

A SYMPLECTIC CASE OF ARTIN'S CONJECTURE

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1. Introduction

Here we will prove a new 4-dimensional symplectic case of Artin's conjecture. Let us first recall Artin's conjecture. Let G be the Galois group of a finite Galois extension of number fields. Let ρ be a representation of G over \mathbb{C} . In this context one can associate to ρ a meromorphic function $L(s, \rho)$ called the Artin L -function associated to ρ .

Artin's Conjecture. *If ρ does not contain the trivial representation, then $L(s, \rho)$ is entire.*

An equivalent form of this conjecture is the following. If ρ is irreducible and non-trivial, then $L(s, \rho)$ is entire. Artin proved this conjecture in the case where ρ is 1-dimensional using his reciprocity law together with a result of Hecke. As L -functions are inductive, this proved the conjecture if ρ is *monomial*, i.e. induced from a degree one representation of some subgroup.

Now consider the case where ρ is 2-dimensional. Let $\bar{\rho} : G \rightarrow \mathrm{PGL}_2(\mathbb{C})$ be the composition of $\rho : G \rightarrow \mathrm{GL}_2(\mathbb{C})$ with the natural projection of $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{PGL}_2(\mathbb{C})$. Let \bar{G} denote the image of G under $\bar{\rho}$. So \bar{G} is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$, which is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. The only finite subgroups of $\mathrm{SO}_3(\mathbb{C})$ are cyclic, dihedral, tetrahedral (isomorphic to A_4), octahedral (isomorphic to S_4) and icosahedral (isomorphic to A_5). We thus classify ρ according the isomorphism type of \bar{G} . If \bar{G} is cyclic or dihedral, then ρ is reducible or monomial and $L(s, \rho)$ is entire by Artin's result. Much later, Langlands applied to this problem the theory of *automorphic representations*, which also have associated L -functions. For *cuspidal* automorphic representations of GL_n the associated L -function is known to be entire [Ja]. Both automorphic and Artin L -functions can be written as Euler products of *local factors* $L(s, \pi) = \prod_v L(s, \pi_v)$ and $L(s, \rho) = \prod_v L(s, \rho_v)$. Langlands formulated the following amazing conjecture [La1].

Strong Artin Conjecture. *Let G be the Galois group of an extension K/F of number fields. Let ρ be an n -dimensional complex representation of G . There exists an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, such that the L -functions agree almost everywhere, i.e. except at a finite number of places v , $L(s, \rho_v) = L(s, \pi_v)$. Moreover, if ρ is irreducible, then π is cuspidal.*

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We say ρ is *modular* if the strong Artin conjecture holds for ρ . Furthermore, if π is as in the conjecture, we will say ρ *corresponds to* π and write $\rho \leftrightarrow \pi$ or $\pi \leftrightarrow \rho$.

The strong Artin (also called Langlands' reciprocity) conjecture is really a stronger statement than Artin's conjecture. For example, nilpotent and supersolvable groups are monomial and hence their representations ρ satisfy Artin's conjecture. Arthur and Clozel recently proved ([AC]) that if ρ has nilpotent image, ρ is also modular. However it has not yet been shown that all ρ with supersolvable image must be modular.

Langlands proved the strong Artin conjecture, and hence Artin's conjecture, in the tetrahedral case [La2], i.e. when $\bar{G} \simeq A_4$ (and $\dim \rho = 2$). In his proof, he used three important elements: the symmetric square lift from GL_2 to GL_3 of Gelbart and Jacquet [GJ], *normal cubic base change* on GL_2 developed by him in [La2], and the structure of the group A_4 . After non-normal cubic base change was proven by Jacquet, Piatetski-Shapiro and Shalika [JPSS], Tunnell used it to extend Langlands' argument and prove the strong Artin conjecture for the octahedral case ($\bar{G} \simeq S_4$) [Tu]. This completed the solvable case in dimension 2. The non-solvable (icosahedral) case is much more difficult. However, partial but outstanding progress has been made recently in this case by Buzzard, Dickinson, Shepherd-Barron and Taylor [BDST]. For modular 2-dimensional ρ , the corresponding symmetric m -th power representations for $m \leq 4$ are also modular by the work of Kim and Shahidi [KS], [Ki].

