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Central L-Values and Toric Periods for GL(2)

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Let π be a cusp form on GL(2) over a number field F and let E be a quadratic extension of F. Denote by π_E the base change of π to E and by Ω a unitary character of $\mathbf{A}_E^\times/E^\times$. We use the relative trace formula to give an explicit formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of period integrals of Gross–Prasad test vectors. We give an application of this formula to equidistribution of geodesics on a hyperbolic 3-fold.

1 Introduction

Let us begin by recalling relevant results on central L-values for GL(2). Let F be a number field and fix a quadratic extension E/F. Denote the norm map from E to F by $N_{E/F}$ and the adeles of a number field K by $\mathbf{A}_K$. We will take π to be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbf{A}_F)$ whose central character ω_π is trivial on $N_{E/F}\mathbf{A}_E^\times$. That is, either $\omega_\pi = 1$ or $\omega_\pi = \eta_{E/F}$, the quadratic character attached to E/F. Now take a unitary character

$$\Omega : \mathbf{A}_E^\times/E^\times \to \mathbf{C}^\times$$

such that $\Omega|_{\mathbf{A}_F^\times} = \omega_\pi$. Assume that the ramifications of π and Ω are disjoint. We will be interested in a formula for the central L-value of the automorphic representation $\pi_E \otimes \Omega$

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of $\mathrm{GL}(2, \mathbf{A}_E)$. For GL(2) L-values, typically one wants a formula in terms of either period integrals or Fourier coefficients, as these are easier to compute. In this paper, we will establish a formula in terms of period integrals. For a formula for $L(1/2, \pi)$ in terms of Fourier coefficients when $\omega_\pi = 1$ and F is totally real, see [24], [1].

Let D be a quaternion algebra defined over F such that

(1) $E \hookrightarrow D$ and

(2) π transfers, in the sense of Jacquet–Langlands, to a representation π^D of $D^\times(\mathbf{A}_F)$.

We allow for the possibility that $D(F) \simeq M_2(F)$, so at least one such D always exists. In this case we take π for π^D.

Given such a quaternion algebra D, we define period integrals

$$P^D(\varphi) = \int_{E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times} \varphi(t)\Omega^{-1}(t)\, dt$$

for $\varphi \in \pi^D$. We note that the integral makes sense because of the compatibility between Ω and ω_π. Waldspurger [25] and Jacquet [14] proved that $L(1/2, \pi_E \otimes \Omega) = 0$ if and only if $P^D(\varphi) = 0$ for all D and $\varphi \in \pi^D$ as above.

Note that the function

$$\varphi \mapsto P^D(\varphi)$$

is an element of $\mathrm{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega)$. It is known that

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega) \leq 1,$$

and moreover, it is clear that it is nonzero if and only if

$$\mathrm{Hom}_{E_v^\times}\left(\pi_v^D, \Omega_v\right) \neq 0$$

for all places v of F.

The L-function $L(s, \pi_E \otimes \Omega)$ is symmetric and when the sign in the functional equation is -1, the period integrals are forced to vanish for local reasons, namely the fact that

$$\mathrm{Hom}_{E_v^\times}\left(\pi_v^D, \Omega_v\right) = 0$$

for some place v of F.

Let us assume from now on that the sign in the functional equation is +1. In this case, there is a unique quaternion algebra D/F such that

$$\mathrm{Hom}_{E_v^\times}\left(\pi_v^D, \Omega_v\right) \neq 0$$

for all places v of F. The algebra D can be characterized by local ε-factors; see [9, Proposition 1.1]. Let us fix this D.

In [25], assuming $\omega_\pi = 1$, Waldspurger proved that for *any* $\varphi \in \pi^D$,

$$L(1/2, \pi_E \otimes \Omega) = \frac{1}{\zeta(2)} \prod_v \alpha_v(E, \varphi, \Omega) \frac{|P^D(\varphi)|^2}{(\varphi, \varphi)},$$

where the $\alpha_v(E, \varphi, \Omega)$'s (almost all 1) are local constants defined in terms of certain integrals. It is convenient to write the formula in terms of $|P^D(\varphi)|^2/(\varphi, \varphi)$ because this quantity is invariant under scaling. There are two points we wish to emphasize about Waldspurger's work. First, Waldspurger uses the theta correspondence to establish his result. Second, the constants $\alpha_v(E, \varphi, \Omega)$ are not as explicit as one would like for applications. In fact, it is not even clear from Waldspurger's formula that $L(1/2, \pi_E \otimes \Omega) \geq 0$. (This is predicted by the generalized Riemann hypothesis. It is immediate from our formula below, and was proven by Guo [10] and Jacquet–Chen [15] using the relative trace formula.)

It seems likely that in order to get the most explicit formula possible, one needs to choose a specific vector $\varphi \in \pi^D$. In the case that the ramifications of π and Ω are disjoint, the work of Gross and Prasad [9] provides a nice test vector $\varphi \in \pi^D$ such that $\ell(\varphi) \neq 0$ for any nonzero $\ell \in \mathrm{Hom}_{\mathbb{A}_E^\times}(\pi^D, \Omega)$. Thus, the result of Waldspurger and Jacquet can be rephrased as $L(1/2, \pi_E \otimes \Omega) = 0$ if and only if $P^D(\varphi) = 0$, with φ equal to the Gross–Prasad test vector. We also remark that a formula in terms of the Gross–Prasad test vector is particularly well suited for certain applications [22].

Subsequent to Waldspurger's work, there has been considerable work devoted to obtaining an explicit formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of $P^D(\varphi)$ for a specific choice of $\varphi \in \pi^D$. We mention four results. First, in [7] Gross obtained a formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of the Gross–Prasad test vector when $F = \mathbb{Q}$, E is an imaginary quadratic field, π is holomorphic of weight 2 and inert prime level, and Ω is unramified. Then in [27], Zhang generalized Gross's formula to F totally real, E/F imaginary quadratic, π holomorphic of parallel weight 2 and arbitrary level N, and Ω unramified above N and where E/F is ramified. However, the test vector φ that Zhang must choose is not necessarily the

Gross–Prasad test vector; it is locally away from the places of F that ramify in E. Xue [26] generalized Zhang's result to π holomorphic of arbitrary even weight, again with ramification conditions. (Again, his test vector is not the one given by Gross–Prasad.) For real quadratic extensions, Popa [19] obtained a formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of the Gross–Prasad test vector when $F = \mathbf{Q}$, E is a real quadratic field, π has even weight with square-free level prime to the discriminant of E, and Ω is unramified. These results are all established using the theta correspondence and the Rankin–Selberg method.

Jacquet developed another method to study period integrals and L-values, known as the relative trace formula. In this paper, we continue the work of [13], [14], [10], and [15] to prove an explicit version of Waldspurger's formula for $L(1/2, \pi_E \otimes \Omega)$, in the generality given in the first paragraph, in terms of $P^D(\varphi)$ when $\varphi \in \pi^D$ is the Gross–Prasad test vector. We would like to point out that the relative trace formula, while perhaps having greater analytic difficulties, is a much more general method for studying L-values and periods than the theta correspondence (see, for example, [16] for exact formulae for period integrals over unitary groups). Even for GL(2), the formula we have obtained is more general than the explicit results obtained to date via the theta correspondence. For instance, F need not be totally real and ω_π need not be trivial.

Let us briefly outline our method. We define $G = D^\times$ and $\sigma = \pi^D$. For $f \in C_c^\infty(G(\mathbf{A}_F))$, consider the distribution

$$f \mapsto J_\sigma(f) = \sum_{\varphi} \int (\sigma(f)\varphi)(t)\Omega(t)^{-1}\, dt \overline{\int \varphi(t)\Omega(t)^{-1}\, dt},$$

where the sum is taken over an orthonormal basis $\{\varphi\}$ of σ and the integrals are taken over $E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times$. By local considerations it is known that the distribution factors into a product of local ones, however this factorization is not unique. The work of Jacquet and Chen [15, Theorem 2] uses the relative trace formula to give a canonical decomposition of this distribution.

Let $f \in C_c^\infty(G(\mathbf{A}_F))$ be of the form $f = \prod_{v_0 \in S_0} f_{v_0} f^{S_0}$, with f^{S_0} the unit in the Hecke algebra of G away from S_0. Then Jacquet and Chen prove that

$$J_\sigma(f) = \frac{1}{2} \prod_{v_0 \in S_0} \tilde{J}_{\sigma_{v_0}}(f_{v_0}) \prod_{\substack{v_0 \in S_0 \\ v_0 \text{ inert}}} \left(\varepsilon(1, \eta_{v_0}, \psi_{v_0}) 2L(0, \eta_{v_0})\right) \frac{L_{S_0}(1, \eta) L^S(1/2, \pi_E \otimes \Omega)}{L^{S_0}(1, \pi, Ad)},$$

where S denotes the places of F and S_0, $\eta = \eta_{E/F}$, and the $\widetilde{J}_{\sigma_{v_0}}$'s are certain local distributions defined in Sections 2 and 3. To obtain our formula, we choose test functions f_{v_0}

such that

$$J_\sigma(f) = |P^D(\varphi)|^2,$$

where φ is the Gross–Prasad test vector. We then compute the local distributions which gives a formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of $P^D(\varphi)$ and $L(1, \pi, Ad)$ (Theorem 4.1). Now $L(1, \pi, Ad)$ is essentially $(\varphi_\pi, \varphi_\pi)$, where φ_π is a newvector for π. Since one may prefer a formula in terms of $(\varphi_\pi, \varphi_\pi)$ for certain applications, we also rewrite our formula for $L(1/2, \pi_E \otimes \Omega)$ in terms of $(\varphi_\pi, \varphi_\pi)$ (Theorem 4.2). The precise statement of these formulas is given in Section 4. We have attempted to make that section self-contained for the convenience of the reader. In Sections 2 and 3, we work out the necessary local calculations.

There are several applications of these Waldspurger-type formulas. In Section 5, we use Theorem 4.1 to obtain results about equidistribution of geodesics on a hyperbolic 3-fold. Brooke Feigon, together with the second author also used this formula to study average L-values [5]. For more arithmetic applications of such formulas, see for example, [2], [21], and [19].

Lastly, we remark that this approach can be used to obtain an explicit Waldspurger-type formula for arbitrary central character and ramification conditions with any reasonable test vector φ. The assumptions above are made for simplicity and are not essential to the methods used here. The central character assumption is present in [15] and the ramification assumption is made for the sake of the Gross–Prasad test vector.

2 Non-Archimedean Calculations

We fix F, a non-Archimedean local field of characteristic zero. We let $\mathcal{O}_F$ denote the ring of integers in F and let $\mathfrak{p}_F$ denote the prime ideal of $\mathcal{O}_F$.

2.1 Split case

We fix an additive character ψ of F of conductor $n(\psi)$, i.e. ψ is trivial on $\mathfrak{p}_F^{-n(\psi)}$ but is nontrivial on $\mathfrak{p}_F^{-n(\psi)-1}$. We take the Haar measure on F which is self-dual with respect to ψ and take the measure

$$d^\times x = L(1, 1_F) \frac{dx}{|x|_F}$$

on $F^\times$. We note that

$$\mathrm{vol}(\mathcal{O}_F, dx) = \mathrm{vol}(U_F, d^\times x) = q^{-\frac{n(\psi)}{2}}.$$

We fix a unitary character Ω of $F^\times$ of conductor $\mathfrak{p}_F^{n(\Omega)}$.

Suppose now that π is an irreducible generic unitary representation of $\mathrm{GL}(2, F)$ with trivial central character. We consider the Whittaker model $\mathcal{W}(\pi, \psi)$ of π with respect to the character ψ. We take the inner product on $\mathcal{W}(\pi, \psi)$, given by

$$(W_1, W_2) = \int_{F^\times} W_1 \begin{pmatrix} a & \\ & 1 \end{pmatrix} \overline{W_2 \begin{pmatrix} a & \\ & 1 \end{pmatrix}} \, d^\times a.$$

Let

$$K_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}_F) : c \in \mathfrak{p}_F^n \right\}.$$

Let $n(\pi)$ be the conductor of π, i.e. the minimal n such that π has a $K_0(n)$-fixed vector. Then the space $\pi^{K_0(n(\pi))}$ is one-dimensional and any nonzero vector in this space is called a newvector.

Let W_π denote the newvector in $\mathcal{W}(\pi, \psi)$ normalized so that $W_\pi(\mathrm{diag}(\varpi^{-n(\psi)}, 1)) = \mathrm{vol}(U_F)^{-1}$, and hence such that

$$Z(s, \pi(\mathrm{diag}(\varpi^{-n(\psi)}, 1)) W_\pi) = L(s, \pi),$$

where $Z(s, W)$ denotes the local zeta integral of $W \in \mathcal{W}(\pi, \psi)$. For future use we record the following lemma. The proof is straightforward, using the fact that one may compute the values of W on the diagonal torus via the relation with $L(s, \pi)$ (cf. [6], [18]).

Lemma 2.1. If π is unramified, then

$$(W_\pi, W_\pi) = \mathrm{vol}(U_F)^{-1} \frac{L(1, \pi, Ad) L(1, 1_F)}{L(2, 1_F)}.$$

If $n(\pi) = 1$, then π is special and

$$(W_\pi, W_\pi) = \mathrm{vol}(U_F)^{-1} \frac{1}{1 - q^{-2}} = \mathrm{vol}(U_F)^{-1} L(1, \pi, Ad).$$

If $n(\pi) > 1$, then

$$(W_\pi, W_\pi) = \mathrm{vol}(U_F)^{-1}. \qquad \square$$

Given $f \in C_c^\infty(\mathrm{GL}(2,F))$, we define

$$\tilde{J}_\pi(f) = \sum_W \int_{F^\times} (\pi(f)W)\begin{pmatrix} a & \\ & 1 \end{pmatrix} \Omega^{-1}(a)\, d^\times a \int_{F^\times} \overline{W\begin{pmatrix} b & \\ & 1 \end{pmatrix} \Omega^{-1}(b)\, d^\times b},$$

with the sum being taken over an orthonormal basis $\{W\}$ of $\mathcal{W}(\pi, \psi)$.

We now compute $\tilde{J}_\pi(f)$ for certain choices of test function f depending on the ramification of Ω.

2.1.1 Ω *unramified*

The following result is well known.

Lemma 2.2. If f is the characteristic function of $K_0(n(\pi))$ divided by its volume, then

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi \otimes \Omega)L(1/2, \pi \otimes \Omega^{-1})}{(W_\pi, W_\pi)}. \qquad \square$$

2.1.2 Ω *ramified*

Let

$$h = \begin{pmatrix} 1 & \varpi^{-n(\Omega)} \\ & 1 \end{pmatrix}.$$

Then the newvector with respect to $hK_0(n(\pi))h^{-1}$ is $W' = \pi(h)W'_\pi$.

