$  if and only if  $M$  is odd and  $M$  has an odd number of prime factors which are  $3 \pmod{4}$ . Otherwise, for sufficiently large  $k$ , the sign of  $\Delta(M^2, k)'$  is  $(-1)^{k/2}\mu(M)$ .*

Thus even though the proof of our theorems suggests the difference in root number bias behavior for cubefree square levels comes from level 1 forms, this difference does not always disappear when we omit twists of level 1 forms. In fact, when  $M$  is even and squarefree, there are no twists from level 1.

From this perspective, it is natural to ask what happens if one excludes all newforms which are twists from smaller levels, i.e., if one restricts to minimal newforms. Since every newform in  $S_k^{\text{new}}(4)$  is minimal, and  $\Delta(4, k) < 0$  whenever  $k \equiv 0 \pmod{4}$  and  $k \geq 12$ , the sum of root numbers over minimal newforms is not always positive. We also exhibit examples of  $p \equiv 3 \pmod{4}$  such that  $S_k^{\text{new}}(p^2)$  has more minimal newforms with root number  $-1$  than  $+1$ . On the other hand, we give a simple local condition which forces root numbers to be perfectly equidistributed for minimal forms in certain (not necessarily square) levels.

**Proposition 1.4.** *Let  $N \geq 1$ , and let  $S_k^{\text{min}}(N)$  be the subspace of  $S_k^{\text{new}}(N)$  generated by newforms which are not twists from smaller levels. Suppose there exists  $p^2 \parallel N$  such that  $(\frac{-N/p^2}{p}) = 1$ . Then  $\dim S_k^{\text{min}}(N)^+ = \dim S_k^{\text{min}}(N)^-$ , i.e., root numbers are perfectly balanced for minimal newforms.*

The congruence conditions on primes dividing  $N$  in these propositions arise from the way that local root numbers behave under twisting by quadratic characters.

**1.3. Further remarks.** As in [Mar18a], our formula for  $\Delta(N, k)$  implies an exact formula for the dimensions

$$\dim S_k^{\text{new}}(N)^\pm = \frac{1}{2} (\dim S_k^{\text{new}}(N) \pm \Delta(N, k)).$$

We do not write down these dimension formulas explicitly, but they immediately follow by comparing with the explicit formula for  $\dim S_k^{\text{new}}(N)$  in [Mar05].

Consequently, our formula for  $\Delta(N, k)$  implies that root numbers of are  $+1$  (or  $-1$ ) for 50% of newforms as  $k + N \rightarrow \infty$ . We are not aware of an explicit proof of this fact in the literature for non-squarefree  $N$ , though proofs were surely known to experts. In any case, our formulas together with standard class number bounds yield equidistribution of root numbers with a very good error estimate.

The original motivation for the dimension formulas for newforms with prescribed Atkin–Lehner signs in [Mar18a] was to use them to obtain mod 2 congruences

in [Mar18b]. Namely, we showed that perfect equidistribution of Atkin–Lehner sign patterns in squarefree levels essentially means that every newform is mod 2 congruent to one with any desired Atkin–Lehner signs at those places. It may be interesting to see whether the cases of perfect equidistribution of root numbers in non-squarefree levels we give here are similarly related to mod 2 congruences.

The work [LPW23] establishes biases of root numbers for Hilbert modular forms for certain levels and base fields. One should be able to imitate [SZ88] to derive a trace formula for Fricke, as well as Atkin–Lehner, involutions of Hilbert modular forms. Then we expect that the strategy used in the present paper (resp. the one in [Mar18a]) should yield distributions of root numbers (resp. Atkin–Lehner sign patterns) in quite general settings (resp. for squarefree levels).

As a check on our formulas, we compared them numerically with newform and root number calculations in Sage [Sage] for various ranges of values of  $(N, k)$ .

2. PRELIMINARIES

**2.1. Notation.** Throughout,  $M$  and  $N$  denote positive integers, and  $k \geq 2$  is even. Let  $v_p(N) = \max\{r \in \mathbb{Z} : p^r \mid N\}$ . For  $N \in \mathbb{N}$ , denote by  $N^{\text{odd}}$  the odd part of  $N$ , i.e.,  $N = 2^{v_2(N)}N^{\text{odd}}$ . Let  $\square \subset \mathbb{N}$  be the multiplicative submonoid of squares. Let  $\mu$  be the Möbius function.

Denote by  $\text{tr } W_N$  the trace of  $W_N$  on the full cusp space  $S_k(N)$ , and by  $\text{tr } W_N^{\text{new}}$  the trace on the new subspace  $S_k^{\text{new}}(N)$ . These traces also depend on the weight  $k$ , but we suppress it in our notation.

For a statement  $*$ , we use  $\delta_*$  to mean 1 if the statement  $*$  is satisfied, and 0 otherwise. E.g.,  $\delta_{k=2}$  represents the Kronecker delta function  $\delta_{k,2}$ .

The following function arises in the trace formula for  $W_N$ . For  $s \in \mathbb{Z}_{\geq 0}$ , set  $p_k(s) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}$  where  $\rho, \bar{\rho}$  are the roots of  $X^2 - sX + 1$  if  $s^2 \neq 4$ , and set  $p_k(2) = k - 1$ .

**2.2. Class number relations.** Let  $-D < 0$  be a discriminant, and  $h(-D)$  be the class number of the quadratic order  $\mathcal{O}_D$  of discriminant  $-D$ . Let  $h'(-D)$  be the class number weighted by  $[\mathcal{O}_D^\times : \mathbb{Z}^\times]^{-1}$ , i.e.,  $h'(-D) = h(-D)$  unless  $D < 5$ , and then  $h'(-4) = \frac{1}{2}$  and  $h'(-3) = \frac{1}{3}$ . Define the Hurwitz class number  $H(-D)$  to be the number of positive definite binary quadratic forms of discriminant  $\Delta$  up to equivalence, where we weight forms of discriminant  $-4$  and  $-3$  by  $\frac{1}{2}$  and  $\frac{1}{3}$ , respectively. (The class numbers  $h(-D)$  and  $h'(-D)$  can be interpreted in terms of counting primitive forms.) We also set  $H(0) = -\frac{1}{12}$ .

Now suppose  $-D$  is a fundamental discriminant, and  $\lambda > 0$ . Then

$$(2.1) \quad h'(-\lambda^2 D) = \lambda \prod_{p|\lambda} \left(1 - \left(\frac{-D}{p}\right) \frac{1}{p}\right) h'(-D) = \lambda \sum_{t|\lambda} \mu(t) \left(\frac{-D}{t}\right) \frac{1}{t} h'(-D).$$

(E.g., see [Coh78, Corollary 15.40].)