There are not many other cases of Artin's conjecture known in higher dimensions. Progress has been primarily made only for essentially self-dual (i.e., orthogonal or symplectic) representations. Let ρ be an essentially self-dual irreducible complex n -dimensional representation of G with solvable image. If n is odd, then ρ maps into $GO_n(\mathbb{C})$ and ρ is monomial [Ra]. Therefore in the odd-dimensional case Artin's conjecture (but not strong Artin) is known. If n is even, ρ has image either in $GO_n(\mathbb{C})$ or $GSp_n(\mathbb{C})$. In the case $\rho : G \rightarrow GO_4(\mathbb{C})$ has solvable image, then Ramakrishnan recently showed ρ satisfies the strong Artin conjecture by proving the modularity of the Asai representation [Ra].

Here we make progress in the case ρ has image in $GSp_4(\mathbb{C})$. We are able to exhibit another group whose 4-dimensional representations can be proven modular in a manner similar to Langlands' tetrahedral argument. Consider $\rho : G \rightarrow GSp_4(\mathbb{C})$. Then $\bar{G} \subseteq PGSp_4(\mathbb{C}) = GSp_4(\mathbb{C})/\mathbb{C}^*$, which is isomorphic to $SO_5(\mathbb{C})$. The finite subgroups of $PGSp_4(\mathbb{C})$ have been classified in [Mi] and [CM]. Let E_{2^4} denote the elementary abelian group of order 2^4 and C_5 the cyclic group of order 5. Then there is a semidirect product $E_{2^4} \rtimes C_5$ with C_5 acting fixed point freely on E_{2^4} contained inside $PGSp_4(\mathbb{C})$. If $\bar{G} \simeq E_{2^4} \rtimes C_5$, we show ρ is modular. Our proof uses the recent construction of Kim [Ki] of the exterior square $\Lambda^2(\pi)$ of an (isobaric) automorphic representation π of GL_4 , which agrees locally almost everywhere with an automorphic representation of GL_6 . We also crucially use *normal quintic base change* ([AC]) and the structure of our group

$E_{2^4} \rtimes C_5$. An investigation of the other subgroups of $\mathrm{PGSp}_4(\mathbb{C})$ will appear in the author's thesis. We now state the main result precisely.

Theorem 1. *Let L/F be a Galois extension of number fields and ρ be an irreducible 4-dimensional representation of $G = \mathrm{Gal}(L/F)$ into $\mathrm{GSp}_4(\mathbb{C})$. Suppose $\tilde{G} = \mathrm{Im}(\bar{\rho}) \simeq E_{2^4} \rtimes C_5$. Then ρ is modular.*

One pre-image of $E_{2^4} \rtimes C_5$ inside $\mathrm{Sp}_4(\mathbb{C})$ is $G = E_{2^5} \rtimes C_5$, where E_{2^5} is the extraspecial group of order 2^5 isomorphic to the central product of Q_8 and D_8 with identified centers (see [Do] for definitions). Consider an irreducible 4-dimensional complex representation ρ of G . As G has no subgroups of index 2 or 4, ρ is primitive, i.e. not induced. Thus Artin's conjecture for ρ does not follow from previous results.

We know that examples of $E_{2^5} \rtimes C_5$ extensions of \mathbb{Q} exist by Shafarevich's Theorem ([NSW]) because $E_{2^5} \rtimes C_5$ is solvable. Though Shafarevich's proof is non-constructive, we can illustrate how to construct such an extension in our case. Let $\alpha_i = \zeta_{11}^i + \zeta_{11}^{-i}$, where ζ_{11} is a complex 11-th root of unity. Let E be the cyclic Galois extension $\mathbb{Q}(\alpha_1)$ of \mathbb{Q} of degree 5. It is known how to construct Q_8 and D_8 extensions of a number field ([JLY]). Let $K = E(\sqrt{1 + A + B + AB})$ and $M = E(\sqrt{\alpha_1 + i\alpha_2}, \sqrt{\alpha_2 + \alpha_4 + 4})$, where $A = (3 + \alpha_5)^{-1/2}$ and $B = (1 + \alpha_1^2 + \alpha_1^2\alpha_3^2)^{-1/2}$. Then $\mathrm{Gal}(K/E) \simeq Q_8$ and $\mathrm{Gal}(M/E) \simeq D_8$. The compositum KM has three normal (over E) subextensions of index 2. Let L/E be the one corresponding to the central product of Q_8 with D_8 . Then L/\mathbb{Q} is Galois with Galois group $E_{2^5} \rtimes C_5$.