Lemma 2.3. If f is the characteristic function of $hK_0(n(\pi))h^{-1}$ divided by its volume, then

$$\tilde{J}_\pi(f) = q^{-n(\Omega)} \frac{L(1, 1_F)^2}{(W_\pi, W_\pi)}.$$

In particular, if π is unramified,

$$\tilde{J}_\pi(f) = q^{-n(\Omega)} L(1, 1_F)^2 \frac{L(1/2, \pi \otimes \Omega) L(1/2, \pi \otimes \Omega^{-1})}{(W_\pi, W_\pi)}. \qquad \square$$

Proof. Note that

$$W' \begin{pmatrix} a & \\ & 1 \end{pmatrix} = \psi\left(a\varpi^{-n(\Omega)}\right) W_\pi \begin{pmatrix} a & \\ & 1 \end{pmatrix}.$$

Hence we have, for s with $\Re s \gg 0$,

$$\begin{aligned} Z(s, W', \Omega) &= \int_{F^\times} \psi\left(a\varpi^{-n(\Omega)}\right) W_\pi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \Omega^{-1}(a) |a|^{s-\frac{1}{2}} \, d^\times a \\ &= \sum_{m=-\infty}^{\infty} W_\pi \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} |\varpi^m|^{s-\frac{1}{2}} \int_{|a|=q^{-m}} \psi\left(a\varpi^{-n(\Omega)}\right) \Omega^{-1}(a) \, d^\times a. \end{aligned}$$

We note that the integral is nonvanishing unless $-m = n(\psi)$, in which case we have

$$\begin{aligned} \int_{|a|=q^{n(\psi)}} \psi\left(a\varpi^{-n(\Omega)}\right) \Omega^{-1}(a) \, d^\times a &= \Omega\left(\varpi^{n(\Omega)}\right)^{-1} \int_{|a|=q^{n(\psi)+n(\Omega)}} \psi(a) \Omega^{-1}(a) \, d^\times a \\ &= \Omega\left(\varpi^{n(\Omega)}\right)^{-1} L(1, 1_F) q^{-(n(\psi)+n(\Omega))} \int_{|a|=q^{n(\psi)+n(\Omega)}} \psi(a) \Omega^{-1}(a) \, da \\ &= \Omega\left(\varpi^{n(\Omega)}\right)^{-1} L(1, 1_F) q^{-(n(\psi)+n(\Omega))} \varepsilon(\Omega, \psi, da). \end{aligned}$$

Hence by [20, (3.4.7)],

$$\left| \int_{|a|=q^{n(\psi)}} \psi\left(a\varpi^{-n(\Omega)}\right) \Omega^{-1}(a) \, d^\times a \right|^2 = L(1, 1_F)^2 q^{-(n(\psi)+n(\Omega))}.$$

Thus we deduce that

$$\tilde{J}_\pi(f) = \frac{1}{\mathrm{vol}(U_F)^2} \frac{L(1, 1_F)^2 q^{-(n(\psi)+n(\Omega))}}{(W_\pi, W_\pi)} = \frac{L(1, 1_F)^2 q^{-n(\Omega)}}{(W_\pi, W_\pi)}.$$

■

2.2 Nonsplit case

We now take E/F to be a quadratic extension of F. Let η denote the quadratic character of $F^\times$ associated to E, and let D denote the quaternion division algebra over F. We fix embeddings of $E^\times$ into $\mathrm{GL}(2, F)$ and $D^\times$.

Let π be an irreducible generic unitary representation of $\mathrm{GL}(2, F)$ with $\omega_\pi \in \{1, \eta\}$. When it exists, denote by π_D the Jacquet–Langlands transfer of π to $D^\times$. We fix inner products on π and π_D.

Fix a unitary character $\Omega : E^\times \to \mathbf{C}^\times$ such that $\Omega|_{F^\times} = \omega_\pi$. We consider the subspaces

$$V(\pi) = \{v \in \pi : \pi(t)v = \Omega(t)v \text{ for all } t \in E^\times\},$$

and

$$V(\pi_D) = \{v \in \pi_D : \pi_D(t)v = \Omega(t)v \text{ for all } t \in E^\times\}$$

of π and π_D, respectively. We know that precisely one of the $V(\pi)$ and $V(\pi_D)$ is isomorphic to $\mathbf{C}$ and the other is zero. We denote by π' the representation such that $V(\pi') \neq 0$ and we fix a nonzero unit vector $e'_T \in V(\pi')$. Let G be the group of which π' is a representation.

Suppose now that $f \in C_c^\infty(G(F))$ and define the distribution

$$\tilde{J}_\pi(f) = \int_{G(F)} f(g)\, \langle \pi'(g)e'_T, e'_T \rangle \, dg.$$

We wish to compute $\tilde{J}_\pi(f)$ for a suitable test function. We do this on a case-by-case basis according to the table below.

π	E/F	Ω
Ramified	Arbitrary	Unramified
Unramified	Unramified	Unramified
Unramified	Unramified	Ramified
Unramified	Ramified	Unramified
Unramified	Ramified	Ramified

2.2.1 π *ramified*

We denote by $R_{n(\pi)}$ an order of reduced discriminant $\mathfrak{p}_F^{n(\pi)}$ containing $\mathcal{O}_E$. It is well defined up to conjugation by $E^\times$. We now take f to be the characteristic function of the subgroup $R_{n(\pi)}^\times$ of $G(F)$ divided by its volume. We note that in this case we have $e_T' \in (\pi')^{R_{n(\pi)}^\times}$, and hence $\tilde{J}_\pi(f) = 1$.

2.2.2 π *unramified*

We now fix a uniformizer ϖ in F. We fix $\tau \in \mathcal{O}_E$ such that $\mathcal{O}_E = \mathcal{O}_F[\tau]$. In the case that E/F is ramified, we further assume that τ is a uniformizer in E. We take

$$a + b\tau \mapsto \begin{pmatrix} a + b\,\mathrm{Tr}\,\tau & b\,\mathrm{N}\,\tau \\ -b & a \end{pmatrix}$$

for the embedding of $E \hookrightarrow \mathrm{GL}(2, F)$, where Tr and N denote the trace and norm maps. Denote by $n = n(\Omega)$ the smallest integer such that Ω is trivial on $(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times$.

Let

$$h = \begin{pmatrix} \varpi^n\,\mathrm{N}\,\tau & \\ & 1 \end{pmatrix}.$$

Then for $\alpha = a + b\tau$, we have

$$h^{-1}\alpha h = \begin{pmatrix} a + b\,\mathrm{Tr}\,\tau & b\varpi^{-n} \\ -b\varpi^n\,\mathrm{N}\,\tau & a \end{pmatrix},$$

and hence $h^{-1}\alpha h \in M_2(\mathcal{O}_F)$ if and only if $a \in \mathcal{O}_F$ and $b \in \varpi^n \mathcal{O}_F$, which is if and only if $\alpha \in \mathcal{O}_F + \varpi^n \mathcal{O}_E$. Thus $R = hM_2(\mathcal{O}_F)h^{-1}$ is a maximal order in $M_2(\mathcal{O}_F)$ optimally containing $\mathcal{O}_F + \varpi^n \mathcal{O}_E$.

We now assume that π is a unitarizable unramified representation of $\mathrm{GL}(2, F)$ with $\omega_\pi \in \{1, \eta\}$ as before. We take the Kirillov model for π with respect to an unramified additive character ψ, and we denote by v_0 the newvector in π normalized by the requirement that $v_0(e) = 1$. We take the inner product on π to be given by

$$\langle v_1, v_2 \rangle = \int_{F^\times} v_1(x)\overline{v_2(x)}\; d^\times x,$$

where the Haar measure on $F^\times$ is normalized to give U_F volume 1. By Lemma 2.1 (and a similar argument when $\omega_\pi = \eta$), we have

$$\langle v_0, v_0 \rangle = \frac{L(1, \pi, Ad)L(1, 1_F)}{L(2, 1_F)}.$$

We note that the set of maximal orders in $M_2(F)$ optimally containing $\mathcal{O}_F + \varpi^n \mathcal{O}_E$ is permuted simply transitively by $E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times$. We set $v = \pi(h)v_0$ and

$$e''_T = \sum_{\alpha \in E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times} \Omega(\alpha)^{-1}\pi(\alpha)v.$$

We let f denote the characteristic function of $R^\times$ divided by its volume. Then e''_T is a nonzero vector such that $\pi(\alpha)e''_T = \Omega(\alpha)e''_T$ for all $\alpha \in E^\times$, and we have

$$\tilde{J}_\pi(f) = \frac{1}{\mathrm{vol}(R^\times)} \int_{R^\times} \frac{\langle \pi(g)e''_T, e''_T \rangle}{\langle e''_T, e''_T \rangle}\, dg.$$

Clearly, we have

$$\langle e''_T, e''_T \rangle = \#(E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times)\langle e''_T, v \rangle,$$

and

$$\frac{1}{\mathrm{vol}(R^\times)} \int_{R^\times} \langle \pi(g)e''_T, e''_T \rangle\, dg = \frac{\langle v, e''_T \rangle \langle e''_T, v \rangle}{\langle v, v \rangle}.$$

Hence

$$\tilde{J}_\pi(f) = \frac{\langle v, e''_T \rangle}{\# E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times \langle v_0, v_0 \rangle}.$$

We also have

$$\begin{aligned}\langle v, e''_T \rangle &= \sum_{\alpha \in E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times} \Omega(\alpha)^{-1} \langle \pi(h^{-1}\alpha h)v_0, v_0 \rangle \\ &= \sum_{m=0}^{\infty} \overline{v_0(\varpi^m)} \sum_{\alpha \in E^\times/F^\times(\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times} \Omega(\alpha)^{-1} \int_{U_F} (\pi(h^{-1}\alpha h)v_0)(u\varpi^m)\, du.\end{aligned}$$

Note that

$$\#E^\times/F^\times(\mathcal{O}_F+\varpi^n\mathcal{O}_E)^\times = \begin{cases} 2q^n, & \text{if } E/F \text{ is ramified;}\\ 1, & \text{if } E/F \text{ is unramified and } n=0;\\ q^n(1+q^{-1}), & \text{if } E/F \text{ is unramified and } n>0.\end{cases}$$

We recall that for $v \in \pi$, we have

$$\left(\pi\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} v\right)(x) = \psi(bx/d)v(ax/d).$$

Suppose we now take $\alpha = a + b\tau \in E^\times$. Then we have

$$h^{-1}\alpha h = \begin{pmatrix} a+b\operatorname{Tr}\tau & b\varpi^{-n}\\ -b\varpi^n\operatorname{N}\tau & a\end{pmatrix}.$$

Hence, when $|a| \le |b\varpi^n \operatorname{N}\tau|$,

$$h^{-1}\alpha h = \begin{pmatrix} \operatorname{N}(\alpha)/(b\varpi^n\operatorname{N}(\tau)) & -(a+b\operatorname{Tr}(\tau))\\ 0 & b\varpi^n\operatorname{N}(\tau)\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & a/(b\varpi^n\operatorname{N}(\tau))\end{pmatrix},$$

and when $|b\varpi^n\operatorname{N}\tau| \le |a|$,

$$h^{-1}\alpha h = \begin{pmatrix} a^{-1}\operatorname{N}(\alpha) & b\varpi^{-n}\\ 0 & a\end{pmatrix}\begin{pmatrix} 1 & 0\\ -a^{-1}b\varpi^n\operatorname{N}(\tau) & 1\end{pmatrix}.$$

Thus, when $|a| \le |b\varpi^n\operatorname{N}\tau|$,

$$(\pi(h^{-1}\alpha h)v_0)(x) = \psi\left(-\frac{(a+b\operatorname{Tr}(\tau))x}{b\varpi^n\operatorname{N}(\tau)}\right)v_0\left(\frac{\operatorname{N}(\alpha)x}{(b\varpi^n\operatorname{N}(\tau))^2}\right),$$

and when $|b\varpi^n\operatorname{N}\tau| \le |a|$,

$$(\pi(h^{-1}\alpha h)v_0)(x) = \psi\left(\frac{bx}{a\varpi^n}\right)v_0\left(\frac{\operatorname{N}(\alpha)x}{a^2}\right).$$

We define

$$e(E/F) = \begin{cases} 1, & \text{if } E/F \text{ is unramified;} \\ 2, & \text{if } E/F \text{ is ramified.} \end{cases}$$

Lemma 2.4. With f as above, we have

$$\tilde{J}_\pi(f) = \frac{1}{e(E/F)} \frac{L(1/2, \pi_E \otimes \Omega) L(2, 1_F)}{L(1, \pi, Ad) L(1, \eta)}$$

if Ω is unramified and

$$\tilde{J}_\pi(f) = \frac{q^{-n(\Omega)}}{e(E/F)} L(1, \eta)^2 \frac{L(1/2, \pi_E \otimes \Omega) L(2, 1_F)}{L(1, \pi, Ad) L(1, \eta)},$$

if Ω is ramified. □

We prove this lemma in the subsequent sections according to the ramification of E/F and Ω.

2.2.3 *E/F unramified and Ω unramified*

In this case, we clearly have $\tilde{J}_\pi(f) = 1$ and

$$\frac{L(1/2, \pi_E \otimes \Omega) L(2, 1_F)}{L(1, \pi, Ad) L(1, \eta)} = 1.$$

2.2.4 *E/F unramified and Ω ramified*

Suppose now that E/F is unramified and $n > 0$. Then we have $\tau \in U_E$ and

$$E^\times / F^\times (\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times = \{1 + b\tau : b \in \mathcal{O}_F / \varpi^n \mathcal{O}_F\} \amalg \{a + \tau : a \in \varpi \mathcal{O}_F / \varpi^n \mathcal{O}_F\}.$$

Thus for $\alpha = a + b\tau \in U_E$ with $v_F(a) \le n$, we have

$$(\pi(h^{-1} \alpha h) v_0)(x) = \psi(a^{-1} b \varpi^{-n} x) v_0(a^{-2} x).$$

Suppose now we fix $m \ge 0$. We wish to compute

$$\sum_{\alpha \in E^\times / F^\times (\mathcal{O}_F + \varpi^n \mathcal{O}_E)^\times} \Omega(\alpha)^{-1} \int_{U_F} (\pi(h^{-1} \alpha h) v_0)(u \varpi^m) \, du.$$

We see that this sum is equal to $f_1(m) + f_2(m) + f_3(m)$, where

$$f_1(m) = \sum_{b \in \mathcal{O}_F/\mathfrak{p}_F^n} \Omega(1+b\tau)^{-1} v_0(\varpi^m) \int_{U_F} \psi(ub\varpi^{m-n})\, du,$$
$$f_2(m) = \sum_{a \in \mathfrak{p}_F/\mathfrak{p}_F^n, a \neq 0} \Omega(a+\tau)^{-1} v_0(a^{-2}\varpi^m) \int_{U_F} \psi(ua^{-1}\varpi^{m-n})\, du,$$

and

$$f_3(m) = \Omega(\tau)^{-1} v_0(\varpi^{m-2n}) \int_{U_F} \psi(u\varpi^{m-2n})\, du.$$

For future use we record the following.