We can write the Hurwitz class number in terms of  $h'$  by

$$(2.2) \quad H(-\lambda^2 D) = \sum_{t|\lambda} h'(-Dt^2).$$

Then

$$(2.3) \quad H(-\lambda^2 D) = \sum_{m|\lambda} \sum_{t|m} \mu(t) \left(\frac{-D}{t}\right) \frac{m}{t} h'(-D) = \sum_{t|\lambda} \mu(t) \left(\frac{-D}{t}\right) \sigma(\lambda/t) h'(-D).$$

### 3. TRACES ON NEWSPACE

Since  $\Delta(N, k) = (-1)^{k/2} \operatorname{tr} W_N^{\text{new}}$  we want to compute  $\operatorname{tr} W_N^{\text{new}}$ . A formula for the trace of a product  $T_n W_M$  of Hecke and Atkin–Lehner operators on  $S_k(N)$ , together with a relation for how the trace on the newspace is related to traces on full cusp spaces was given by Yamauchi [Yam73]. Unfortunately, Yamauchi’s paper contains some clerical errors. A corrected formulation of these traces was later given by Skoruppa and Zagier [SZ88], and we will use their formulation.

We just need the case that  $n = 1$  and  $M = N$ , but some effort is still required to get from the Skoruppa–Zagier formula to a relatively simple expression for  $\Delta(N, k)$ . We will first find a simple expression for  $\operatorname{tr} W_N$ , and then use it to get our desired formula for  $\operatorname{tr} W_N^{\text{new}}$ .

As explained in [SZ88, pp. 132–133], a newform  $f \in S_k^{\text{new}}(M)$  with root number  $(-1)^{k/2} w_f$  contributes  $w_f$  to  $\operatorname{tr} W_N$  if  $N/M \in \square$  and 0 otherwise. Thus

$$\operatorname{tr} W_N = \sum_{M|N, N/M \in \square} \operatorname{tr} W_M^{\text{new}},$$

and so by Möbius inversion,

$$(3.1) \quad \operatorname{tr} W_N^{\text{new}} = \sum_{Q^2|N} \mu(Q) \operatorname{tr} W_{N/Q^2}.$$

Now a special case of [SZ88, (2.7)] yields

$$(3.2) \quad \operatorname{tr} W_N = -\frac{1}{2} \sum_{M|N, N/M \in \square} \mu(\sqrt{N/M}) \sum_s I_s(k, M) - \frac{1}{2} \delta_{N \in \{1,4\}} + \delta_{k=2},$$

where

$$I_s(k, M) = p_k(s/\sqrt{M})H(s^2 - 4M),$$

and  $0 \leq s \leq 2\sqrt{M}$  such that  $\sqrt{MN} \mid s$ . The only way that  $s > 0$  is possible then is if  $N \leq 4$ . Note that  $p_k(0) = (-1)^{(k-2)/2}$ , so when  $s = 0$  we get

$$I_0(k, M) = (-1)^{(k-2)/2} H(-4M).$$

#### 3.1. Small levels.

We first deal with levels  $N \leq 4$ .

The Fricke involution is trivial if  $N = 1$ . In particular we have

$$\operatorname{tr} W_1 = \dim S_k(1) = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor - 1 + \delta_{k=2} & k \equiv 2 \pmod{12}. \end{cases}$$

The cases  $N = 2, 3$  are included in the case of  $N$  squarefree (e.g., see [Mar18a, Thm 2.2]). Explicitly, we have

$$\operatorname{tr} W_2 = \operatorname{tr} W_2^{\text{new}} = \begin{cases} (-1)^{k/2} (1 - \delta_{k=2}) & k \equiv 0, 2 \pmod{8} \\ 0 & k \equiv 4, 6 \pmod{8}, \end{cases}$$

and

$$\operatorname{tr} W_3 = \operatorname{tr} W_3^{\text{new}} = \begin{cases} (-1)^{k/2} (1 - \delta_{k=2}) & k \equiv 0, 2, 6, 8 \pmod{12} \\ 0 & k \equiv 4, 10 \pmod{12}. \end{cases}$$

If  $N = 4$ , we have

$$\operatorname{tr} W_4 = \frac{1}{2} (I_0(k, 1) + I_2(k, 1) - I_0(k, 4) - I_4(k, 4)) - \frac{1}{2} + \delta_{k=2}.$$

Note that  $I_0(k, 1) - I_0(k, 4) = (-1)^{(k-2)/2}(H(-4) - H(-16)) = (-1)^{k/2}$ , and  $I_2(k, 1) = p_k(2)H(0) = I_4(k, 4)$ . Thus

$$\text{tr } W_4 = \begin{cases} 0 & k \equiv 0 \pmod{4} \\ -1 + \delta_{k=2} & k \equiv 2 \pmod{4}. \end{cases}$$

Then

$$\text{tr } W_4^{\text{new}} = \text{tr } W_4 - \text{tr } W_1 = \begin{cases} -\lfloor \frac{k}{12} \rfloor & k \equiv 0, 2, 4, 8 \pmod{12} \\ -\lfloor \frac{k}{12} \rfloor - 1 & k \equiv 6, 10 \pmod{12}. \end{cases}$$

**3.2. Generic levels: trace on full cusp space.** Now suppose  $N > 4$ . Then the only  $s$ -terms in (3.2) are  $s = 0$ , and we have

$$\text{tr } W_N = (-1)^{k/2} \frac{1}{2} \sum_{M|N, N/M \in \square} \mu(\sqrt{N/M})H(-4M) + \delta_{k=2}.$$

We can uniquely write  $N = N_1N_2^2$  where  $N_1$  is squarefree. If  $N/M$  is a square, then we can write  $M = N_1M_2^2$  where  $M_2 | N_2$ . Hence

$$\text{tr } W_N = (-1)^{k/2} \frac{1}{2} \sum_{M_2|N_2} \mu(N_2/M_2)H(-4N_1M_2^2) + \delta_{k=2}.$$

**Proposition 3.1.** *Suppose  $N > 4$ , and write  $N = N_1N_2^2$  with  $N_1$  squarefree as above. Then*

$$(-1)^{k/2} \text{tr } W_N + \delta_{k=2} = \begin{cases} \frac{1}{2}h'(-4N) & \text{if } N_1 \equiv 1, 2 \pmod{4}, \\ \frac{1}{2}\left(3 - \left(\frac{-N_1}{2}\right)\right)h'(-N) & \text{if } N_1 \equiv 0, 3 \pmod{4}, \quad N_2 \text{ odd}, \\ h'(-N) & \text{if } N_1 \equiv 0, 3 \pmod{4}, \quad N_2 \text{ even}. \end{cases}$$

See [ZZ22] for an alternative proof, which also allows for quadratic nebentypus.