Before we go on, let us be a little more explicit about the similarities between Langlands' tetrahedral case and Theorem 1. We will first outline Langlands' argument. Let ρ be a representation of G into $\mathrm{GL}_2(\mathbb{C})$ with image A_4 . Then G has a normal subgroup H of index 3. Then H is dihedral. All representations of dihedral groups are modular ([La2]), so ρ_H is modular. Let Π be an automorphic representation corresponding to ρ_H . Normal cubic base change for GL_2 tell us that there are three representations π_i whose *base change* $\pi_{i,H}$ (the automorphic version of restricting a representation) to H is Π . So we think one of these π_i should correspond to ρ . Using the determinant of ρ , Langlands proved there is a unique π among the π_i 's with its central character ω_π corresponding to $\det(\rho)$, such that $\mathrm{Sym}^2(\rho) \leftrightarrow \mathrm{Sym}^2(\pi)$. Langlands then showed that if $\rho_H \leftrightarrow \pi_H$ and $\mathrm{Sym}^2(\rho) \leftrightarrow \mathrm{Sym}^2(\pi)$, but ρ does not correspond to π , then $\tilde{G} \simeq A_4$ must have an element of order 6. But A_4 has no elements of order 6, therefore ρ must correspond to π , i.e. ρ is modular.

The basic approach to proving Theorem 1 is similar to the tetrahedral argument, but with several differences. The first difficulty encountered in our case is that the determinant alone does not give enough information to choose the appropriate π . More precisely, suppose ρ is a representation of G into $\mathrm{GSp}_4(\mathbb{C})$ with projective image $E_{2^4} \rtimes C_5$. Then G has a normal subgroup H of index 5. Since H is nilpotent, ρ_H is modular. Say $\rho_H \leftrightarrow \Pi$. Normal quintic base change tells us that there are five representations π_i whose base change $\pi_{i,H}$ to H equals

II. Using the fact that $\Lambda^2(\rho)$ has an invariant line, we are able to pick a unique π from the π_i 's such that $\Lambda^2(\rho) \leftrightarrow \Lambda^2(\pi)$. Then we deduce either $\rho \leftrightarrow \pi$ or \bar{G} has an element of order 10, which it does not. Therefore ρ must be modular.

In fact, Langlands' conjectures actually predict ρ is *modular of symplectic type*, i.e. ρ corresponds to an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. We discuss this briefly in the final section.

2. Preliminaries

In this section, we will review the theories of L -functions and base change. Those already familiar with them may wish to skip to the proof in the next section.

First we will begin with some notation. Let F be a number field and L be a finite Galois extension of F with Galois group G . Let ρ be a representation of G into $\mathrm{GL}_n(\mathbb{C})$. We will refer to restriction and induction of representations with the corresponding fields. More precisely, let E be a subextension of L/F and $H = \mathrm{Gal}(L/E)$. Then ρ_E denotes the restriction ρ_H of ρ to the subgroup H . For a representation σ of H , $I_E^F \sigma$ denotes the induced representation $I_H^G \sigma$ of σ from H to G . We will use $\mathfrak{N}_{E/F}$ to denote the norm map from E to F .

Let v be a place of F and Fr_v be the corresponding Frobenius class. Denote by q_v the size of the residue field \mathcal{O}_F/v . Recall the Artin L -function is defined on some right-half plane by $L(s, \rho) = \prod_v L(s, \rho_v)$ where the product is over all places v of F and if v is unramified,

$$L(s, \rho_v) = \frac{1}{\det(1 - \rho(\mathrm{Fr}_v)q_v^{-s})}.$$

Brauer showed for any ρ , $L(s, \rho)$ extends to a meromorphic function on all of \mathbb{C} . In the case ρ is the trivial representation $\mathbf{1}$, the L -function is the Dedekind zeta function $\zeta_F(s)$ of F . For a complete definition of Artin L -functions, see [Ro], [Ne] or [Ma].

One can also define L -functions for automorphic representations. Let \mathbb{A}_F be the adèles of the number field F .

Theorem 2.2 ([Ja]). *Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then we can associate local factors $L(s, \pi_v)$ to each v and define a holomorphic L -function $L(s, \pi) = \prod L(s, \pi_v)$ in some right-half plane. Moreover $L(s, \pi)$ extends to a meromorphic function on \mathbb{C} , which is actually entire if π is cuspidal and non-trivial.*

For more information on automorphic representations and their L -functions, see [Ge1], [Kn], or [Ge2]. The strong Artin conjecture asserts that given a Galois representation ρ as before, there exists an automorphic representation π corresponding to ρ (see Introduction). If $\rho \leftrightarrow \pi$ and ρ is irreducible, then the following result implies π is cuspidal.