Lemma 2.5.

$$\int_{U_F} \psi(\varpi^k u)\, du = \begin{cases} 0, & \text{if } k < -1; \\ -\frac{1}{q-1}, & \text{if } k = -1; \\ 1, & \text{if } k \geq 0. \end{cases}$$

□

Lemma 2.6.

$$\sum_{b \in \mathcal{O}_F/\mathfrak{p}_F^n} \Omega(1+b\tau)^{-1} = \begin{cases} -\Omega(\tau), & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

For $0 < k \leq n$,

$$\sum_{b \in \mathfrak{p}_F^k/\mathfrak{p}_F^n} \Omega(1+b\tau)^{-1} = \begin{cases} 0, & \text{if } k < n; \\ 1, & \text{if } k = n. \end{cases}$$

□

Proof. It suffices to observe that

$$\sum_{b \in \mathcal{O}_F/\mathfrak{p}_F^n} \Omega(1+b\tau)^{-1} = -\sum_{a \in \mathfrak{p}_F/\mathfrak{p}_F^n} \Omega(a+\tau),$$

that $\{a + b\tau : a \in U_F, b \in \mathfrak{p}_F^k\}$ is a subgroup of U_E for $k > 0$, and that Ω is trivial on U_F. ■

Applying this lemma gives the following formulae for $f_i(m)$.

Lemma 2.7. We have

$$f_1(0) = \begin{cases} \frac{1}{1-q^{-1}} + \Omega(\tau)\frac{1}{q-1}, & \text{if } n = 1; \\ \frac{1}{1-q^{-1}}, & \text{if } n > 1; \end{cases}$$

and

$$f_1(m) = \begin{cases} -v_0(\varpi^m)\Omega(\tau), & \text{if } n = 1; \\ 0, & \text{if } n > 1 \end{cases}$$

when $m > 0$. Also,

$$f_2(m) = \begin{cases} 0, & \text{if } m < 2n; \\ \Omega(\tau)v_0(\varpi^{m-2n}), & \text{if } m \geq 2n. \end{cases}$$

If $n = 1$, then $f_3(m) = 0$ for all m; otherwise, if $n > 1$ and $m > 0$, then

$$f_3(m) = \begin{cases} 0, & \text{if } m < 2n-2; \\ \Omega(\tau)\frac{1}{q-1}, & \text{if } m = 2n-2; \\ -\Omega(\tau)v_0(\varpi^{m-2n+2}), & \text{if } m > 2n-2. \end{cases}$$ □

Thus we see that $\langle v, e''_T \rangle$ is equal to the sum of

$$\frac{1}{1-q^{-1}} + \Omega(\sqrt{d})\frac{1}{q-1}\omega_\pi(\varpi^{n-1})\overline{v_0(\varpi^{2n-2})},$$

and

$$\Omega(\sqrt{d}) \sum_{m=2n}^{\infty} \left(\omega_\pi(\varpi^n)v_0(\varpi^{m-2n})\overline{v_0(\varpi^m)} - \omega_\pi(\varpi^{n-1})v_0(\varpi^{m-2n+1})\overline{v_0(\varpi^{m-1})}\right).$$

Let $\{\beta_1, \beta_2\}$ denote the Satake parameters of π. We have $\beta_2 = \omega_\pi(\varpi)\beta_1^{-1}$ and

$$v_0(\varpi^m) = q^{-\frac{m}{2}} \frac{\beta_1^{m+1} - \beta_2^{m+1}}{\beta_1 - \beta_2}$$

for $m \geq 0$. Moreover, since π is unitary, we have

$$\overline{v_0(\varpi^m)} = \omega_\pi(\varpi)^{-m} v_0(\varpi^m).$$

Thus

$$\omega_\pi(\varpi^n) v_0(\varpi^{m-2n}) \overline{v_0(\varpi^m)} - \omega_\pi(\varpi^{n-1}) v_0(\varpi^{m-2n+1}) \overline{v_0(\varpi^{m-1})}$$

is equal to

$$-q^{-(m-n)} \frac{\omega_\pi(\varpi^n)}{(\beta_1 - \beta_2)^2} \left(\beta_1 \beta_2^{-2n+1} + \beta_1^{-2n+1} \beta_2 - \beta_2^{-2n+2} - \beta_1^{-2n+2}\right),$$

which equals

$$-q^{-m+2n-1} \omega_\pi(\varpi^{-n+1}) v_0(\varpi^{2n-2}).$$

So we see that

$$\sum_{m=2n}^{\infty} \left(\omega_\pi(\varpi^n) v_0(\varpi^{m-2n}) \overline{v_0(\varpi^m)} - \omega_\pi(\varpi^{n-1}) v_0(\varpi^{m-2n+1}) \overline{v_0(\varpi^{m-1})}\right)$$
$$= -\frac{1}{q-1} \omega_\pi(\varpi^{-n+1}) v_0(\varpi^{2n-2}).$$

Hence

$$\langle v, e_T'' \rangle = \sum_\alpha \Omega(\alpha)^{-1} \langle \pi(h^{-1} \alpha h) v_0, v_0 \rangle = \frac{1}{1 - q^{-1}},$$

and we may conclude that

$$\tilde{J}_\pi(f) = \frac{1}{1 - q^{-1}} \frac{1}{q^n (1 + q^{-1})} \frac{L(2, 1_F)}{L(1, \pi, Ad) L(1, 1_F)}$$
$$= q^{-n} L(1, \eta)^2 \frac{L(1/2, \pi_E \otimes \Omega) L(2, 1_F)}{L(1, \pi, Ad) L(1, \eta)}.$$

2.2.5 *E/F ramified*

We now assume that E/F is ramified. In this case, a set of representatives for $E^\times/F^\times(\mathcal{O}_F + \varpi^n\mathcal{O}_E)^\times$ is

$$\left\{1 + b\tau : b \in \mathcal{O}_F/\mathfrak{p}_F^n\right\} \amalg \left\{a\varpi + \tau : a \in \mathcal{O}_F/\mathfrak{p}_F^n\right\}.$$

For $\alpha = 1 + b\tau$ in the first set with $v_F(b) \leq n$, we have

$$(\pi(h^{-1}\alpha h)v_0)(x) = \psi(b\varpi^{-n}x)v_0(x),$$

and for $\alpha = a\varpi + \tau$ in the second set with $v_F(a) \leq n$, we have

$$(\pi(h^{-1}\alpha h)v_0)(x) = \psi(a^{-1}\varpi^{-n-1}x)v_0(\varpi a^{-2}x).$$

We wish to compute, for $m \geq 0$,

$$\sum_{\alpha\in E^\times/F^\times(\mathcal{O}_F+\varpi^n\mathcal{O}_E)^\times} \Omega(\alpha)^{-1} \int_{U_F} (\pi(h^{-1}\alpha h)v_0)(u\varpi^m)\, du.$$

The contribution from the first set of representatives is

$$f_1(m) = \sum_{b\in\mathcal{O}_F/\mathfrak{p}_F^n} \Omega(1 + b\tau)^{-1} v_0(\varpi^m) \int_{U_F} \psi(ub\varpi^{m-n})\, du,$$

whereas the contribution from the second set is the sum of

$$f_2(m) = \sum_{a\in\mathcal{O}_F/\mathfrak{p}_F^n, a\neq 0} \Omega(a\varpi + \tau)^{-1} v_0(\varpi^{m-1}a^{-2}) \int_{U_F} \psi(ua^{-1}\varpi^{m-n-1})\, du$$

and

$$f_3(m) = \Omega(\varpi^{n+1} + \tau)^{-1} v_0(\varpi^{m-2n-1}) \int_{U_F} \psi(u\varpi^{m-2n-1})\, du.$$

Lemma 2.8.

$$\sum_{a\in\mathcal{O}_F/\mathfrak{p}_F^n, v_F(a)=k} \Omega(1 + a\tau) = \begin{cases} 0, & \text{if } 0 \leq k < n-1; \\ -1, & \text{if } k = n-1; \\ 1, & \text{if } k = n. \end{cases}$$

□

The above lemma gives the following.

Lemma 2.9. We have $f_1(m) = v_0(\varpi^m)$ if $n = 0$ and if $n > 0$,

$$f_1(m) = \begin{cases} \frac{1}{1-q^{-1}}, & \text{if } m = 0; \\ 0, & \text{if } m > 0. \end{cases}$$

If $n = 0$, $f_2(m) \equiv 0$; otherwise

$$f_2(m) = \begin{cases} 0, & \text{if } m < 2n-1; \\ \Omega(\tau)^{-1}\frac{1}{q-1}, & \text{if } m = 2n-1; \\ -\Omega(\tau)^{-1} v_0(\varpi^{m-2n+1}), & \text{if } m > 2n-1. \end{cases}$$

Lastly,

$$f_3(m) = \begin{cases} 0, & m < 2n+1; \\ \Omega(\tau)^{-1} v_0(\varpi^{m-2n-1}), & m \geq 2n+1. \end{cases}$$

□

First suppose $n = 0$. Then we have

$$\langle v, e''_T \rangle = \sum_{m=0}^{\infty} v_0(\varpi^m)\overline{v_0(\varpi^m)} + \Omega(\tau)^{-1} \sum_{m=1}^{\infty} v_0(\varpi^{m-1})\overline{v_0(\varpi^m)}.$$

We note that in this case we have $\Omega(\tau) = \pm 1$. We denote by $\{\alpha, \alpha^{-1}\}$ the Satake parameters of π. Then we have

$$v_0(\varpi^m) = q^{-\frac{m}{2}} \frac{\alpha^{m+1} - \alpha^{-(m+1)}}{\alpha - \alpha^{-1}},$$

and hence

$$v_0(\varpi^{m+1}) = \alpha q^{-\frac{1}{2}} v_0(\varpi^m) + q^{-\frac{m+1}{2}} \alpha^{-m-1}.$$

Therefore

$$\langle v, e''_T \rangle = \left(1 + \Omega(\tau)\alpha q^{-\frac{1}{2}}\right)\langle v_0, v_0 \rangle + \frac{\Omega(\tau)}{\alpha q^{\frac{1}{2}}(1 - \alpha^{-1}\bar{\alpha} q^{-1})(1 - \alpha^{-1}\bar{\alpha}^{-1} q^{-1})}.$$

We recall from Lemma 2.1 that

$$\langle v_0, v_0\rangle = \frac{L(1,\pi, Ad)L(1, 1_F)}{L(2, 1_F)} = \frac{(1-q^{-2})}{(1-\alpha^2 q^{-1})(1-\alpha^{-2}q^{-1})(1-q^{-1})^2},$$

which yields

$$\langle v, e''_T\rangle = \frac{L(1/2, \pi_E\otimes\Omega)}{1-q^{-1}}.$$

When $n > 0$, we have

$$\begin{aligned}\langle v, e''_T\rangle &= \frac{1}{1-q^{-1}} + \Omega(\tau)^{-1}\frac{\overline{v_0(\varpi^{2n-1})}}{q-1}\\ &\quad +\Omega(\tau)^{-1}\sum_{m=2n}^{\infty}\left(v_0(\varpi^{m-2n})\overline{v_0(\varpi^{m+1})} - v_0(\varpi^{m-2n+1})\overline{v_0(\varpi^{m})}\right).\end{aligned}$$

As in the case that E/F is unramified, one has

$$\sum_{m=2n}^{\infty}\left(v_0(\varpi^{m-2n})\overline{v_0(\varpi^{m+1})} - v_0(\varpi^{m-2n+1})\overline{v_0(\varpi^{m})}\right) = -\frac{v_0(\varpi^{2n-1})}{q-1},$$

and hence

$$\langle v, e''_T\rangle = \frac{1}{1-q^{-1}} = \frac{L(1/2, \pi_E\otimes\Omega)}{1-q^{-1}},$$

since $L(s, \pi_E\otimes\Omega)\equiv 1$.

Thus, in all cases one has

$$\begin{aligned}\tilde{J}_\pi(f) &= \frac{L(1/2, \pi_E\otimes\Omega)}{2q^n(1-q^{-1})}\frac{L(2, 1_F)}{L(1,\pi, Ad)L(1, 1_F)}\\ &= \frac{1}{2q^n}\frac{L(1/2, \pi_E\otimes\Omega)L(2, 1_F)}{L(1,\pi, Ad)}.\end{aligned}$$

This concludes the proof of Lemma 2.4.

3 Archimedean Calculations

For $F=\mathbf{R}$ or $\mathbf{C}$, let μ_{Leb} denote the Lebesgue measure. When $F=\mathbf{R}$, let $dx=\mu_{\mathrm{Leb}}$ and $d^\times x = L(1,1_{\mathbf{R}})\frac{dx}{|x|_{\mathbf{R}}} = \frac{dx}{|x|}$. When $F=\mathbf{C}$, let $dz=2\mu_{\mathrm{Leb}}$ and $d^\times z = L(1,1_{\mathbf{C}})\frac{dz}{|z|_{\mathbf{C}}} = \frac{2}{\pi}\frac{\mu_{\mathrm{Leb}}}{z\bar{z}}$. We fix additive characters of $\mathbf{R}$ and $\mathbf{C}$, given by $\psi(x)=e^{2\pi i \operatorname{Tr}_{F/\mathbf{R}} x}$.