*Proof.* First suppose  $-4N_1$  is a fundamental discriminant, i.e.,  $-N_1 \equiv 2, 3 \pmod{4}$ . We compute that

$$\begin{aligned} (-1)^{k/2} \text{tr } W_N + \delta_{k=2} &= \frac{1}{2} \sum_{M_2|N_2} \sum_{t|M_2} \mu(N_2/M_2)\sigma(M_2/t)\mu(t)\left(\frac{-4N_1}{t}\right)h'(-4N_1) \\ &= \frac{1}{2} \sum_{t|N_2} \frac{N_2}{t} \mu(t)\left(\frac{-4N_1}{t}\right)h'(-4N_1) \\ &= \frac{1}{2}h'(-4N). \end{aligned}$$

Here we got from the first equation to the second by interchanging the order of summation and observing that for fixed  $t | N_2$ ,

$$\sum_{t|M_2|N_2} \mu(N_2/M_2)\sigma(M_2/t) = \sum_{M'_2|(N_2/t)} \mu((N_2/t)/M'_2)\sigma(M'_2),$$

where we have written  $M_2 = tM'_2$ . Then observe that this expression is just the Dirichlet convolution  $(\mu * \sigma)(N_2/t) = \text{Id}(N_2/t) = N_2/t$ . We will use this latter fact again in the remaining cases.

For the rest of the proof, assume  $-N_1$  is a fundamental discriminant. Then (3.3)

$$(-1)^{k/2} \operatorname{tr} W_N + \delta_{k=2} = \frac{1}{2} \sum_{M_2|N_2} \sum_{t|2M_2} \mu(N_2/M_2) \sigma(2M_2/t) \mu(t) \left(\frac{-N_1}{t}\right) h'(-N_1).$$

First consider the case that  $N_2$  is odd. Note that for  $t$  odd and  $M_2 \mid N_2$ ,  $\sigma(2M_2/t) = 3\sigma(M_2/t)$ . Consequently,

$$\begin{aligned} & \sum_{M_2|N_2} \sum_{t|2M_2} \mu(N_2/M_2) \sigma(2M_2/t) \mu(t) \left(\frac{-N_1}{t}\right) \\ &= \sum_{t|M_2} \sum_{M_2|N_2} \mu(N_2/M_2) \left( \sigma(2M_2/t) \mu(t) \left(\frac{-N_1}{t}\right) + \sigma(M_2/t) \mu(2t) \left(\frac{-N_1}{2t}\right) \right) \\ &= \sum_{t|N_2} \sum_{t|M_2|N_2} \mu(N_2/M_2) \sigma(M_2/t) \mu(t) \left(\frac{-N_1}{t}\right) \left( 3 - \left(\frac{-N_1}{2}\right) \right) \\ &= \sum_{t|N_2} (\mu * \sigma)(N_2/t) \mu(t) \left(\frac{-N_1}{t}\right) \left( 3 - \left(\frac{-N_1}{2}\right) \right) \\ &= \sum_{t|N_2} \frac{N_2}{t} \left(\frac{-N_1}{t}\right) \mu(t) \left( 3 - \left(\frac{-N_1}{2}\right) \right). \end{aligned}$$

Plugging this into (3.3) and using (2.1) yields the second case of the proposition.

Finally suppose that  $N_2$  is even, so we can write  $N_2 = 2^e N_2^{\text{odd}}$  where  $e \geq 1$ . For an  $M_2 \mid N_2$  appearing in the double sum in (3.3), we must have  $M_2 = 2^f M_2^{\text{odd}}$  where  $f \in \{e - 1, e\}$  and  $M_2^{\text{odd}} \mid N_2^{\text{odd}}$  due to the factor  $\mu(N_2/M_2)$ . Furthermore the  $\mu(t)$  factor means that we can restrict to  $t \mid 2M_2^{\text{odd}}$  in the same double sum. Hence (3.3) is equal to  $\frac{1}{2} h'(-N_1)$  times

$$\begin{aligned} & \sum_{M_2^{\text{odd}}|N_2^{\text{odd}}} \sum_{t|2M_2^{\text{odd}}} \left( \mu(2N_2^{\text{odd}}/M_2^{\text{odd}}) \sigma(2^e M_2^{\text{odd}}/t) \mu(t) \left(\frac{-N_1}{t}\right) + \right. \\ & \quad \left. \mu(N_2^{\text{odd}}/M_2^{\text{odd}}) \sigma(2^{e+1} M_2^{\text{odd}}/t) \mu(t) \left(\frac{-N_1}{t}\right) \right) = \\ & \sum_{M_2^{\text{odd}}|N_2^{\text{odd}}} \sum_{t|2M_2^{\text{odd}}} \mu(N_2^{\text{odd}}/M_2^{\text{odd}}) (\sigma(2^{e+1} M_2^{\text{odd}}/t) - \sigma(2^e M_2^{\text{odd}}/t)) \mu(t) \left(\frac{-N_1}{t}\right). \end{aligned}$$

Note that  $\sigma(2^j m) = (2^{j+1} - 1)\sigma(m)$  for  $m > 0$  odd and  $j \geq 0$ . Now considering the  $t$  odd and  $t$  even terms separately, we rewrite the above as

$$\begin{aligned} & 2^e \sum_{M_2^{\text{odd}}|N_2^{\text{odd}}} \sum_{t|M_2^{\text{odd}}} \mu(N_2^{\text{odd}}/M_2^{\text{odd}}) \sigma(M_2^{\text{odd}}/t) \mu(t) \left(\frac{-N_1}{t}\right) \left( 2 - \left(\frac{-N_1}{2}\right) \right) = \\ & \quad 2^e \left( 2 - \left(\frac{-N_1}{2}\right) \right) \sum_{t|N_2^{\text{odd}}} (\mu * \sigma)(N_2^{\text{odd}}/t) \mu(t) \left(\frac{-N_1}{t}\right) = \\ & \quad 2^e \left( 2 - \left(\frac{-N_1}{2}\right) \right) \sum_{t|N_2^{\text{odd}}} \frac{N_2^{\text{odd}}}{t} \mu(t) \left(\frac{-N_1}{t}\right). \end{aligned}$$

Multiplying this expression by  $\frac{1}{2}h'(-N_1)$  and applying both equalities in (2.1) yields

$$2^{e-1} \left( 2 - \left( \frac{-N_1}{2} \right) \right) h'(-N_1(N_2^{\text{odd}})^2) = h'(-N).$$