Theorem 2.3 ([JS1]). *Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\tilde{\pi}$ its contragredient. Then $L(s, \pi \times \tilde{\pi})$ has a simple pole at $s = 1$ if and only if π is cuspidal.*

For if ρ is irreducible and $\check{\rho}$ its contragredient, then $L(s, \rho \otimes \check{\rho})$ has a simple pole at $s = 1$. Moreover if $\rho \leftrightarrow \pi$, then $L(s, \rho \otimes \check{\rho})$ and $L(s, \pi \times \tilde{\pi})$ agree almost everywhere locally. So by Theorem 2.3, π is indeed cuspidal.

Now to show that the strong Artin conjecture actually implies Artin's conjecture we only need the following standard fact.

Proposition 2.1. *If π is cuspidal and $L(s, \pi_v) = L(s, \rho_v)$ for almost all v , then in fact $L(s, \pi) = L(s, \rho)$.*

So if ρ is irreducible and non-trivial and corresponds to ρ , then by Theorem 2.3 ρ is cuspidal (and non-trivial). The proposition tells us that in fact $L(s, \pi) = L(s, \rho)$. By Theorem 2.2 we know that $L(s, \pi)$ is entire, i.e. $L(s, \rho)$ is entire, i.e. Artin's conjecture is true for ρ .

If π and π' are cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_F)$ respectively, then one can form their *isobaric sum* $\pi \boxplus \pi'$ which is an automorphic representation of $\mathrm{GL}_{m+n}(\mathbb{A}_F)$ [JS1]. If $\rho \leftrightarrow \pi$ and $\rho' \leftrightarrow \pi'$, then $\rho \oplus \rho' \leftrightarrow \pi \boxplus \pi'$. So every automorphic representation which corresponds to a Galois representation will be *isobaric*, i.e. a finite isobaric sum of cuspidal representations.

A powerful tool to prove the strong Artin conjecture in certain instances is the theory of base change. Base change is an operation on automorphic representations which corresponds to the restriction of Galois representations. We now list the important properties of base change that we need.

Theorem 2.4 ([AC]). *Let L/F be a Galois extension of number fields. Let E/F be a normal cyclic subextension of prime degree. For each isobaric representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, there exists a unique automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ called the base change of π to E and denoted by π_E such that:*

- (i) (descent) a cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is the base change π_E of some π if and only if Π is Galois invariant (in particular, if $\Pi \leftrightarrow \rho_E$ where ρ is some representation of $\mathrm{Gal}(L/F)$);
- (ii) if π' is also an isobaric representation of $\mathrm{GL}_n(\mathbb{A}_F)$ then $\pi_E = \pi'_E$ if and only if $\pi' = \pi \otimes \delta$ for some idele class character δ of $F^* \backslash \mathfrak{N}_{E/F}(\mathbb{A}_E^*) \backslash \mathbb{A}_F^* \simeq \mathrm{Gal}(E/F)$;
- (iii) (compatibility with reciprocity) if ρ is a representation of $\mathrm{Gal}(L/F)$, $\rho \leftrightarrow \pi$ and ρ_E is modular, then $\rho_E \leftrightarrow \pi_E$; and
- (iv) (compatibility with twisting) if χ is an idele class character of F and $\chi_E = \chi \circ N_{E/F}$, then

$$(\pi \otimes \chi)_E = \pi_E \otimes \chi_E.$$

The complementary construction to base change is *automorphic induction*, which corresponds to induction of Galois representations.

Theorem 2.5 ([AC],[HH]). *Let L/F be a Galois extension of number fields. Let E/F be a normal cyclic subextension of prime degree. Let ρ be a complex representation of $\text{Gal}(L/E)$ and suppose $\rho \leftrightarrow \pi$, for some automorphic representation π . Then there exists an induced automorphic representation, denoted $I_E^F \pi$ such that $I_E^F \rho \leftrightarrow I_E^F \pi$.*

3. Proof of Theorem 1

Let L/F be a (finite) Galois extension of number fields with Galois group G . Suppose ρ is an (injective) representation of G into $\text{GSp}_4(\mathbb{C})$ such that $\bar{G} \simeq E_{24} \rtimes C_5$. Let E be the normal quintic subextension of L/F corresponding to the pre-image of E_{24} .