3.1 *L*-factors

Let us first recall the definition of Archimedean L-factors. The real and complex gamma factors are defined, for $s\in\mathbf{C}$, by

$$G_1(s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),\ \ G_2(s)=2(2\pi)^{-s}\Gamma(s).$$

If μ is a character of $\mathbf{R}^\times$, then one can write this character as

$$\mu(x)=|x|_{\mathbf{R}}^r \operatorname{sgn}^m(x)$$

with $r\in\mathbf{C}$ and $m\in\{0,1\}$. In this case, one defines

$$L(s,\mu)=G_1(s+r+m).$$

On the other hand, if μ is a character of $\mathbf{C}^\times$, then we may write μ in the form

$$\mu(z)=|z|_{\mathbf{C}}^r\left(\frac{z}{\bar{z}}\right)^m$$

with $m\in\frac{1}{2}\mathbf{Z}$. Here the local L-factor is defined as

$$L(s,\mu)=G_2(s+r+|m|).$$

3.1.1 *Principal series representations*

Suppose now that π is an admissible representation of $\mathrm{GL}(2,F)$ with $F=\mathbf{R}$ or $\mathbf{C}$. If π is in the principal series, then we can write $\pi=\pi(\mu_1,\mu_2)$ for a pair of characters μ_1 and μ_2, and the standard and adjoint local L-factors are

$$L(s,\pi) = L(s,\mu_1)L(s,\mu_2),$$
$$L(s,\pi,Ad) = L\big(s,\mu_1\mu_2^{-1}\big)L(s,1_F)L\big(s,\mu_1^{-1}\mu_2\big).$$

3.1.2 *Discrete series representations*

Suppose now that π lies in the discrete series of $\mathrm{GL}(2,\mathbf{R})$ with weight k. Then π is of the form $\pi = \sigma(\mu_1,\mu_2)$ with $\mu_1(t) = |t|^{s_1}$ and $\mu_2(t) = |t|^{s_2}\,\mathrm{sgn}^m(t)$ with $s_1 - s_2 = k-1$, and

$$m = \begin{cases} 0, & \text{if } k \text{ is even;} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

We denote by λ the character of $\mathbf{C}^\times$ given by $\lambda(z) = z^{s_1}\bar{z}^{s_2}$, so that π corresponds to the two-dimensional representation of $W_\mathbf{R}$ induced from the character λ of $W_\mathbf{C}$. Then

$$L(s,\pi) = L(s,\lambda) = G_2(s+s_1),$$

and

$$L(s,\pi,Ad) = G_1(s+1)G_2(s+k-1).$$

3.2 **Whittaker functions**

Suppose $F = \mathbf{R}$ or $\mathbf{C}$ and let π be an irreducible generic unitary representation of $\mathrm{GL}_2(F)$. We consider the Whittaker model $\mathcal{W}(\pi,\psi)$ of π with respect to the character ψ fixed above. We let

$$W(a) := W\begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

for $W \in \mathcal{W}(\pi,\psi)$. We let K denote the standard maximal compact subgroup of $\mathrm{GL}(2,F)$, and we let T denote the diagonal torus in $\mathrm{GL}(2,F)$. We take the inner product on $\mathcal{W}(\pi,\psi)$ to be given by

$$(W_1,W_2) = \int_{F^\times} W_1(a)\overline{W_2(a)}\,d^\times a.$$

We let $\chi : F^\times \to \mathbf{C}^\times$ be a character of $F^\times$ which we view as a character of T by

$$\chi : \begin{pmatrix} a & \\ & b \end{pmatrix} \mapsto \chi(ab^{-1}).$$

Let $\mathcal{W}$ be a K-type of π. Denote by $\mathcal{W}^{T,\chi}$ the subspace of $\mathcal{W}$ on which $T \cap K$ acts by χ^{-1}. Then $\mathcal{W}^{T,\chi}$ is at most one-dimensional [18, Proposition 3]. Suppose that $\mathcal{W}$ is the minimal K-type of π such that $\mathcal{W}^T \neq 0$. Then for $\Re(s)$ large,

$$L(s, \pi \otimes \chi) = \int_0^\infty W(a)\chi(a)|a|_F^{s-1/2} d^\times a$$

for some $W \in \mathcal{W}^{T,\chi}$. To see this, observe that we may reduce to the $\chi = 1$ case by considering the representation $\pi'(g) = (\pi \otimes \chi)(g) := \pi(g)\chi(\det g)$. When $\chi = 1$, this is precisely Proposition 4 of [18]. We denote the element W by $W_{\pi,\chi}$. When χ is trivial, we write W_π for $W_{\pi,1}$.

3.3 Split case

Suppose $F = \mathbf{R}$ or $\mathbf{C}$ and $E = F \oplus F$. Let π be an irreducible generic unitary representation of $\mathrm{GL}(2, F)$ with trivial central character. Regard $T = E^\times$ as the diagonal torus A in $\mathrm{GL}_2(F)$ and let $\Omega : T \to \mathbf{C}^\times$ be a character such that $\Omega|_{F^\times}$ is trivial.

Let $\mathcal{W}(\pi, \psi)$ be a Whittaker model for π and let $\{W_i\}$ be an orthornormal basis. We consider the distribution

$$\widetilde{J}_\pi(f) = \sum_{W_i} \int_{F^\times} \pi(f) W_i(a)\Omega^{-1}(a) d^\times a \overline{\int_{F^\times} W_i(a)\Omega^{-1}(a) d^\times a}.$$

Having fixed π and Ω, we set $W = W_{\pi,\Omega}$. We can and will choose $f \in C_c^\infty(G)$ such that $\pi(f)$ is the orthogonal projection onto $\langle W \rangle$. Then we have

$$\widetilde{J}_\pi(f) = \frac{\left| \int_{F^\times} W(a)\Omega^{-1}(a) d^\times a \right|^2}{(W, W)}.$$

Note that our choice of measures and the fact that $|W(ua)| = |W(a)|$ for $|u| = 1$ give

$$(W, W) = \int_{F^\times} |W(a)|^2 d^\times a = c_F \int_0^\infty |W(a)|^2 d^\times a,$$

where c_F is 2 if $F = \mathbf{R}$ and 4 if $F = \mathbf{C}$. Thus we have

$$\widetilde{J}_\pi(f) = c_F^2 \frac{L(1/2, \pi \otimes \Omega) L(1/2, \pi \otimes \Omega^{-1})}{(W, W)}. \tag{1}$$

Presently, we will rewrite (W, W) in terms of the adjoint L-value $L(1, \pi, Ad)$ and obtain expressions for $\widetilde{J}_\pi(f)$ in terms of L-values. To compute (W, W), we will make use of the following result (cf. Lemmata 17.3.2 and 18.2.1 of [12]).

Lemma 3.1 (Barnes's lemma). Let $F = \mathbf{R}$ or $\mathbf{C}$ and set $i = 1$ if $F = \mathbf{R}$ and $i = 2$ if $F = \mathbf{C}$. Let W_1 and W_2 be Whittaker functions on F such that

$$\int_0^\infty W_1(a)|a|_F^{s-1/2} d^\times a = G_i(s+\alpha) G_i(s+\beta)$$
$$\int_0^\infty W_2(a)|a|_F^{s-1/2} d^\times a = G_i(s+\gamma) G_i(s+\delta),$$

for $\Re(s)$ sufficiently large. Then

$$\int_0^\infty W_1(a) W_2(a) |a|_F^{s-1} d^\times a$$
$$= (2\pi)^{i-1} c_F \frac{G_i(s+\alpha+\gamma) G_i(s+\alpha+\delta) G_i(s+\beta+\gamma) G_i(s+\beta+\delta)}{G_i(2s+\alpha+\beta+\gamma+\delta)},$$

for $\Re(s)$ sufficiently large. □

3.3.1 *Real case*

Suppose $F = \mathbf{R}$.

We fix a unitary character Ω of $\mathbf{R}^\times$, we write Ω in the form

$$\Omega(x) = |x|^{it} \operatorname{sgn}^n(x),$$

with $t \in \mathbf{R}$ and $n \in \{0, 1\}$.

First suppose π is a principal series representation for $\mathrm{GL}_2(\mathbf{R})$ with trivial central character. Then it must be of the form $\pi(|\cdot|^r \operatorname{sgn}^m, |\cdot|^{-r} \operatorname{sgn}^m)$ with $m \in \{0, 1\}$. In this case, we have

$$L(s, \pi \otimes \Omega^{-1}) = G_1(s + r - it + \varepsilon_{m,n}) G_1(s - r - it + \varepsilon_{m,n}),$$

where $\varepsilon_{m,n} = 1 - \delta_{m,n}$.

With W as above, we see that

$$\int_0^\infty W(a)|a|^{s-1/2}d^\times a = G_1(s+r+\varepsilon_{m,n})G_1(s-r+\varepsilon_{m,n}).$$

By Lemma 3.1 and the fact that r is either purely real or imaginary, we have

$$(W,W) = 4\frac{G_1(1+2r+2\varepsilon_{m,n})G_1(1-2r+2\varepsilon_{m,n})G_1(1+2\varepsilon_{m,n})^2}{G_1(2+4\varepsilon_{m,n})}.$$

We may simplify this by considering the two cases $\varepsilon_{m,n}=0$ and $\varepsilon_{m,n}=1$ separately. In the first case, one evidently gets $(W,W)=4\pi G_1(1+2r)G_1(1-2r)$. If $\varepsilon_{m,n}=1$, the relation $G_1(z+2)=\frac{z}{2\pi}G_1(z)$ yields

$$(W,W) = \frac{\pi}{2}G_1(3+2r)G_1(3-2r) = \frac{1-4r^2}{8\pi}G_1(1+2r)G_1(1-2r).$$

Using $L(1,\pi,Ad)=G_1(1+2r)G_1(1-2r)$, we may write both cases together in the equation

$$(W,W) = 4\pi\left(\frac{1-4r^2}{32\pi^2}\right)^{\varepsilon_{m,n}} L(1,\pi,Ad).$$

When Ω is trivial, we obtain

$$(W_\pi,W_\pi) = 4\pi\left(\frac{1-4r^2}{32\pi^2}\right)^{m} L(1,\pi,Ad).$$

For future comparison, we will also want to write $\tilde{J}_\pi(f)$ with a factor $\frac{L(2,1_F)}{L(1,1_F)}$, which equals $\frac{1}{\pi}$ in this case. The inner-product formulas, together with equation (1) yield the following lemma.

Lemma 3.2. For π in the principal series with parameters as above and with f chosen as above, we have

$$\begin{aligned}\tilde{J}_\pi(f) &= \left(\frac{32\pi^2}{1-4r^2}\right)^{\varepsilon_{m,n}} \frac{L(2,1_F)}{L(1,1_F)}\frac{L(1/2,\pi\otimes\Omega^{-1})L(1/2,\pi\otimes\Omega)}{L(1,\pi,Ad)} \\ &= 4\left(\frac{1-4r^2}{32\pi^2}\right)^{m-\varepsilon_{m,n}} \frac{L(1/2,\pi\otimes\Omega^{-1})L(1/2,\pi\otimes\Omega)}{(W_\pi,W_\pi)}.\end{aligned}$$

□

Now suppose π is a discrete series $\sigma(|\cdot|^{s_1}, |\cdot|^{-s_1})$ with trivial central character. (In particular, $k = 2s_1 + 1$ is even.) Then $L(s, \pi) = G_1(s + s_1)G_1(s + s_1 + 1)$. In a similar manner as above, we see that

$$(W, W) = 2^{1-2s_1} G_1(1 + 2s_1)G_1(2 + 2s_1) = 2^{2-k} G_2(k) = 2^{2-k}\pi L(1, \pi, Ad).$$

Alternatively, one may compute the inner product directly in this case as one has an explicit expression $W(t) = 2|t|^{s_1+1/2}e^{-2\pi|t|}$.

Lemma 3.3. For π discrete series of weight k and with f chosen as above, we have

$$\begin{aligned}\tilde{J}_\pi(f) &= 2^k \frac{L(2, 1_F)}{L(1, 1_F)} \frac{L(1/2, \pi \otimes \Omega^{-1})L(1/2, \pi \otimes \Omega)}{L(1, \pi, Ad)} \\ &= 4\frac{L(1/2, \pi \otimes \Omega^{-1})L(1/2, \pi \otimes \Omega)}{(W_\pi, W_\pi)}.\end{aligned}$$

□

3.3.2 *Complex case*

Suppose $F = \mathbf{C}$.

We fix a unitary character Ω which we write as

$$\Omega(z) = z^{\frac{n}{2}+it}\bar{z}^{-\frac{n}{2}+it} = |z|_{\mathbf{C}}^{it}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}$$

with $n \in \mathbf{Z}$ and $t \in \mathbf{R}$. Again, since $\omega_\pi = \eta_{E/F} = 1$, we may write $\pi = \pi(\mu, \mu^{-1})$ where

$$\mu(z) = |z|_{\mathbf{C}}^{r}\left(\frac{z}{\bar{z}}\right)^{\frac{m}{2}}$$

with $r \in \mathbf{C}$ and $m \in \mathbf{Z}$. Then $L(s, \pi \otimes \Omega^{-1})$ equals

$$L(s, \mu\Omega^{-1})L(s, \mu^{-1}\Omega^{-1}) = G_2(s + r - it + |m - n|/2)G_2(s - r - it + |m + n|/2).$$

So

$$\int_0^\infty W(a)|a|_{\mathbf{C}}^{s-1/2}d^\times a = G_2(s + r + |m - n|/2)G_2(s - r + |m + n|/2).$$

Hence by Lemma 3.1, we may write (W, W) as

$$32\pi \frac{G_2(1+2\Re r+|m-n|)G_2(1-2\Re r+|m+n|)G_2(1+2i\Im r+l)G_2(1-2i\Im r+\ell)}{G_2(2+2\ell)},$$

where $\ell = (|m-n|+|m+n|)/2 = \max\{|m|,|n|\}$.

Both in the case of complementary series ($r \in \mathbf{R}$ and $m = 0$, so $\ell = |n|$) and non-complementary series ($r \in i\mathbf{R}$), we obtain

$$(W, W) = \frac{64\pi}{(1+2\ell)} \binom{2\ell}{|m-n|}^{-1} G_2(1+2r+\ell)G_2(1-2r+\ell).$$

Since

$$L(1, \pi, Ad) = \frac{1}{\pi} G_2(1+2r+|m|)G_2(1-2r+|m|) = \frac{1+2|m|}{64\pi^2} \binom{2|m|}{|m|} (W_\pi, W_\pi)$$

and $\frac{L(2,1_F)}{L(1,1_F)} = \frac{G_2(1)}{G_2(2)} = \frac{1}{2\pi}$, we have the following result.

Lemma 3.4.

$$\widetilde{J}_\pi(f) = \frac{(1+2\ell)}{2} \binom{2\ell}{|m-n|} \times \prod_{j=|m|+1}^{\ell} \frac{4\pi^2}{j^2-4r^2} \times \frac{L(2,1_F)}{L(1,1_F)} \frac{L(1/2,\pi\otimes\Omega^{-1})L(1/2,\pi\otimes\Omega)}{L(1,\pi,Ad)},$$

or alternatively,

$$\widetilde{J}_\pi(f) = 16 \frac{1+2\ell}{1+2|m|} \binom{2\ell}{|m-n|} \binom{2|m|}{|m|}^{-1} \times \prod_{j=|m|+1}^{\ell} \frac{4\pi^2}{j^2-4r^2} \times \frac{L(1/2,\pi\otimes\Omega^{-1})L(1/2,\pi\otimes\Omega)}{(W_\pi, W_\pi)}.$$

□

3.4 Nonsplit case

In this case, we have $F = \mathbf{R}$ and Ω a unitary character of $\mathbf{C}^\times$. We write $\Omega(z) = (z\bar{z}^{-1})^n$ with $n \in \frac{1}{2}\mathbf{Z}$. In this case, we study the distribution

$$\tilde{J}_\pi(f) = \int_{G(F)} f(g) \langle \pi(g)e'_T, e'_T \rangle \, dg,$$

where e'_T is a unit vector in π such that $\pi(\alpha)e'_T = \Omega(\alpha)e'_T$ for all $\alpha \in \mathbf{C}^\times$. We wish to pick out the vector of weight $2n$ in π or π'. In this case, we just take f to be some smooth compactly supported function such that $f(k_1 g k_2) = \Omega^{-1}(k_1)\Omega^{-1}(k_2) f(g)$ and $\pi(f)e'_T = e'_T$. Clearly, then $\tilde{J}_\pi(f) = 1$.