This completes the last case of the proposition. □

**3.3. Alternating class number sums.** Let  $N \geq 1$  and write  $N = N_1 N_2^2 > 4$  with  $N_1$  squarefree. Then by (3.1), we have

$$(3.4) \quad \text{tr } W_N^{\text{new}} = \sum_{Q|N_2} \mu(N_2/Q) \text{tr } W_{N_1 Q^2}.$$

Comparing this with Proposition 3.1, to get a simplified formula for  $\text{tr } W_N^{\text{new}}$ , we essentially just need to evaluate

$$(3.5) \quad \sum_{Q|N_2} \mu(N_2/Q) h'(-DQ^2),$$

where  $-D = -4N_1$  or  $-D = -N_1$ , whichever is a fundamental discriminant. We will also need to include ‘‘correction terms’’ when  $k = 2$  or when  $N_1 Q^2 \leq 4$ , and distinguish the  $Q$  odd from  $Q$  even cases when  $-D = -N_1$ , but let us first evaluate (3.5).

We can rewrite the sum in (3.5) as  $\sum_{Q|N_2} \sum_{t|Q} \mu(N_2/Q) \frac{Q}{t} \mu(t) \left( \frac{-D}{t} \right) h'(-D)$ , which is equal to  $\sum_{t|N_2} \left( \sum_{Q'|N_2/t} \mu((N_2/t)/Q') Q' \right) \mu(t) \left( \frac{-D}{t} \right) h'(-D)$ . The inner sum is the Dirichlet convolution  $(\mu * \text{Id})(N_2/t) = \phi(N_2/t)$ . Hence

$$(3.6) \quad \sum_{Q|N_2} \mu(N_2/Q) h'(-DQ^2) = c(-D, N_2) h'(-D),$$

where

$$(3.7) \quad \begin{aligned} c(-D, n) &= \sum_{t|n} \phi(n/t) \mu(t) \left( \frac{-D}{t} \right) \\ &= \prod_{p||n} \left( p - 1 - \left( \frac{-D}{p} \right) \right) \prod_{p^2|n} p^{v_p(n)-2} (p-1) \left( p - \left( \frac{-D}{p} \right) \right). \end{aligned}$$

The second equality follows from observing  $c(-D, n)$  is a Dirichlet convolution of multiplicative functions in  $n$ , and thus multiplicative in  $n$ , and then noting that  $c(-D, p^r) = \phi(p^r) - \phi(p^{r-1}) \left( \frac{-D}{p} \right)$ .

We make two remarks. First,  $c(-D, N_2) \geq 0$  for all  $N$ , and  $c(-D, N_2) = 0$  if and only if  $2 \parallel N_2$  and  $\left( \frac{-D}{2} \right) = 1$ . Second, if  $N_2 \mid D$  (which, when  $-D$  is odd, is equivalent to  $v_p(N) \neq 2$  for all  $p$ ), then we simply have that  $c(-D, N_2) = \phi(N_2)$ .

**3.4. Main and correction terms.** As above, write  $N = N_1 N_2^2$  ( $N \geq 1$ ) with  $N_1$  squarefree, and let  $-D$  be the discriminant of  $\mathbb{Q}(\sqrt{-N})$ . Set

$$b(N) = \begin{cases} \frac{1}{2} & -N_1 \equiv 2, 3 \pmod{4}, \\ \frac{1}{2} \left( 3 - \left( \frac{-N_1}{2} \right) \right) & -N_1 \equiv 1 \pmod{4}, \quad N_2 \text{ odd}, \\ 1 & -N_1 \equiv 1 \pmod{4}, \quad N_2 \text{ even}. \end{cases}$$

Then Proposition 3.1 asserts that

$$(3.8) \quad \text{tr } W_{N_1 Q^2} = (-1)^{k/2} b(N_1 Q^2) h'(-DQ^2) + \delta_{k=2},$$

when  $N_1Q^2 > 4$ . Combining this with (3.4) implies that

$$(3.9) \quad \text{tr } W_N^{\text{new}} = A(N, k) + \xi_0(N, k) + \xi_1(N, k),$$

where  $A(N, k)$  is the ‘‘main term’’

$$A(N, k) = (-1)^{k/2} \sum_{Q|N_2} \mu(N_2/Q) b(N_1Q^2) h'(-DQ^2)$$

and  $\xi_0(N, 1)$  and  $\xi_1(N, 1)$  are the correction terms given as

$$\begin{aligned} \xi_0(N, k) &= \sum_{Q|N_2, N_1Q^2 \leq 4} \mu(N_2/Q) \left( \text{tr } W_{N_1Q^2} - (-1)^{k/2} b(N_1Q^2) h'(-DQ^2) \right), \\ \xi_1(N, k) &= \delta_{k=2} \sum_{Q|N_2, N_1Q^2 > 4} \mu(N_2/Q). \end{aligned}$$

First we rewrite  $A(N, k)$ . Note that  $b(N_1Q^2) = b(N)$  for all  $Q | N_2$  unless  $-N_1$  is a fundamental discriminant,  $N_2$  is even,  $Q$  is odd and  $(\frac{-N_1}{2}) \neq 1$ . In the latter instance,  $b(N) = 1$  and for  $Q | N_2$ ,  $b(N_1Q^2)$  is 1 or 2 when  $Q$  is even or odd, respectively. Bearing this in mind, using (3.6) we can write

$$(3.10) \quad A(N, k) = (-1)^{k/2} b(N) c'(-D, N_2) h'(-D),$$

where

$$c'(-D, N_2) = \begin{cases} \frac{1}{2} c(-D, N_2) & N_1 \equiv 3 \pmod{8}, 2 \parallel N_2 \\ c(-D, N_2) & \text{else.} \end{cases}$$

The  $\frac{1}{2}c(-D, N_2)$  case arises as  $c(-D, N_2) - c(-D, N_2^{\text{odd}})$ , and then observing that  $c(-D, N_2) = 2c(-D, N_2^{\text{odd}})$  when  $N_1 \equiv 3 \pmod{8}$  and  $2 \parallel N_2$ . Consequently, the function  $c'(-D, N_2)$  also has image in  $\mathbb{Z}_{\geq 0}$  and is 0 precisely when  $c(-D, N_2)$  is. Note that in general we can write

$$b(N) c'(-D, N_2) = \frac{1}{2} \beta(N) c(-D, N_2)$$

where  $\beta(N)$  is as in (1.1).

Now we simplify the correction terms.

Terms with  $N_1Q^2 \leq 4$  occur if and only if (i)  $Q = 1$ ,  $N_2$  is squarefree and  $N_1 \leq 3$ ; or (ii)  $Q = 2$ ,  $N_2$  is twice a squarefree number and  $N_1 = 1$ . Hence,

$$\xi_0(N, k) = \begin{cases} \mu(N_2) (\text{tr } W_1 - (-1)^{k/2} \cdot \frac{1}{4}) & N_1 = 1, N_2 \text{ odd squarefree} \\ \mu(N_2) (\text{tr } W_1 - \text{tr } W_4 + (-1)^{k/2} \cdot \frac{1}{4}) & N_1 = 1, N_2 \text{ even squarefree} \\ \mu(N_2/2) (\text{tr } W_4 - (-1)^{k/2} \cdot \frac{1}{2}) & N_1 = 1, N_2/4 \text{ odd squarefree} \\ \mu(N_2) (\text{tr } W_2 - (-1)^{k/2} \cdot \frac{1}{2}) & N_1 = 2, N_2 \text{ squarefree} \\ \mu(N_2) (\text{tr } W_3 - (-1)^{k/2} \cdot \frac{2}{3}) & N_1 = 3, N_2 \text{ squarefree,} \end{cases}$$

and  $\xi_0(N, k) = 0$  otherwise.

For the other correction term, note that

$$\sum_{Q|N_2} \mu(N_2/Q) \cdot 1 = \sum_{Q|N_2} \mu(Q) = \delta_{N_2=1}.$$

Hence

$$\xi_1(N, k) = \delta_{k=2} (\delta_{N_2=1} + \varepsilon(N))$$

where  $\varepsilon(N) = -\sum_{Q|N_2, N_1 Q^2 \leq 4} \mu(N_2/Q)$ . Explicitly, we have

$$\varepsilon(N) = \begin{cases} -\mu(N_2) & N_1 = 1, N_2 \text{ odd squarefree} \\ -\mu(N_2/2) & N_1 = 1, N_2/4 \text{ odd squarefree} \\ -\mu(N_2) & N_1 = 2, 3, N_2 \text{ squarefree} \\ 0 & \text{else.} \end{cases}$$

**3.5. Formulas for traces on newspaces.** Combining the explicit expressions for the main and correction terms in the previous section gives the following.

**Proposition 3.2.** *Let  $N \geq 1$ , and let  $k \geq 2$  be even. Write  $N = N_1 N_2^2$  where  $N_1$  is squarefree, and let  $-D \in \{-N_1, -4N_1\}$  be the discriminant of  $\mathbb{Q}(\sqrt{-N})$ .*

(1) *If  $N$  is not 1, 2, 3 or 4 times the square of a squarefree number, then*

$$\text{tr } W_N^{\text{new}} = \frac{1}{2}(-1)^{k/2} \beta(N) c(-D, N_2) h'(-D) + \delta_{k=2} \delta_{N_2=1}.$$

(2) *Suppose  $N = N_2^2$ , where  $N_2$  is squarefree. If  $N_2$  is odd, then*

$$\text{tr } W_N^{\text{new}} = \frac{1}{4}(-1)^{k/2} (c(-4, N_2) - \mu(N_2)) + \mu(N_2) \left( \left\lfloor \frac{k}{12} \right\rfloor - \kappa_1^{\text{odd}} \right) + \delta_{k=2} \delta_{N_2=1},$$

where  $\kappa_1^{\text{odd}} = 1$  if  $k \equiv 2 \pmod{12}$  and  $\kappa_1^{\text{odd}} = 0$  otherwise. If  $N_2$  is even, then

$$\text{tr } W_N^{\text{new}} = \frac{1}{4}(-1)^{k/2} (c(-4, N_2) + \mu(N_2)) + \mu(N_2) \left( \left\lfloor \frac{k}{12} \right\rfloor + \kappa_1^{\text{even}} \right),$$

where  $\kappa_1^{\text{even}} = 1$  if  $k \equiv 6, 10 \pmod{12}$  and  $\kappa_1^{\text{even}} = 0$  otherwise.

(3) *If  $N = N_2^2$ , where  $N_2$  is twice an even squarefree number, then*

$$\text{tr } W_N^{\text{new}} = \frac{1}{4}(-1)^{k/2} c(-4, N_2) - \frac{1}{2} \mu(N_2/2).$$

(4) *If  $N = 2N_2^2$ , where  $N_2$  is squarefree, then*

$$\text{tr } W_N^{\text{new}} = \frac{1}{2}(-1)^{k/2} (c(-8, N_2) + \kappa_2 \mu(N_2)) + \delta_{k=2} \delta_{N_2=1},$$

where  $\kappa_2 = 1$  if  $k \equiv 0, 2 \pmod{8}$  and  $\kappa_2 = -1$  if  $k \equiv 4, 6 \pmod{8}$ .

(5) *If  $N = 3N_2^2$ , where  $N_2$  is squarefree, then*

$$\text{tr } W_N^{\text{new}} = \frac{1}{3}(-1)^{k/2} \left( \frac{1}{2} \beta(N) c(-3, N_2) + \kappa_3 \mu(N_2) \right) + \delta_{k=2} \delta_{N_2=1},$$

where  $\kappa_3 = -2$  if  $k \equiv 4, 10 \pmod{12}$  and  $\kappa_3 = 1$  otherwise.

*Proof.* From the previous section, we see that

$$\text{tr } W_N^{\text{new}} = \frac{1}{2}(-1)^{k/2} \beta(N) c(-D, N_2) h'(-D) + \xi_0(N, k) + \delta_{k=2} (\delta_{N_2=1} + \varepsilon(N)).$$

In case (1),  $\xi_0(N, k) = \varepsilon(N) = 0$ , which yields the formula.