We note that if $\omega_\pi = 1$ then we have $n \in \mathbf{Z}$, and if $\omega_\pi = \mathrm{sgn}$ then we have $n \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$. For later use we record the L-values

$$L(2, 1_F) = L(1, \mathrm{sgn}) = G_1(2) = \frac{1}{\pi}.$$

We recall the definition of the beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

First we assume that $\pi = \pi(\mu, \mu^{-1})$ is a principal series representation or that $\pi = \pi(\mu, \mu^{-1}\,\mathrm{sgn})$. We write $\mu(t) = |t|^r \mathrm{sgn}^m(t)$. We define $\lambda = 1/4 - r^2$. Then $\pi_E = \pi(\mu_E, \mu_E^{-1})$ with $\mu_E(z) = z^r \bar{z}^r$. Hence we have

$$\begin{aligned}\frac{L(1/2, \pi_E \otimes \Omega^{-1})}{L(1, \pi, Ad)} &= \frac{G_2(1/2 + r + |n|)G_2(1/2 - r + |n|)}{G_1(1+2r)G_1(1-2r)} \\ &= 2(2\pi)^{-2|n|}\frac{\Gamma(1/2 + r + |n|)\Gamma(1/2 - r + |n|)}{\Gamma(1/2+r)\Gamma(1/2-r)}.\end{aligned}$$

We note that when ω_π is trivial so that $n \in \mathbf{Z}$, then we have

$$\begin{aligned}\frac{\Gamma(1/2 + r + |n|)\Gamma(1/2 - r + |n|)}{\Gamma(1/2+r)\Gamma(1/2-r)} &= \prod_{j=0}^{|n|-1}(1/2 + r + j)(1/2 - r + j) \\ &= \prod_{j=0}^{|n|-1}(\lambda + j(j+1)).\end{aligned}$$

When ω_π is trivial, we have

$$\begin{aligned}(W_\pi, W_\pi) &= 4\frac{G_1(1+2r+2m)G_1(1-2r+2m)G_1(1+2m)^2}{G_1(2+4m)} \\ &= 4\frac{\Gamma(1/2+r+m)\Gamma(1/2-r+m)\Gamma(1/2+m)^2}{\pi^{1+2m}\Gamma(1+2m)}.\end{aligned}$$

Hence when $m = 0$,

$$(W_\pi, W_\pi) = 4\Gamma(1/2 + r)\Gamma(1/2 - r),$$

and when $m = 1$,

$$(W_\pi, W_\pi) = \frac{\lambda}{2\pi^2}\Gamma(1/2 + r)\Gamma(1/2 - r).$$

Hence we get the following result.

Lemma 3.5. For π in the principal series with trivial central character and with f as above,

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi_E \otimes \Omega)L(1, \mathrm{sgn})}{L(1, \pi, Ad)L(2, 1_F)} \times \frac{2^{-1}(2\pi)^{2|n|}}{\prod_{j=0}^{|n|-1}(\lambda + j(j+1))}$$

and

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi_E \otimes \Omega)}{(W_\pi, W_\pi)} \times \frac{2^{1-m}\lambda^m(2\pi)^{2|n|-2m}}{\prod_{j=0}^{|n|-1}(\lambda + j(j+1))}.$$

□

In the case that $\omega_\pi = \mathrm{sgn}$, we have

$$\begin{aligned}(W_\pi, W_\pi) &= 4\frac{G_1(2 + 2r)G_1(2 - 2r)G_1(1)G_1(3)}{G_1(4)} \\ &= \frac{2}{\pi}\Gamma(1 + r)\Gamma(1 - r).\end{aligned}$$

Hence

$$\begin{aligned}\frac{L(1/2, \pi_E \otimes \Omega)}{(W_\pi, W_\pi)} &= (2\pi)^{-2|n|}\frac{\Gamma(1/2 + r + |n|)\Gamma(1/2 - r + |n|)}{\Gamma(1 + r)\Gamma(1 - r)} \\ &= (2\pi)^{-2|n|}\prod_{j=0}^{|n|-\frac{3}{2}}(1 + r + j)(1 - r + j) \\ &= (2\pi)^{-2|n|}\prod_{j=0}^{|n|-\frac{3}{2}}((1 + j)^2 - r^2).\end{aligned}$$

Lemma 3.6. For π as above with $\omega_\pi = \mathrm{sgn}$ and f chosen as above, we have

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi_E \otimes \Omega) L(1, \mathrm{sgn})}{L(1, \pi, Ad) L(2, 1_F)} \times \frac{\Gamma(1/2 + r)\Gamma(1/2 - r)}{\Gamma(1 + r)\Gamma(1 - r)} \frac{(2\pi)^{2|n|}}{2 \prod_{j=0}^{|n|-\frac{3}{2}} ((1+j)^2 - r^2)}$$
$$= \frac{L(1/2, \pi_E \otimes \Omega)}{(W_\pi, W_\pi)} \times \frac{(2\pi)^{2|n|}}{\prod_{j=0}^{|n|-\frac{3}{2}} ((1+j)^2 - r^2)}. \qquad \square$$

Next we take π to be discrete series of weight k. In this case, π corresponds to $\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} (z\bar{z}^{-1})^{\frac{k-1}{2}}$. Hence we have $\pi_E = \pi((z\bar{z}^{-1})^{\frac{k-1}{2}}, (z\bar{z}^{-1})^{-\frac{k-1}{2}})$. Thus we have

$$L(1/2, \pi_E \otimes \Omega^{-1}) = \begin{cases} G_2(k/2 + |n|) G_2(-k/2 + 1 + |n|), & \text{if } |n| \geq \frac{k-1}{2}; \\ G_2(k/2 + |n|) G_2(k/2 - |n|), & \text{if } |n| \leq \frac{k-1}{2}. \end{cases}$$

On the other hand,

$$L(1, \pi, Ad) = G_1(2) G_2(k) = \frac{1}{\pi} G_2(k)$$

and

$$(W_\pi, W_\pi) = 2^{2-k} G_2(k).$$

Hence we get

$$\frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \pi, Ad)} = 2\pi B(k/2 + |n|, k/2 - |n|),$$

if $|n| \leq \frac{k-1}{2}$, and

$$\frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \pi, Ad)} = (2\pi)^{-(2|n|-k)} \frac{2|n|!}{k!} B(k/2 + |n|, 1 - k/2 + |n|),$$

if $|n| \geq \frac{k-1}{2}$.

Lemma 3.7. For π in the discrete series of weight k and with f as above,

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi_E \otimes \Omega) L(1, \mathrm{sgn})}{L(1, \pi, Ad) L(2, 1_F)} \times \frac{1}{2\pi B(k/2 + |n|, k/2 - |n|)}$$
$$= \frac{L(1/2, \pi_E \otimes \Omega)}{(W_\pi, W_\pi)} \times \frac{2}{2^k B(k/2 + |n|, k/2 - |n|)},$$

if $|n| \le \frac{k-1}{2}$, and

$$\tilde{J}_\pi(f) = \frac{L(1/2, \pi_E \otimes \Omega)L(1, \mathrm{sgn})}{L(1, \pi, Ad)L(2, 1_F)} \times \frac{(2\pi)^{2|n|-k}k!}{2n!B(k/2+|n|, 1-k/2+|n|)}$$

$$= \frac{L(1/2, \pi_E \otimes \Omega)}{(W_\pi, W_\pi)} \times \frac{2^{2|n|-2k+1}\pi^{2|n|-k+1}k!}{n!B(k/2+|n|, 1-k/2+|n|)},$$

if $|n| \ge \frac{k-1}{2}$. □

4 Global Result

Suppose now that F is a number field and E/F is a quadratic extension. We denote by Δ_F (resp. Δ_E) the discriminant of F (resp. E) and by $d_{E/F}$ the absolute norm of the relative discriminant of E/F. We take π to be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbf{A}_F)$ such that ω_π is either trivial or η, the quadratic character of $F^\times \backslash \mathbf{A}_F^\times$ associated to E/F by class field theory. Let

$$\Omega : E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$$

be a unitary character such that $\Omega|_{\mathbf{A}_F^\times} = \omega_\pi$. We assume that π and Ω have disjoint ramifications.

Let π_E denote the base change of π to an automorphic representation of $\mathrm{GL}(2, \mathbf{A}_E)$. The L-function of $\pi_E \otimes \Omega$ satisfies a functional equation

$$L(s, \pi_E \otimes \Omega) = \varepsilon(s, \pi_E \otimes \Omega)L(1-s, \pi_E \otimes \Omega).$$

We assume that $\varepsilon(1/2, \pi_E \otimes \Omega) = +1$. In this case, there is a unique quaternion algebra D/F such that

- $E \hookrightarrow D$;
- π transfers to π^D on $D^\times(\mathbf{A}_F)$; and
- $\mathrm{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega) \neq 0$.

Regard $G = D^\times$ as an algebraic group over F. Let $f = \prod_v f_v \in C_c^\infty(G(\mathbf{A}_F))$. Define

$$J_{\pi^D}(f) = \sum_\varphi \int_{E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times} (\pi^D(f)\varphi)(t)\Omega(t)^{-1}\, dt \overline{\int_{E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times} \varphi(t)\Omega(t)^{-1}\, dt},$$

where the sum is taken over an orthonormal basis of the space of π^D.

We fix an additive character $\psi : F\backslash \mathbf{A}_F \to \mathbf{C}^\times$. Let S denote a finite set of places (including the infinite places) of F outside of which everything is unramified. Let $f \in C_c^\infty(G(\mathbf{A}_F))$ be a function of the form $f = \left(\prod_{v\in S} f_v\right) f^S$, where f^S is the characteristic function of K^S, a fixed maximal compact subgroup of $G(\mathbf{A}_F^S)$. Following [15, Theorem 2], and the appendix to this paper when π is dihedral with respect to E, an explicit factorization of the distribution $J_{\pi^D}(f)$ is given by

$$J_{\pi^D}(f) = \frac{1}{2}\prod_{v\in S}\tilde{J}_{\pi_v}(f_v) \times \prod_{\substack{v\in S \\ \text{inert in } E}} \varepsilon(1,\eta_v,\psi_v)2L(0,\eta_v) \times \frac{L_S(1,\eta_v)L^S(1/2,\pi_E\otimes\Omega)}{L^S(1,\pi,Ad)},$$

where the distributions $\tilde{J}_{\pi_v}(f_v)$ are defined as in the previous sections. Here the measures on $G(\mathbf{A}_F)$ and $\mathbf{A}_F^\times E^\times\backslash \mathbf{A}_E^\times$ are fixed as in [15, Section 3.1]. On $\mathbf{A}_E^\times$ and $\mathbf{A}_F^\times$ we take the product of the local Tamagawa measures, on $E^\times$ we take the counting measure, and on $G(\mathbf{A}_F)$ we take the product of the local Tamagawa measures multiplied by $L^S(2, 1_F)$.

Take $\psi = \psi_0 \circ \mathrm{tr}_{F/\mathbf{Q}}$ where ψ_0 denotes the standard character on $\mathbf{Q}\backslash\mathbf{A}_\mathbf{Q}$, so that

$$\varepsilon(1,\eta_v,\psi_v) = \begin{cases} 1, & \text{if } v \text{ is Archimedean;} \\ q_v^{-\frac{n(\eta_v)+n(\psi_v)}{2}}, & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Here $n(\eta_v)$ (resp. $n(\psi_v)$) denotes the conductor of η_v (resp. ψ_v). Similarly for a finite place v of F, we define $n(\pi_v)$ to be the conductor of π_v and we define $n(\Omega_v)$ to be the smallest integer such that Ω_v is trivial on $(\mathcal{O}_{F_v} + \varpi_v^{n(\Omega_v)}\mathcal{O}_{E_v})^\times$, where ϖ_v denotes a uniformizer in F_v. We note that

$$\prod_{v<\infty} q_v^{n(\Omega_v)} = \sqrt{c(\Omega)},$$

where $c(\Omega)$ denotes the absolute norm of the conductor of Ω.

4.1 Test function

We now define a test function

$$f = \prod_v f_v \in C_c^\infty(G(\mathbf{A}_F)).$$

At a finite place v of F, we take $R(\pi_v)$ to be an order of reduced discriminant $\mathfrak{p}_{F_v}^{n(\pi_v)}$ such that $R(\pi_v) \cap E_v = \mathcal{O}_{F_v} + \varpi_v^{n(\Omega_v)}\mathcal{O}_{E_v}$ (see [8, Section 3]). We then take f_v to be the characteristic function of $R(\pi_v)^\times$ divided by its volume.

At an infinite place v of F, let K_v be a maximal compact subgroup of $D_v^\times$ such that $K_v \cap E_v^\times$ is a maximal compact subgroup of $E_v^\times \hookrightarrow D_v^\times$. Now let φ_v be a vector of minimal weight such that $\pi_v(t)\varphi_v = \Omega_v(t)\varphi_v$ for t in $K_v \cap E_v^\times$. Then φ_v is determined up to a scalar factor. We choose f_v such that $\pi_v(f_v)$ is an orthogonal projection onto the space $\langle \varphi_v \rangle$.

Thus, for such f we have

$$J_{\pi^D}(f) = \frac{\left|\int \varphi(t)\Omega(t)^{-1}\, dt\right|^2}{[\varphi, \varphi]}, \tag{2}$$

where $\varphi \in \pi^D$ is a nonzero vector which is invariant under $R(\pi_v)^\times$ at each finite place v. Furthermore, at places v where E_v/F_v is ramified and $n(\pi_v) \geq 2$, we make the requirement that $E_v^\times$ acts on φ by Ω_v. At the infinite places of F, we have that $K_v \cap E_v^\times$ acts on φ by Ω_v and φ lies in the minimal such K_v-type.