For the remaining cases, one just explicates  $\xi_0(N, k) + \delta_{k=2} \varepsilon(N)$  using the special cases for  $\text{tr } W_N$  in Section 3.1. For instance, in case (2) with  $N_2$  odd, we have

$-D = -4$  so  $h'(-4) = \frac{1}{2}$  and

$$\begin{aligned} \xi_0(N, k) + \delta_{k=2}\varepsilon(N) &= \mu(N_2) \left( \text{tr } W_1 - (-1)^{k/2} \frac{1}{4} - \delta_{k=2} \right) \\ &= \mu(N_2) \left( \left\lfloor \frac{k}{12} \right\rfloor - \kappa_1^{\text{odd}} - (-1)^{k/2} \frac{1}{4} \right). \end{aligned}$$

□

**Corollary 3.3.** *Keep the notation of Proposition 3.2.*

- (1) *Suppose  $N$  is not 1, 2, 3 or 4 times the square of a squarefree number. Then  $\Delta(N, k) \geq 0$ . Furthermore,  $\Delta(N, k) = 0$  exactly when (i)  $k = 2$  and either  $\dim S_k^{\text{new}}(N) = 0$  or  $N \in \{37, 58\}$ ; or (ii)  $2 \parallel N_2$  and  $N_1 \equiv 7 \pmod{8}$ .*
- (2) *Suppose  $N = N_2^2$  where  $N_2$  is squarefree. Then  $\Delta(N, k) \geq 0$  for all  $N$  if  $k \leq 10$  or  $k = 14$ . For any fixed  $k$ ,  $\Delta(N, k) \geq 0$  for  $N$  sufficiently large. For fixed  $N$  and sufficiently large  $k$ ,  $|\Delta(N, k)| > 0$  and the sign of  $\Delta(N, k)$  is  $(-1)^{k/2}\mu(N_2)$ .*
- (3) *Suppose  $N = N_2^2$  where  $N_2$  is twice an even squarefree number. Then  $\Delta(N, k) \geq 0$ . Further,  $\Delta(N, k) = 0$  if and only if  $N = 16$  and  $k \equiv 2 \pmod{4}$ .*
- (4) *Suppose  $N = 2N_2^2$ , where  $N_2$  is squarefree. Then  $\Delta(N, k) \geq 0$ . Further  $\Delta(N, k) = 0$  if and only if (i)  $N \in \{8, 18\}$  and  $k \equiv 0, 2 \pmod{8}$ ; (ii)  $N \in \{2, 72\}$  and  $k \equiv 4, 6 \pmod{8}$ ; or (iii)  $(N, k) = (2, 2)$ .*
- (5) *Suppose  $N = 3N_2^2$ , where  $N_2$  is squarefree. Then  $\Delta(N, k) \geq 0$ . Further  $\Delta(N, k) = 0$  if and only if (i)  $N = \{3, 108\}$  and  $k \equiv 4, 10 \pmod{12}$ ; (iii)  $N = 12$  and  $k \not\equiv 4, 10 \pmod{12}$ ; or (iii)  $(N, k) = (3, 2)$ .*

*Proof.* Recall that we always have  $b(N) \geq \frac{1}{2}$ ,  $c(-D, N_2) \in \mathbb{Z}_{\geq 0}$  and  $h'(-D) \geq \frac{1}{3}$ . Moreover  $c(-D, N_2) = 0$  if and only if either  $2 \parallel N_2$  is even and  $\left(\frac{-D}{p}\right) = 1$ .

Hence the statement that  $\Delta(N, k) \geq 0$  in (1) is obvious when  $k \neq 2$ . When  $k = 2$ , the only possible negative term in our formula for  $(-1)^{k/2} \text{tr } W_N^{\text{new}}$  is  $-\delta_{N_2=1}$ . However this only occurs when  $N$  is squarefree, in which case we already know  $\Delta(N_1, 2) \geq 0$  with equality if and only if  $\dim S_2^{\text{new}}(N) = 0$  or  $N \in \{37, 58\}$  by [Mar18a]. When  $N$  is not squarefree,  $\Delta(N, k) = 0$  if and only if  $c(-D, N_2) = 0$ . This finishes (1).

Suppose we are in case (2) now. Note  $c(-4, N_2) = \prod_{p|N_2^{\text{odd}}}(p - 1 - \left(\frac{-4}{p}\right)) \geq 3^{\omega(N_2^{\text{odd}})}$ . Here  $\omega(n)$  denotes the number of prime divisors of  $n$ . For fixed  $k$ ,  $\Delta(N, k)$  is thus dominated by  $c(-4, N_2)$  as  $N_2 \rightarrow \infty$ . For fixed  $N$ ,  $\Delta(N, k)$  is dominated by  $(-1)^{k/2}\mu(N_2)\lfloor k/12 \rfloor$  as  $k \rightarrow \infty$ . If  $k < 12$  and  $k = 14$ , the only possible negative terms in our formulas for  $(-1)^{k/2} \text{tr } W_N^{\text{new}}$  will be a  $-1$  (depending on  $\mu(N_2)$  and  $k$ ). However, in these cases is also a strictly positive term, and since the trace is an integer,  $(-1)^{k/2} \text{tr } W_N^{\text{new}} \geq 0$ .

In case (3), we have  $c(-4, N_2) = 2 \prod_{p|N_2^{\text{odd}}}(p - 1 - \left(\frac{-1}{p}\right)) \geq 2$ , with equality if and only if  $N = 16$ . This easily gives the assertions in (3).

Consider case (4). When  $N_2 = 1$ , by [Mar18a]  $\Delta(2, k) \geq 0$  with equality if and only if  $k \equiv 4, 6 \pmod{8}$ . Suppose  $N_2 > 1$ . Then  $\Delta(N, k) \geq 0$  if and only if  $c(-8, N_2) \geq -\kappa_2\mu(N_2)$ . Note  $c(-8, N_2) = \prod_{p|N_2}(p - 1 - \left(\frac{-8}{p}\right)) \geq 1$  and equals 1 if and only if  $N_2|6$ . Then the assertions readily follow.

Finally consider case (5). Again, the squarefree case  $N = 3$  is treated in [Mar18a], so suppose  $N_2 > 1$ . Now  $\beta(N)c(-3, N_2)$  equals  $\prod_{p|N_2}(p - 1 - \left(\frac{-3}{p}\right)) \geq 2^{\omega(N_2)}$  if  $N_2$

is odd and  $\frac{1}{2} \prod_{p|N_2^{\text{odd}}} (p - 1 - \frac{-3}{p}) \geq 2^{\omega(N_2^{\text{odd}} - 1)}$  if  $N_2$  is even. Comparing with the proposition now finishes the proof.  $\square$

This proves the theorems stated in the introduction.

#### 4. EXCLUDING TWISTS FOR CERTAIN LEVELS

Motivated by the different behavior in root number bias for cubefree square levels, we briefly investigate what happens when we exclude certain twists from smaller levels. Our main goal is to prove the propositions in Section 1.2.

First we recall some facts about twists of modular forms of cube-free levels. We will explain things from the point of view of associated local representations, as this perspective makes things more transparent to us. However, many of the facts that we recall are also well known from the more classical “global” approach (see [AL70], [AL78], [HPS90]).

Given a newform  $f \in S_k^{\text{new}}(N, \psi)$  with nebentypus  $\psi$ , there is an associated cuspidal automorphic representation  $\pi = \pi_f$  of  $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ . We have a decomposition  $\pi = \bigotimes_p \pi_p \otimes \pi_{\infty}$ , where each  $\pi_p$  is an infinite-dimensional irreducible admissible complex representation of  $\text{GL}_2(\mathbb{Q}_p)$ . We say  $\pi_p$  is the local representation at  $p$  associated to  $f$ . The local conductor  $c(\pi_p)$  is the exponent  $v_p(N)$  in the level of  $f$ .

Assume now that  $\psi$  is trivial, i.e.,  $f \in S_k^{\text{new}}(N)$ . Then  $\pi$ , and each  $\pi_p$ , has trivial central character. Moreover,  $\pi_p$  is an unramified principal series representation (i.e.,  $c(\pi_p) = 0$ ) if and only if  $p \nmid N$ . Also,  $\pi_p$  is an unramified quadratic twist (possibly trivial twist) of the Steinberg representation  $\text{St}_p$  of  $\text{GL}_2(\mathbb{Q}_p)$  (i.e.,  $c(\pi_p) = 1$ ) if and only if  $p \parallel N$ .

Now there are 3 possibilities when  $c(\pi_p) = 2$ :  $\pi_p$  can be a ramified principal series, a ramified quadratic twist of Steinberg, or supercuspidal. However, the former two possibilities do not happen for  $p = 2$  (e.g., see [Pac13, Corollary 4.1]). In fact there are 2 possibilities for  $\pi_p$  being a ramified principal series: either it is a ramified quadratic twist of an unramified principal series, or a minimal twist (a twist minimizing the local conductor) of  $\pi_p$  is a ramified principal series of conductor 1 (necessarily the central character is nontrivial and has conductor 1). In any event, only the supercuspidal representations are minimal.

Suppose now that  $N = M^2$  where  $M > 1$  is squarefree. Then the above description of local representations means exactly one of the following is true for a newform  $f \in S_k^{\text{new}}(N)$ :

- (1)  $f$  is minimal, i.e., no twist (possibly with nebentypus) has smaller level. Here  $\pi_p$  is supercuspidal for each  $p \mid N$ .
- (2)  $f$  is a quadratic twist of a level 1 newform. Here  $\pi_p$  is a ramified quadratic twist of an unramified principal series for each  $p \mid N$ . Necessarily  $N$  is odd.
- (3) A minimal twist  $f'$  of  $f$  (possibly with nebentypus) has level  $N'$  strictly between 1 and  $N$ . Necessarily  $v_2(N') = v_2(N)$ .

The utility of the local representation theory perspective for us comes both from a straightforward description of the level of twists and the fact that root numbers can be read off of the local representations. Namely, the root number of  $f$  is the product over  $p$  of the local representation root numbers  $w_p(\pi_p)$  times  $(-1)^{k/2}$  (which is the local root number of  $\pi_{\infty}$ ).

Recall  $S_k^{\text{new}}(N)'$  (resp.  $S_k^{\text{min}}(N)$ ) is defined to be the subspace of  $S_k^{\text{new}}(N)$  generated by newforms of types (1) and (3) above (resp. of type (1) above). Define

$\text{tr } W'_N$  and  $\text{tr } W_N^{\min}$  to be the traces of  $W_N$  restricted to the spaces  $S_k^{\text{new}}(N)'$  and  $S_k^{\min}(N)$ , respectively. First we describe  $\text{tr } W'_N$ .

**Lemma 4.1.** *We have*

$$\text{tr } W'_N = \begin{cases} \text{tr } W_N^{\text{new}} - (-1)^{k/2} \prod_{p|N} \left(\frac{-1}{p}\right) \cdot \dim S_k(1) & N \text{ odd} \\ \text{tr } W_N^{\text{new}} & N \text{ even.} \end{cases}$$

*Proof.* Since  $S_k^{\text{new}}(N)' = S_k^{\text{new}}(N)$  if  $N$  is even, assume  $N$  is odd. Then we simply need to subtract off from  $\text{tr } W_N^{\text{new}}$  the trace of  $W_N$  restricted to the subspace of  $S_k^{\text{new}}(N)$  generated by forms of type (2). Now we simply use the fact that if  $\pi_p$  is a ramified quadratic twist of an unramified principal series, then  $w_p(\pi_p) = \left(\frac{-1}{p}\right)$  (see [Pac13, Theorem 3.2] in terms of local representations, or [AL70, Theorem 6] in terms of modular forms).  $\square$

Let  $\Delta(N, k)' = (-1)^{k/2} \text{tr } W'_N$ , which is the difference between the number of newforms in  $S_k^{\text{new}}(N)'$  with root number  $+1$  and  $-1$ .

*Proof of Proposition 1.3.* When  $N$  is even, we simply have  $\Delta(N, k)' = \Delta(N, k)$ , so assume  $N$  is odd. Then Proposition 3.2 combined with the above lemma shows  $\Delta(N, k)'$  is

$$\frac{1}{4}(c(-4, M) - \mu(M)) - \left( \prod_{p|M} \left(\frac{-1}{p}\right) - (-1)^{k/2} \mu(M) \right) \dim S_k(1) + \mu(M) \delta_{k=2}.$$

The sum of the first and the last term is always non-negative. Note the coefficient of  $\dim S_k(1)$  is negative if and only if  $\mu(M) = -(-1)^{k/2}$  and  $\prod_{p|M} \left(\frac{-1}{p}\right) = 1$ . In this case,  $\Delta(N, k)' < 0$  if  $k$  is sufficiently large.  $\square$

In general to describe  $\text{tr } W_N^{\min}$ , one needs to isolate the forms which are supercuspidal at each ramified place. One can presumably do this in a similar way as Lemma 4.1, by subtracting off the contributions from twists from smaller level. However, we will restrict ourselves to giving a couple of examples to show that  $(-1)^{k/2} \text{tr } W_N^{\min}$  can be negative, and then prove that often there is a quadratic twist which forces  $\text{tr } W_N^{\min} = 0$  to get Proposition 1.4.

The example of  $N = 4$  was already given in Section 1.2. The following examples are taken from the LMFDB [LMFDB].

**Example 4.2.** The space  $S_{10}^{\text{new}}(9)$  has 3 newforms, all rational. Two are twists from  $S_{10}^{\text{new}}(3)$  by the quadratic character  $\left(\frac{-3}{\cdot}\right)$ , and they have root number  $+1$  (though the corresponding forms on  $S_{10}^{\text{new}}(3)$  have opposite root numbers). The other newform in  $S_{10}^{\text{new}}(9)$ , which is CM and minimal, has root number  $-1$ . Hence the sum of root numbers of minimal newforms in  $S_{10}(9)$  is  $-1$ , though  $\Delta(10, 9) = +1$ .