4.2 **Local constants**

We consider the Whittaker model for π with respect to the character $\psi = \psi_0 \circ \mathrm{tr}_{F/\mathbb{Q}}$. Explicitly for $\varphi_\pi \in \pi$, one defines

$$W_{\varphi_\pi}(g) = \int_{F\backslash \mathbb{A}_F} \varphi_\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)\, dx.$$

The Whittaker functions factor and we take for each place v of F the Whittaker function $W_{\pi_v} \in \mathcal{W}(\pi_v, \psi_v)$ defined above. Take $\varphi_\pi \in \pi$, so that

$$W_{\varphi_\pi} = \prod_v W_{\pi_v}.$$

Then we have [15, p. 53]

$$(\varphi_\pi, \varphi_\pi) = 2L^S(1, \pi, Ad) \prod_{v \in S} \frac{\left(W_{\pi_v}, W_{\pi_v}\right)_v}{L\left(1, 1_{F_v}\right)}. \tag{3}$$

4.2.1 *Non-Archimedean constants*

First suppose that v is non-Archimedean. The calculations in Section 2 give the following. When v splits in E, we have

$$L(1, 1_{F_v})\tilde{J}_{\pi_v}(f_v) = \frac{L\left(1/2, \pi_{E_v} \otimes \Omega_v\right) L\left(1, 1_{F_v}\right)}{\left(W_{\pi_v}, W_{\pi_v}\right)} \times \begin{cases} 1, & \text{if } \Omega_v \text{ is unramified;} \\ q_v^{-n(\Omega_v)} L(1, \eta_v)^2, & \text{if } \Omega_v \text{ is ramified.} \end{cases}$$

When v is inert and unramified in E, we have

$$L(1,\eta_v)q_v^{-\frac{n(\psi_v)}{2}}\tilde{J}_{\pi_v}(f_v)=\frac{L\big(1/2,\pi_{E_v}\otimes\Omega_v\big)L\big(1,1_{F_v}\big)}{(W_{\pi_v},W_{\pi_v})}\times\begin{cases}1, & \text{if } \pi_v \text{ and } \Omega_v \text{ are unramified;}\\ q_v^{-n(\Omega_v)}L(1,\eta_v)^2, & \text{if } \Omega_v \text{ is ramified;}\\ \frac{L\left(1,\eta_v\right)}{L\left(1,1_{F_v}\right)}, & \text{if } \pi_v \text{ is ramified.}\end{cases}$$

When v is ramified in E and π_v is unramified, we have

$$2q_v^{-\frac{n(\psi_v)}{2}}\tilde{J}_{\pi_v}(f_v)=\frac{L\big(1/2,\pi_{E_v}\otimes\Omega_v\big)L\big(1,1_{F_v}\big)}{\big(W_{\pi_v},W_{\pi_v}\big)}\times\begin{cases}1, & \text{if } \pi_v \text{ and } \Omega_v \text{ are unramified;}\\ q_v^{-n(\Omega_v)}L(1,\eta_v)^2, & \text{if } \Omega_v \text{ is ramified.}\end{cases}$$

When v is ramified in E and π_v is ramified, we have

$$2q_v^{-\frac{n(\psi_v)}{2}}\tilde{J}_{\pi_v}(f_v)=\frac{L\big(1,1_{F_v}\big)}{\big(W_{\pi_v},W_{\pi_v}\big)}\times\begin{cases}2\big(1+q_v^{-1}\big)^{-1}, & \text{if } n(\pi_v)=1;\\ 2\big(1-q_v^{-1}\big), & \text{if } n(\pi_v)\geq 2.\end{cases}$$

We also define certain subsets of the finite places of F,

$$S(\Omega)=\{\text{places of } F \text{ above which } \Omega \text{ ramifies}\},$$
$$S_1(\pi,E)=\{\text{places of } F \text{ where } \pi \text{ ramifies but } E \text{ does not}\},$$
$$S_2(\pi,E)=\{\text{places of } F \text{ where both } \pi \text{ and } E \text{ ramify}\}.$$

For $v\in S_2(\pi,E)$, we define

$$C'(\pi_v)=\begin{cases}2\big(1+q_v^{-1}\big)^{-1}, & \text{if } n(\pi_v)=1;\\ 2\big(1-q_v^{-1}\big), & \text{if } n(\pi_v)\geq 2.\end{cases}$$

We also set

$$\mathrm{Ram}(\pi)=\{\text{finite places } v \text{ of } F \text{ such that } \pi_v \text{ is ramified}\},$$
$$S'(\pi)=\{v\in\mathrm{Ram}(\pi) \text{ such that } n(\pi_v)\geq 2 \text{ or } n(\pi_v)=1 \text{ and } v \text{ ramifies in } E\},$$

and $e_v(E/F)$ to be the ramification degree of E/F at v.

4.2.2 *Archimedean constants*

We write the infinite places of F as

$$\Sigma_\infty^F = \Sigma_{\mathbf{R},sp}^F \amalg \Sigma_{\mathbf{R},in}^F \amalg \Sigma_{\mathbf{C}}^F,$$

with the sets on the right-hand side being the places which are, respectively, real and split in E, real and inert in E, and complex. For each place $v \in \Sigma_\infty^F$, write

$$C_v(E,\pi,\Omega) = e_v(E/F)\widetilde{J}_{\pi_v}(f_v)\frac{L(1,\pi_v,Ad)L(1,\eta_v)}{L(1/2,\pi_{E,v}\otimes\Omega_v)L(2,1_{F_v})}$$

and

$$C'_v(E,\pi,\Omega) = e_v(E/F)\widetilde{J}_{\pi_v}(f_v)\frac{\left(W_{\pi_v},W_{\pi_v}\right)}{L(1/2,\pi_{E,v}\otimes\Omega_v)L\left(1,1_{F_v}\right)},$$

where $e_v(E/F) = 2$ if v is inert in E, and $e_v(E/F) = 1$ otherwise. The expressions below for $C_v(E,\pi,\Omega)$ and $C'_v(E,\pi,\Omega)$ all come immediately from the lemmata in Section 3.

Suppose first $v \in \Sigma_{\mathbf{R},sp}^F$, so $E_v = \mathbf{R}\oplus\mathbf{R}$. Write Ω_v in the form

$$\Omega_v(x_1,x_2) = \left(|x_1|^{it}\operatorname{sgn}^{n_v}(x_1), |x_2|^{-it}\operatorname{sgn}^{n_v}(x_2)\right)$$

with $t \in \mathbf{R}$ and $n_v \in \{0,1\}$. Then

$$C_v(E,\pi,\Omega) = \left(\frac{8\pi^2}{\lambda_v}\right)^{\epsilon_v},\ C'_v(E,\pi,\Omega) = 4\left(\frac{\lambda_v}{8\pi^2}\right)^{m-\varepsilon_v}$$

if $\pi_v = \pi(\mu_v,\mu_v^{-1})$ is a principal series, with Laplacian eigenvalue λ_v and $\epsilon_v \in \{0,1\}$ according to $\mu_v\Omega_v = |\cdot|^r\operatorname{sgn}^{\epsilon_v}$. If π_v is a discrete series of weight k_v, then

$$C_v(E,\pi,\Omega) = 2^{k_v},\ C'_v(E,\pi,\Omega) = 4.$$

Now suppose $v \in \Sigma_{\mathbf{C}}^F$, so $E_v = \mathbf{C}\oplus\mathbf{C}$. We may write

$$\Omega_v(z_1,z_2) = \left((z_1\bar{z}_1)^{it}\left(\frac{z_1}{\bar{z}_1}\right)^{n_v}, (z_2\bar{z}_2)^{-it}\left(\frac{z_2}{\bar{z}_2}\right)^{-n_v}\right)$$

with $t \in \mathbf{R}$ and $n_v \in \frac{1}{2}\mathbf{Z}$, well defined up to a sign. (The constants below do not depend upon this sign). Say π_v is a principal series of weight m_v with Laplacian eigenvalue λ_v and let $\ell_v = \max(m_v, |n_v|)$. Then

$$C_v(E, \pi, \Omega) = \left(\frac{1}{2} + \ell_v\right) \binom{2\ell_v}{|m_v - n_v|} \prod_{j=m_v+1}^{\ell_v} \frac{4\pi^2}{4\lambda_v + j^2 - 1}$$

and

$$C_v'(E, \pi, \Omega) = 16\pi \frac{1 + 2\ell_v}{1 + 2m_v} \binom{2\ell_v}{|m_v - n_v|} \binom{2m_v}{m_v}^{-1} \prod_{j=m_v+1}^{\ell_v} \frac{4\pi^2}{4\lambda_v + j^2 - 1}.$$

Finally, consider $v \in \Sigma_{\mathbf{R},in}^F$. Then $E_v = \mathbf{C}$ and we write

$$\Omega_v : z \mapsto \left(\frac{z}{\bar{z}}\right)^{n_v}$$

with $n_v \in \frac{1}{2}\mathbf{Z}$, well defined up to a sign. First suppose π_v is a principal series with ω_{π_v} trivial. We write $\pi_v = \pi(\mu_v, \mu_v^{-1})$ with $\mu_v = |\cdot|_v^r \operatorname{sgn}^{m_v}$ such that $m_v \in \{0, 1\}$, and we set $\lambda_v = \frac{1}{4} - r_v^2$. Then

$$C_v(E, \pi, \Omega) = (2\pi)^{2|n_v|} \prod_{j=0}^{|n_v|-1} (\lambda_v + j(j+1))^{-1},$$

$$C_v'(E, \pi, \Omega) = 2^{2-m_v} \lambda^{m_v} (2\pi)^{2|n_v|-2m_v} \prod_{j=0}^{|n_v|-1} (\lambda_v + j(j+1))^{-1}.$$

If $\omega_\pi = \operatorname{sgn}$, then

$$C_v(E, \pi, \Omega) = 2 \frac{\Gamma(1/2 + r_v)\Gamma(1/2 - r_v)}{\Gamma(1 + r_v)\Gamma(1 - r_v)} \frac{(2\pi)^{2|n_v|}}{\prod_{j=1}^{|n_v|-\frac{1}{2}} (j^2 - r_v^2)},$$

$$C_v'(E, \pi, \Omega) = 2(2\pi)^{2|n_v|} \prod_{j=1}^{|n_v|-\frac{1}{2}} \left(j^2 - \lambda_v - \frac{1}{4}\right)^{-1},$$

where $\lambda_v = \frac{1}{4} - r_v^2$. If π_v is a discrete series of weight k_v and $B(x, y)$ denotes the beta function, then

$$C_v(E, \pi, \Omega) = (\pi B(k_v/2 + |n_v|, k_v/2 - |n_v|))^{-1},$$
$$C'_v(E, \pi, \Omega) = (2^{k_v - 2} B(k_v/2 + |n_v|, k_v/2 - |n_v|))^{-1}$$

when $|n_v| < \frac{k-1}{2}$, and

$$C_v(E, \pi, \Omega) = \frac{(2\pi)^{2|n_v| - k_v} k_v!}{n_v! B(k_v/2 + |n_v|, 1 - k_v/2 + |n_v|)},$$

$$C'_v(E, \pi, \Omega) = \frac{(2\pi)^{2|n_v| - k_v + 1} k_v!}{2^{k_v - 1} n_v! B(k_v/2 + |n_v|, 1 - k_v/2 + |n_v|)}$$

when $|n_v| \geq \frac{k-1}{2}$.

4.3 Final formulas

Let $S'(\pi)$ be the complement of the set of finite places of F where either π is unramified, or else $n(\pi_v) = 1$ and v is unramified in E. Then the above calculations give the following results.

Theorem 4.1. The quantity

$$\frac{\left|\int \varphi(t) \Omega^{-1}(t)\, dt\right|^2}{(\varphi, \varphi)}$$

is equal to

$$\frac{L^{S'(\pi)}(1/2, \pi_E \otimes \Omega)}{L^{S'(\pi)}(1, \pi, Ad)} \times \frac{\sqrt{\Delta_F}}{2\sqrt{c(\Omega)\Delta_E}} \times L_{S(\Omega)}(1, \eta)^2 \times \prod_{v \in \mathrm{Ram}(\pi)} e_v(E/F) L(1, \eta_v) \times \prod_{v \in \Sigma_\infty^F} C_v(E, \pi, \Omega).$$

Here the measure on the group $G(\mathbf{A}_F)$ is taken to be the product of the local Tamagawa measures multiplied by $L^{\mathrm{Ram}(\pi)}(2, 1_F)$. □

Theorem 4.2. The quantity

$$\frac{\left|\int \varphi(t) \Omega^{-1}(t)\, dt\right|^2}{(\varphi, \varphi)}$$

is equal to

$$\frac{L^{S_2(\pi,E)}(1/2,\pi_E\otimes\Omega)}{(\varphi_\pi,\varphi_\pi)}\times\frac{1}{\sqrt{d_{E/F}c(\Omega)}}\times L_{S(\Omega)}(1,\eta)^2\times\frac{L_{S_1(\pi,E)}(1,\eta)}{L_{S_1(\pi,E)}(1,1_F)}$$
$$\times\prod_{v\in S_2(\pi,E)}C'(\pi_v)\times\prod_{v\in\Sigma_\infty^F}C'_v(E,\pi,\Omega).$$

Here the measures on the groups $G(\mathbf{A}_F)$ and $\mathrm{GL}(2,\mathbf{A}_F)$ are taken to be the product of the local Tamagawa measures. □

5 Equidistribution

One application of central-value formulas is to prove statements about equidistribution using subconvexity bounds for L-functions. The relevant subconvexity bounds, in the case of a general number field, have been established by Venkatesh [23] for GL(2) L-functions, and announced by Michel and Venkatesh [17] for twisted GL(2) L-functions. (We refer to the latter paper for an introduction to equidistribution and subconvexity.) While it is known that equidistribution results follow in principle from Waldspurger's formula (see [3] for one instance), the necessary details have not been written down in most cases.

In any event, an explicit formula such as Theorem 4.1 allows a more immediate derivation of equidistribution from subconvexity. This has been already carried out in several situations. For example, see [11] for "sparse" equidistribution of Heegner points on Shimura curves and [19] for equidistribution of individual geodesics on a modular curve. These results use, respectively, the explicit central-value formulas in [27] and [19] when $F=\mathbf{Q}$ and E/F is imaginary quadratic and real quadratic.