**Example 4.3.** The space  $S_{14}^{\min}(49)$  has 3 Galois orbits of newforms: one CM orbit of size 1 and root number  $+1$ , one Galois orbit of size 6 and root number  $+1$ , and one Galois orbit of size 12 and root number  $-1$ . Hence the sum of root numbers for minimal forms is  $1 \cdot 1 + 1 \cdot 6 - 1 \cdot 12 = -5$ . On the other hand,  $\Delta(49, 14) = 2$ . We remark that except for the first form, all newforms are non-CM, and all of the newforms have nontrivial inner twists.

*Proof of Proposition 1.4.* Here we allow any  $N > 1$ , i.e.,  $N$  need not be a square or cubefree. Suppose  $p^2 \parallel N^{\text{odd}}$ , and let  $\chi$  be the quadratic character of conductor

$\pm p$ . Then twisting any minimal newform  $f \in S_k^{\min}(N)$  gives another newform of level  $N$ . The local argument for this is that twisting a supercuspidal representation  $\pi_p$  with  $c(\pi_p) = 2$  by  $\chi_p$  yields a supercuspidal representation  $\pi_p \otimes \chi_p$  also of local conductor 2.

Hence twisting by  $\chi$  is an involution on the set of minimal newforms. Now any supercuspidal representation  $\pi_p$  with  $c(\pi_p) = 2$  is necessarily minimal, and thus by [Tun78, Proposition 3.5], is induced from the unramified quadratic extension of  $\mathbb{Q}_p$ . Hence by [Pac13, Theorem 3.2], the local root number of  $\pi_p \otimes \chi_p$  is  $-\left(\frac{-1}{p}\right)$  times the local root number of  $\pi_p$ . Moreover, for  $q \mid p^{-2}N$ , twisting by  $\chi_p$  multiplies the root number of  $\pi_q$  by  $\left(\frac{q}{p}\right)^{v_q(N)}$ . In particular, if  $\left(\frac{-N/p^2}{p}\right) = 1$ , then twisting by  $\chi$  flips the sign of the root number of minimal newforms, which implies  $\text{tr } W_N^{\min} = 0$ .  $\square$

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#### REFERENCES

- [AL70] A. O. L. Atkin and J. Lehner, *Hecke operators on  $\Gamma_0(m)$* , Math. Ann. **185** (1970), 134–160, DOI 10.1007/BF01359701. MR268123 ↑3733, 3734
- [AL78] A. O. L. Atkin and Wen Ch'ing Winnie Li, *Twists of newforms and pseudo-eigenvalues of  $W$ -operators*, Invent. Math. **48** (1978), no. 3, 221–243, DOI 10.1007/BF01390245. MR508986 ↑3733
- [Coh78] Harvey Cohn, *A classical invitation to algebraic numbers and class fields*, Universitext, Springer-Verlag, New York-Heidelberg, 1978. With two appendices by Olga Taussky: “Artin’s 1932 Göttingen lectures on class field theory” and “Connections between algebraic number theory and integral matrices”. MR506156 ↑3725
- [HPS89] Hiroaki Hijikata, Arnold K. Pizer, and Thomas R. Shemanske, *The basis problem for modular forms on  $\Gamma_0(N)$* , Mem. Amer. Math. Soc. **82** (1989), no. 418, vi+159, DOI 10.1090/memo/0418. MR960090 ↑3723
- [HPS90] Hiroaki Hijikata, Arnold K. Pizer, and Thomas R. Shemanske, *Twists of newforms*, J. Number Theory **35** (1990), no. 3, 287–324, DOI 10.1016/0022-314X(90)90119-C. MR1062336 ↑3733
- [LMFDB] The LMFDB Collaboration, *The  $L$ -functions and Modular Forms Database*, 2022. <http://www.lmfdb.org>. ↑3723, 3734
- [LPW23] Zhilin Luo, Qinghua Pi, and Han Wu, *Bias of root numbers for Hilbert newforms of cubic level*, J. Number Theory **243** (2023), 62–116, DOI 10.1016/j.jnt.2022.08.008. MR4492607 ↑3721, 3723, 3725
- [Mar05] Greg Martin, *Dimensions of the spaces of cusp forms and newforms on  $\Gamma_0(N)$  and  $\Gamma_1(N)$* , J. Number Theory **112** (2005), no. 2, 298–331, DOI 10.1016/j.jnt.2004.10.009. MR2141534 ↑3724
- [Mar18a] Kimball Martin, *Refined dimensions of cusp forms, and equidistribution and bias of signs*, J. Number Theory **188** (2018), 1–17, DOI 10.1016/j.jnt.2018.01.015. MR3778620 ↑3721, 3722, 3723, 3724, 3725, 3726, 3732
- [Mar18b] Kimball Martin, *Congruences for modular forms mod 2 and quaternionic  $S$ -ideal classes*, Canad. J. Math. **70** (2018), no. 5, 1076–1095, DOI 10.4153/CJM-2017-019-1. MR3831915 ↑3725
- [Mar20] Kimball Martin, *The basis problem revisited*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 4523–4559, DOI 10.1090/tran/8077. MR4127854 ↑3723

- [Pac13] Ariel Pacetti, *On the change of root numbers under twisting and applications*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2615–2628, DOI 10.1090/S0002-9939-2013-11532-7. MR3056552 ↑3733, 3734, 3735
- [PQ21] Qinghua Pi and Zhi Qi, *Bias of root numbers for modular newforms of cubic level*, Proc. Amer. Math. Soc. **149** (2021), no. 12, 5035–5047, DOI 10.1090/proc/15530. MR4327413 ↑3721, 3722, 3723
- [Sage] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 9.4)*, 2021. <https://www.sagemath.org>. ↑3725
- [SZ88] Nils-Peter Skoruppa and Don Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988), no. 1, 113–146, DOI 10.1007/BF01394347. MR958592 ↑3722, 3725, 3726
- [Tun78] Jerrold B. Tunnell, *On the local Langlands conjecture for  $GL(2)$* , Invent. Math. **46** (1978), no. 2, 179–200, DOI 10.1007/BF01393255. MR476703 ↑3735
- [Yam73] Masatoshi Yamauchi, *On the traces of Hecke operators for a normalizer of  $\Gamma_0(N)$* , J. Math. Kyoto Univ. **13** (1973), 403–411, DOI 10.1215/kjm/1250523380. MR340179 ↑3722, 3726
- [ZZ22] Yichao Zhang and Yang Zhou, *Dimension formulas for modular form spaces with character for Fricke groups*, Acta Arith. **206** (2022), no. 4, 291–311, DOI 10.4064/aa211126-17-11. MR4528421 ↑3722, 3727

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