The generality of Theorem 4.1 allows one to consider equidistribution of toric orbits in a variety of situations. However, to keep details to a minimum, we will only deduce equidistribution results in a specific example of a hyperbolic 3-fold. Specifically, let $F=\mathbf{Q}(i)$ and K be the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbf{A}_F)$. The hyperbolic 3-fold we will consider is

$$X=\mathrm{PSL}_2(\mathbf{Z}[i])\backslash\mathbb{H}^3=Z(\mathbf{A}_F)\,\mathrm{GL}_2(F)\backslash\,\mathrm{GL}_2(\mathbf{A}_F)/K.$$

Now fix a square-free $d\in\mathcal{O}_F=\mathbf{Z}[i]$ and let $E=E_d=F(\sqrt{d})$. Then we may take T_d to be a standard torus obtained by an optimal embedding of $E_d^\times$ in GL_2/F. The key point here is that $\mathcal{O}_{E_v}^\times \simeq T_d(\mathcal{O}_{F_v})$ embeds into K_v for each finite place of F. For $v = \infty$, let $z_d \in \mathrm{GL}_2(\mathbf{A}_F)$ such that $K_{d,\infty} = z_d U(2) z_d^{-1} \cap T_{d,\infty}(\mathbf{A}_F)$ is the maximal compact subgroup of $T_{d,\infty}(\mathbf{A}_F)$. Then

$$K_d = z_d K z_d^{-1} = \prod_{v<\infty} \mathcal{O}_{E_v}^\times \times K_{d,\infty}$$

is the maximal compact subgroup of $T_d(\mathbf{A}_F)$.

The relative discriminant ideal $\Delta_{E/F}$ is generated by $\sigma_d d$ where σ_d depends only upon the congruence class of d mod 4. In particular, $|\Delta_E|$ is a bounded multiple of $|d|$. We define the *geodesics of discriminant* d in X to be the components of

$$\begin{aligned} X_d &= Z(\mathbf{A}_F) T_d(F) \backslash T_d(\mathbf{A}_F) z_d / (K \cap T_d(\mathbf{A}_F) z_d) \\ &= (Z(\mathbf{A}_F) T_d(F) \backslash T_d(\mathbf{A}_F) / K_d) z_d \subseteq X. \end{aligned}$$

A consequence of the requirement that $T_d \hookrightarrow \mathrm{GL}_2$ be optimal is then that the number of such geodesics is the class number h_E of E. More precisely, we can write

$$X_d = \left(E^\times \backslash \mathbf{A}_E^\times / \left(\prod_{v<\infty} \mathcal{O}_{E_v}^\times \times K_{d,\infty} \right) \right) z_d = \bigcup_{\mathfrak{a} \in H_E} \gamma_{\mathfrak{a}},$$

where H_E denotes the ideal class group of E and the individual geodesic $\gamma_{\mathfrak{a}}$ is the fiber above $\mathfrak{a}$ in the quotient on the left-hand side (identifying $E^\times \backslash \mathbf{A}_{E,\mathrm{fin}}^\times / \prod_{v<\infty} \mathcal{O}_{E_v}^\times$ with H_E as usual).

Fix a Haar measure on $G = \mathrm{GL}_2(\mathbf{A}_F)$. This gives a natural choice of measures on subspaces and quotients.

Theorem 5.1. Let X_d be the collection of geodesics of discriminant d in X. As $|d| \to \infty$ along square-free Gaussian integers d, the family X_d becomes equidistributed on X. □

To prove this theorem, by Weyl's equidistribution criterion it suffices to show that the Weyl sums

$$W(\varphi, d) = \frac{1}{\mathrm{vol}(X_d)} \int_{X_d} \varphi$$

tend to 0 as $|d| \to \infty$ for φ running through a dense subspace of $C_c^\infty(Z\backslash G/K)$. Since cusp forms and wave packets of Eisenstein series span a dense subspace of $C_c^\infty(Z\backslash G/K)$, it suffices to check Weyl's criterion for φ running through a basis of eigenforms in $L^2(Z\backslash G/K)$.

We remark that this theorem follows from the work of Clozel–Ullmo [3] and Venkatesh [23], though to the best of our knowledge it was not previously stated. Clozel and Ullmo establish the necessary bounds for $W(\varphi, d)$, assuming subconvexity results when φ is a cusp form. Shortly thereafter, Venkatesh showed the necessary subconvexity results.

Proof. Suppose $\varphi \in L^2(Z\backslash G)^K$ is a cuspidal eigenform occurring in the representation π which is normalized, so that $(\varphi, \varphi) = 1$. Note that $\pi(z_d)\varphi$ is a newform for π, satisfying the conditions in Section 4.1. By construction,

$$W(\varphi, d) = \frac{1}{\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F))} \int_{Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F)} \pi(z_d)\varphi(t)\,dt.$$

Using Theorem 4.1 with $\Omega = 1$ gives

$$|W(\varphi, d)|^2 = c(\pi) \frac{L(1/2, \pi_E)}{\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F))^2 \sqrt{|d|}},$$

where $c(\pi)$ is a constant depending only on π. Since

$$L(1/2, \pi_E) = L(1/2, \pi)L(1/2, \pi \otimes \chi_d)$$

where $\chi_d = \eta_{E/F}$, we have

$$|W(\varphi, d)|^2 \ll \frac{L(1/2, \pi \otimes \chi_d)}{\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F))^2 \sqrt{|d|}}.$$

Now note that

$$\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F)) \asymp \mathrm{res}_{s=1}\zeta_E(s) \asymp L_{\mathrm{fin}}(1, \chi_d),$$

where L_{fin} denotes the finite part of the L-function, and $\asymp$ means equality up to an absolutely bounded nonzero constant. Then Siegel's lower bound gives

$$\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F)) \gg |d|^{-\epsilon}$$

for any $\epsilon > 0$. Hence the subconvexity result [23]

$$L(1/2, \pi \otimes \chi_d) \asymp L_{\text{fin}}(1/2, \pi \otimes \chi_d) \ll |d|^{1/2-1/24}$$

yields

$$|W(\varphi, d)|^2 \ll |d|^{2\epsilon-1/24} \to 0,$$

as $|d| \to \infty$.

For φ an Eisenstein form, we refer to [3]; the spirit of the argument is similar. ■

Theorem 5.2. Let γ_d be a geodesic of discriminant d on X. Suppose one has the subconvexity result

$$L(1/2, \pi \otimes \pi') \ll |\mathfrak{c}(\pi')|^{1/2-\delta},$$

where π is a fixed automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$, π' is an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with (finite) conductor $\mathfrak{c}(\pi')$, and δ is a positive constant. Let $\epsilon_0 > 0$. For a sequence of $d \to \infty$ along square-free Gaussian integers such that $h_E \ll |d|^{\delta/2-\epsilon_0}$, the family γ_d becomes equidistributed on X. □

Such a subconvexity result as is required by the theorem has been announced in [17]. (In fact, we only need the subconvexity result for representations π' that are induced from characters along quadratic extensions.) We remark that in general one needs some condition, such as the one above on the growth of the class number, to ensure equidistribution of individual geodesics (see [4]).

Proof. As before, it will suffice to show that

$$\frac{1}{\mathrm{vol}(\gamma_d)} \int_{\gamma_d} \varphi \to 0,$$

as $d \to \infty$ for φ ranging over an orthonormal basis for $L^2(Z\backslash G/K)$. Suppose $\varphi \in L^2_{\text{cusp}}(Z\backslash G)^K$ is an eigenform with $(\varphi, \varphi) = 1$. Since the ideal class group acts transitively on the components of X_d, all geodesics of discriminant d have the same volume, that is,

$$\mathrm{vol}(\gamma_d) = \frac{1}{h_E} \mathrm{vol}(X_d) = \frac{\mathrm{vol}(Z(\mathbf{A}_F)T_d(F)\backslash T_d(\mathbf{A}_F))}{\mathrm{vol}(K_d)h_E} \asymp \frac{L_{\text{fin}}(1, \chi_d)}{\mathrm{vol}(K_d)h_E}.$$

Let $\hat{H}_E$ be the group of ideal class characters of E. Via class field theory, we may view $\chi \in \hat{H}_E$ as a character on the torus $T_d(F)\backslash T_d(\mathbf{A}_F) \simeq E^\times\backslash\mathbf{A}_E^\times$ which in fact factors through X_d. More precisely, χ may be viewed as a locally constant function on X_d such that $\chi(t) = \chi(\mathfrak{a})$ for $t \in \gamma_{\mathfrak{a}}$. Let $\mathfrak{c} \in H_E$ such that $\gamma_d = \gamma_{\mathfrak{c}}$. Then

$$1_{\gamma_d}(t) = \frac{1}{h_E} \sum_{\chi \in \hat{H}_E} \chi(\mathfrak{c}^{-1}t)$$

for $t \in X_d$, where 1_{γ_d} denotes the characteristic function of γ_d. Note that

$$\left| \int_{X_d} \varphi(t)\chi(\mathfrak{c}^{-1}t)dt \right| = \left| \int_{X_d} \varphi(t)\chi(t)dt \right|.$$

Hence

$$\begin{aligned} \frac{1}{\mathrm{vol}(\gamma_d)} \left| \int_{\gamma_d} \varphi(t)dt \right| &\asymp \frac{\mathrm{vol}(K_d)}{L_{\mathrm{fin}}(1, \chi_d)} \left| \sum_{\chi \in \hat{H}_E} \int_{X_d} \varphi(t)\chi(\mathfrak{c}^{-1}t)dt \right| \\ &\ll \frac{\mathrm{vol}(K_d)}{L_{\mathrm{fin}}(1, \chi_d)} \sum_{\chi \in \hat{H}_E} \left| \int_{X_d} \varphi(t)\chi(t)dt \right|. \end{aligned}$$

Suppose φ occurs in the cuspidal representation π. As before, we consider the translate $\pi(z_d)\varphi$. Since χ is unramified, it is a newform satisfying the conditions in Section 4.1 with $\Omega = \chi^{-1}$. Then Theorem 4.1 implies something good. Using the fact that χ^{-1} is finite order, one gets

$$\left| \int_{X_d} \varphi(t)\chi(t)dt \right|^2 \asymp \frac{L(1/2, \pi_E \otimes \chi^{-1})}{\mathrm{vol}(K_d)^2\sqrt{|d|}}.$$

Note that $L(s, \pi_E \otimes \chi^{-1}) = L(s, \pi \otimes \pi_{\chi^{-1}})$, where $\pi_{\chi^{-1}}$ denotes the automorphic induction of χ^{-1} to $\mathrm{GL}_2(\mathbf{A}_F)$. Furthermore, the conductor of $\pi_{\chi^{-1}}$ is just the conductor of χ_d. Hence the subconvexity assumption gives

$$\left| \int_{X_d} \varphi(t)\chi(t)dt \right|^2 \ll |d|^{-\delta}.$$

Putting everything together with Siegel's lower bound for $L_{\mathrm{fin}}(1, \chi_d)$, we have

$$\frac{1}{\mathrm{vol}(\gamma_d)} \left| \int_{\gamma_d} \varphi(t) dt \right| \ll h_E |d|^{\epsilon - \delta/2}$$

for any $\epsilon > 0$.

One may bound the integrals for φ, an Eisenstein form similarly. ■

Appendix

The results of this paper are obtained via a factorization of the distribution $J_\sigma(f)$ into a product of local distributions. In the case that π is not dihedral with respect to the quadratic extension E/F (so that the base change of π to E remains cuspidal), such a factorization was obtained in [15, Theorem 2]. In this appendix, we obtain the same result for cuspidal representations π which are dihedral with respect to E; we refer to [15, Section 8] for further details.

Let $E = F(\sqrt{\delta})$ be a quadratic extension of number fields and η the associated character of $F^\times \backslash \mathbf{A}_F^\times$. We denote by σ the nontrivial element of $\mathrm{Gal}(E/F)$. Let $H \subset \mathrm{GL}(2, E)$ be the unitary similitude group associated to the matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with similitude character κ. Set $K_H = H(\mathbf{A}_F) \cap K$, where K is the standard maximal compact subgroup of $\mathrm{GL}(2, \mathbf{A}_E)$.

We fix an additive character $\psi : F \backslash \mathbf{A}_F \to \mathbf{C}^\times$. On the groups $\mathrm{GL}(2, \mathbf{A}_E)$ and $H(\mathbf{A}_F)$, we take the product of the local measures defined in [15, Section 2]. To define a measure on the compact group K_v, where v is a place of E or F, we make use of the Iwasawa decomposition

$$\mathrm{GL}(2, E_v) = T(E_v) N(E_v) K_v,$$

where T denotes the diagonal torus in GL(2) and N the upper-triangular unipotent subgroup. The measures dt on $T(E_v)$ and dn on $N(E_v)$ are defined via the obvious isomorphisms $T(E_v) \cong E_v^\times \times E_v^\times$ and $N(E_v) \cong E_v$. Having defined a measure dg on $\mathrm{GL}(2, E_v)$, the measure dk on K_v is taken to be such that

$$dg = dt\, dn\, dk.$$

Similarly for a place v of F,

$$H(F_v) = T_H(F_v)N_H(F_v)K_{H_v},$$

where

$$T_H(F_v) = \left\{ \begin{pmatrix} a & \\ & b\bar{a}^{-1} \end{pmatrix} : a \in E_v^\times, b \in F_v^\times \right\}$$

and

$$N_H(F_v) = \left\{ \begin{pmatrix} 1 & x\sqrt{\delta} \\ & 1 \end{pmatrix} : x \in F_v \right\}.$$

We use the isomorphism $T_H(F_v) \cong E_v^\times \times F_v^\times$ to define a measure on $T_H(F_v)$ and take the Haar measure $|\delta|_{F_v}^{\frac{1}{2}} dx$ on $N_H(F_v)$. The measure dk on K_{H_v} is chosen as before. With these choices,

$$\mathrm{vol}(K_v, dk) = \mathfrak{d}_{E_v}^{\psi_{E_v}} L(2, 1_{E_v})^{-1},$$

for a place v of E, and for a place v of F,

$$\mathrm{vol}\left(K_{H_v}, dk\right) = \mathfrak{d}_{F_v}^{\psi_v} L\left(2, 1_{F_v}\right)^{-1},$$

where $\mathfrak{d}_{E_v}^{\psi_{E_v}}$ and $\mathfrak{d}_{F_v}^{\psi_v}$ are defined as in [16, Section 2.1].

We now fix a unitary character $\chi : E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$ such that $\chi|_{\mathbf{A}_F^\times} \in \{1_F, \eta\}$. We assume that χ^2 is nontrivial, so that the induction of χ to an automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_F)$ is cuspidal. As is well known, $\omega_\pi = \eta\chi|_{\mathbf{A}_F^\times}$. We let Π denote the base change of π to $\mathrm{GL}(2, \mathbf{A}_E)$. Taking the character

$$\tilde{\chi} : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi(a)\chi^{-1}(d)$$

of $B(\mathbf{A}_E)$, we realize Π on the space of smooth functions $f : \mathrm{GL}(2, \mathbf{A}_E) \to \mathbf{C}$ such that

$$f(bg) = \tilde{\chi}(b) f(g)$$

for all $b \in B(\mathbf{A}_E)$. The action of Π is given by

$$(\Pi(g)f)(x) = e^{\langle\rho, H(g)\rangle} f(xg),$$

where $e^{\langle\rho,H(g)\rangle} = |a_1 a_2^{-1}|^{\frac{1}{2}}$ for

$$g = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k$$

with $k \in K$. The inner product on Π is given by

$$\begin{aligned}(\varphi_1, \varphi_2) &= \int_{E^\times\backslash \mathbf{A}_E^1} \int_K \varphi_1\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) \overline{\varphi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right)}\, d^\times a\, dk \\ &= \operatorname{res}_{s=1} L(s, 1_E) \int_K \varphi_1(k)\overline{\varphi_2(k)}\, dk.\end{aligned}$$

We now fix a unitary character $\Omega : E^\times\backslash\mathbf{A}_E^\times \to \mathbf{C}^\times$ such that $\Omega|_{\mathbf{A}_F^\times} = \omega_\pi$, which forces $\Omega \neq \chi$. We assume, as we may, that $\varepsilon(1/2, \chi\Omega) = +1$ and $\varepsilon(1/2, \chi^{-1}\Omega) = +1$, since otherwise $L(1/2, \Pi \otimes \Omega) = 0$ and we know that the relevant period integrals vanish.

Let $f \in C_c^\infty(\mathrm{GL}(2, \mathbf{A}_E))$. We recall, for $x, y \in \mathrm{GL}(2, E)Z(\mathbf{A}_E)\backslash \mathrm{GL}(2, \mathbf{A}_E)$,

$$K_{f,\Pi}(x, y) = \frac{1}{2}\int_{-\infty}^{\infty} \sum_\varphi E(x, \Pi(f)\varphi; it, \Pi)\overline{E(y, \varphi; it, \Pi)}\, dt,$$

where the sum is taken over an orthonormal basis $\{\varphi\}$ of Π, and the Eisenstein series are defined by the analytic continuation of

$$E(g, \varphi; \lambda, \Pi) = \sum_{\gamma\in B(E)\backslash \mathrm{GL}(2,E)} \varphi(\gamma g) e^{\langle\lambda+\rho, H(\gamma g)\rangle}.$$

For $T_1, T_2 > 0$, we consider

$$\Theta_{\Pi,T_1,T_2}(f) = \int_{E^\times\backslash\mathbf{A}_E^\times} \int_{H(F)Z(\mathbf{A}_E)\backslash H(\mathbf{A}_F)} \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_{f,\Pi}\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, h\right) \Omega^{-1}(a) d^\times a\, \omega_\pi\eta(\kappa(h))\, dh,$$

as in [15] and define

$$\Theta_\Pi(f) = \lim_{T_1\to\infty} \lim_{T_2\to\infty} \Theta_{\Pi,T_1,T_2}(f).$$

Following [15, Section 8] and taking care of the normalization of measures,

$$\Theta_\Pi(f) = \frac{\mathrm{vol}\left(F^\times \backslash \mathbf{A}_F^1\right)}{4} \sum_\varphi \mu(\Pi(f)\varphi)\overline{\mathcal{P}_c(\varphi)}.$$

Here the sum is over an orthonormal basis $\{\varphi\}$ of Π,

$$\mathcal{P}_c(\varphi) = \int_{K_H} \varphi(k)\chi(\kappa(k))\, dk,$$

and $\mu(\varphi)$ is defined to be the value at $\lambda = 0$ of the analytic continuation of

$$\mu(\varphi, \lambda) = \int_{\mathbf{A}_E^\times} \varphi(w\nu a)\Omega^{-1}(a_1)e^{\langle \lambda+\rho, H(w\nu a)\rangle}\, d^\times a_1$$

with

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For a place v of F and $\varphi_v \in \Pi_v$, we define $\mu_v(\varphi_v)$ and $\mathcal{P}_{c,v}(\varphi_v)$ analogously, and for $f_v \in C_c^\infty(\mathrm{GL}(2, E_v))$,

$$\Theta_{\Pi_v}(f_v) = \sum_{\varphi_v} \mu_v(\Pi_v(f_v)\varphi_v)\overline{\mathcal{P}_{c,v}(\varphi_v)},$$

with the sum taken over an orthonormal basis of Π_v with respect to the inner product

$$(\varphi_{1,v}, \varphi_{2,v}) = \int_{K_v} \varphi_{1,v}(k)\overline{\varphi_{2,v}(k)}\, dk.$$

Clearly, the distribution $\Theta_\Pi(f)$ factors and if we write $f = f^S \prod_{v\in S} f_v$ where S is a finite set of places of F outside of which everything is unramified and f^S denotes the characteristic function of K^S, then

$$\Theta_\Pi(f) = \frac{1}{4L(1,\eta)} \frac{L^S(1/2, \Pi\otimes\Omega)L^S(1,\eta)}{L^S(1,\pi, Ad)L^S(2, 1_F)} \prod_{v\in S} \Theta_{\Pi_v}(f_v).$$

We shall now compare the distributions $\Theta_{\Pi_v}(f_v)$ with the ones defined in terms of Whittaker models, as is done for the cuspidal spectrum in [15, Section 4]. Having fixed the character ψ, we take the Whittaker model $\mathcal{W}(\Pi, \psi_E)$ for Π to be given by the analytic continuation to $\lambda = 0$ of

$$W_\varphi(g, \lambda) = \int_{\mathbf{A}_E} \varphi(wn(x)g)e^{\langle \lambda+\rho, H(wn(x)g)\rangle} \psi_E(-x)\, dx,$$

where

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

We note, for future reference, that by a simple change of variables,

$$W_\varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \lambda\right) = |a|_{\mathbf{A}_E}^{\frac{1}{2}-\lambda} \chi^{-1}(a) \int_{\mathbf{A}_E} \varphi\left(wn(x)\right) e^{\langle \lambda+\rho, H(wn(x))\rangle} \psi_E(-ax)\, dx$$

for all $a \in \mathbf{A}_E^\times$.

For a place v of F, the inner product on $\mathcal{W}(\Pi_v, \psi_{E_v})$ is taken to be

$$(W_1, W_2) = \int_{E_v^\times} W_1 \begin{pmatrix} a & \\ & 1 \end{pmatrix} \overline{W_2 \begin{pmatrix} a & \\ & 1 \end{pmatrix}}\, d^\times a.$$

For a place v of F and $W_v \in \mathcal{W}(\Pi_v, \psi_{E_v})$,

$$\lambda_v(W_v) = \int_{E_v^\times} W_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Omega_v^{-1}(a)\, d^\times a$$

and

$$\mathcal{P}_v(W_v) = \int_{F_v^\times} W_v \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \chi_v(b)\, d^\times b.$$

We define a distribution for $f \in C_c^\infty(\mathrm{GL}(2, E_v))$ by

$$\Theta_{\Pi_v}^W(f) = \sum_{W_v} \lambda_v(\Pi_v(f)W_v)\overline{\mathcal{P}_v(W_v)},$$

with the sum taken over an orthonormal basis of $\mathcal{W}(\Pi_v, \psi_{E_v})$.

We now compare $\Theta^W_{\Pi_v}(f)$ with $\Theta_{\Pi_v}(f)$. The following lemma can be taken from [16, Proposition 1].

Lemma 1. For a place v of F and all $\varphi_1, \varphi_2 \in \Pi_v$,

$$(\varphi_1, \varphi_2) = \frac{1}{L(1, 1_{E_v})^2}(W_{\varphi_1}, W_{\varphi_2}). \qquad \square$$

Next we compare the distributions μ and λ.

Lemma 2. For a place v of F and $\varphi \in \Pi_v$,

$$\mu(\varphi) = \varepsilon\big(1/2, (\chi\Omega)_v^{-1}, \overline{\psi_{E_v}}\big)^{-1}\lambda(W_\varphi). \qquad \square$$

Proof. We have

$$W_\varphi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}, \lambda\right) = |a|_E^{\frac{1}{2}-\lambda}\chi^{-1}(a)\int_E \varphi\left(wn(x)\right)e^{\langle\lambda+\rho, H(wn(x))\rangle}\psi_E(-ax)\,dx,$$

and hence

$$\begin{aligned}\lambda(W_\varphi, \lambda) &= \int_{E^\times} W_\varphi\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}, \lambda\right)\Omega^{-1}(a)\,d^\times a\\ &= \int_{E^\times}|a|_E^{\frac{1}{2}-\lambda}(\chi\Omega)^{-1}(a)\int_E \varphi\left(wn(x)\right)e^{\langle\lambda+\rho, H(wn(x))\rangle}\psi_E(-ax)\,dx\,d^\times a.\end{aligned}$$

By the Tate functional equation, we have

$$\begin{aligned}\lambda(W_\varphi, \lambda) &= \gamma\big(1/2-\lambda, (\chi\Omega)^{-1}, \overline{\psi_E}\big)\int_{E^\times}|a|_E^{\frac{1}{2}+\lambda}(\chi\Omega)(a)\varphi\begin{pmatrix} 0 & 1\\ 1 & a\end{pmatrix}e^{\left\langle\lambda+\rho, H(wn(a))\right\rangle}\,d^\times a\\ &= \gamma\big(1/2-\lambda, (\chi\Omega)^{-1}, \overline{\psi_E}\big)\int_{E^\times}\Omega(a)\varphi\begin{pmatrix} 0 & 1\\ a^{-1} & 1\end{pmatrix}e^{\left\langle\lambda+\rho, H\left(\begin{smallmatrix} 0 & 1\\ a^{-1} & 1\end{smallmatrix}\right)\right\rangle}\,d^\times a\\ &= \gamma\big(1/2-\lambda, (\chi\Omega)^{-1}, \overline{\psi_E}\big)\int_{E^\times}\Omega(a)^{-1}\varphi\begin{pmatrix} 0 & 1\\ a & 1\end{pmatrix}e^{\left\langle\lambda+\rho, H\left(\begin{smallmatrix} 0 & 1\\ a & 1\end{smallmatrix}\right)\right\rangle}\,d^\times a\\ &= \gamma\big(1/2-\lambda, (\chi\Omega)^{-1}, \overline{\psi_E}\big)\mu(\varphi, \lambda).\end{aligned}$$

Finally, since $\Omega\chi|_{F^\times} = \eta_{E/F}$, so $\gamma(1/2, (\chi\Omega)^{-1}, \overline{\psi_E}) = \varepsilon(1/2, (\chi\Omega)^{-1}, \overline{\psi_E})$. ■

Finally, we compare $\mathcal{P}_c$ and $\mathcal{P}$.

Lemma 3. For a place v of F and $\varphi \in \Pi_v$,

$$\mathcal{P}_c(\varphi) = \frac{1}{L(1, 1_{F_v})^2} \mathcal{P}(W_\varphi).$$

□

Proof. To begin with,

$$\begin{aligned}\mathcal{P}_c(\varphi) &= \int_{K_H} \varphi(k)\chi(\kappa(k))\, dk \\ &= \frac{|\delta|^{\frac{1}{2}}}{L(1, 1_F)} \int_F \varphi(wn(\sqrt{\delta}x)) e^{\langle \rho, H(wn(\sqrt{\delta}x))\rangle}\, dx\end{aligned}$$

by applying [16, (4)] to $H(F) \cong Z(E)\,\mathrm{GL}(2, F)$. On the other hand, in the sense of analytic continuation,

$$\begin{aligned}\mathcal{P}(W_\varphi) &= \int_{F^\times} W_\varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi(a)\, d^\times a \\ &= \int_{F^\times} \int_E |a|_F \varphi(wn(x)) e^{\langle \rho, H(wn(x))\rangle} \psi(-a(x+\bar{x}))\, dx\, d^\times a \\ &= L(1, 1_F) \int_F \int_E \varphi(wn(x)) e^{\langle \rho, H(wn(x))\rangle} \psi(-a(x+\bar{x}))\, dx\, da \\ &= |4\delta|_F^{\frac{1}{2}} L(1, 1_F) \int_F \int_F \int_F \varphi(wn(x_1 + x_2\sqrt{\delta})) e^{\langle \rho, H(wn(x_1+x_2\sqrt{\delta}))\rangle} \psi(-2ax_1)\, dx_1\, dx_2\, da \\ &= |\delta|_F^{\frac{1}{2}} L(1, 1_F) \int_F \int_F \int_F \varphi(wn(x_1 + x_2\sqrt{\delta})) e^{\langle \rho, H(wn(x_1+x_2\sqrt{\delta}))\rangle} \psi(-ax_1)\, dx_1\, dx_2\, da \\ &= |\delta|_F^{\frac{1}{2}} L(1, 1_F) \int_F \varphi(wn(x_2\sqrt{\delta})) e^{\langle \rho, H(wn(x_2\sqrt{\delta}))\rangle}\, dx_2\end{aligned}$$

by the Fourier inversion formula. ■

Combining the above lemmata we have, for any place v of F and $f_v \in C_c^\infty(\mathrm{GL}(2, E_v))$,

$$\Theta_{\Pi_v}(f_v) = \varepsilon\big(1/2, (\chi\Omega)_v^{-1}, \overline{\psi_{E_v}}\big)^{-1} L(1, \eta_v)^2 \Theta^W_{\Pi_v}(f_v).$$

This gives the following corollary.

Corollary 1. For $f = f^S \prod_{v \in S} f_v \in C_c^\infty(\mathrm{GL}(2, \mathbf{A}_E))$ as above,

$$\Theta_\Pi(f) = \frac{1}{4} \prod_{v \in S} \Theta^W_{\Pi_v}(f_v) \frac{L_S(1,\eta) L^S(1/2, \Pi \otimes \Omega)}{L^S(1, \pi, Ad)}.$$

□

The upshot of the relative trace formula comparison is an identity of the form

$$\Theta_\Pi(f) + \Theta_{\Pi'}(f) = \theta_{\sigma_\varepsilon}(f_\varepsilon)$$

as in [15, p. 41], where $\Theta_{\Pi'}$ denotes the contribution to the trace formula from the character $\tilde{\chi}^{-1}$. There is only one term on the right-hand side in this case, since $\pi \otimes \eta = \pi$. Thus for $f = f^S \prod_{v \in S} f_v \in C_c^\infty(\mathrm{GL}(2, \mathbf{A}_E))$,

$$\frac{1}{2} \prod_{v \in S} \Theta^W_{\Pi_v}(f_v) \frac{L_S(1,\eta) L^S(1/2, \Pi \otimes \Omega)}{L^S(1, \pi, Ad)} = \theta_{\sigma_\varepsilon}(f_\varepsilon).$$

We can now apply the purely local arguments of [15, Section 5] and deduce the statement of [15, Theorem 2] for π dihedral with respect to E.

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