Lemma 3.1. *The representations ρ_E and $\Lambda^2(\rho)$ are modular.*

Proof. As $\text{Gal}(L/E)$ is a cyclic central extension of a 2-group, it is a direct product of a 2-group P_2 with a cyclic group C of odd order. Therefore $\text{Gal}(L/E)$ is nilpotent. By a theorem of Arthur and Clozel, all representations of nilpotent groups are modular [AC]. In particular ρ_E is modular.

Since ρ is of symplectic type, $\Lambda^2(\rho)$ has an invariant line. Write $\Lambda^2(\rho) = \nu \oplus r$ where ν is 1-dimensional and r is 5-dimensional. Note r is irreducible because it factors through $E_{24} \rtimes C_5$, which only has 1- and 5- dimensional irreducible representations. We claim r is induced from E . As P_2 is a 2-group, every irreducible representation of P_2 has dimension 2^j for some j . Therefore the same is true for $\text{Gal}(L/E) \simeq P_2 \times C$. Hence in the decomposition of r_E into its irreducible components, we must have a 1-dimensional representation λ . In particular, $r = I_E^F \lambda$. Since E is a normal subextension r is modular by Theorem 2.5, whence $\Lambda^2(\rho)$ is also. □

Let us say $\rho_E \leftrightarrow \Pi$. We claim ρ_E is irreducible. Indeed, ρ irreducible implies that $\text{Gal}(E/F) \simeq C_5$ acts transitively on the irreducible components of ρ_E . This action has order dividing 5. Thus if there is more than one irreducible component of ρ_E , there must be five or a multiple thereof. However $\dim \rho_E = 4$, so that is impossible. Then by Theorem 2.3, Π is cuspidal.

We can apply Theorem 2.4(i) to get an automorphic representation π_0 of $\text{GL}_4(\mathbb{A}_F)$ whose base change $\pi_{0,E}$ corresponds to ρ_E . Let $\delta = \delta_{E/F}$ be a non-trivial idele class character of $F^* \backslash \mathcal{N}_{E/F}(\mathbb{A}_E^*) \backslash \mathbb{A}_F^* \simeq \text{Gal}(E/F) \simeq C_5$. Let $\pi_i = \pi_0 \otimes \delta^i$ for $i = 1, 2, \dots, 4$. Since $\delta_E = \mathbf{1}$, all the π_i 's base change to $\pi_{i,E} \simeq (\pi_0 \otimes \delta)_E \simeq \pi_{0,E} \simeq \Pi$ by Theorem 2.4(iv). In fact, part (ii) of the same theorem tells us that these are all the cuspidal representations of $\text{GL}_4(\mathbb{A}_F)$ whose base change to E is Π .

Lemma 3.2. *There is a unique $i \in \{0, 1, 2, \dots, 4\}$ such that $\Lambda^2(\pi_i) \leftrightarrow \Lambda^2(\rho)$.*

Proof. All the representations $\Lambda^2(\pi_i)$ base change to $\Lambda^2(\pi_0 \otimes \delta^i)_E = \Lambda^2(\pi_0)_E$ by Theorem 2.4(iv). They are all distinct because they have distinct central characters $\omega_{\Lambda^2(\pi_i)} = \omega_{\Lambda^2(\pi_0)} \delta^{2i}$.

Theorem 2.4(ii) then yields that the $\Lambda^2(\pi_i)$ are the only representations which base change to $\Lambda^2(\pi_0)_E$. But by part (iii) of this theorem, the automorphic representation on $\mathrm{GL}_6(\mathbb{A}_F)$ which corresponds to $\Lambda^2(\rho)$ must also base change to $\Lambda^2(\pi_0)_E$. Thus for some i , $\Lambda^2(\pi_i) \leftrightarrow \Lambda^2(\rho)$. \square

Denote the π_i of the lemma by π . We claim now that in fact $\rho \leftrightarrow \pi$. It will suffice to show for all unramified places that $\rho_v \leftrightarrow \pi_v$. Say ρ_v has Frobenius eigenvalues $\{a, b, c, d\}$ and π_v has Satake parameters $\{e, f, g, h\}$. We want to show $\{a, b, c, d\} = \{e, f, g, h\}$. Let D be a diagonal element of GL_4 . Then $\Lambda^2(D) = 1$ if and only if $D = \pm I$. Hence $\Lambda^2(\rho_v) \leftrightarrow \Lambda^2(\pi_v)$ implies

$$(3.1) \quad \{a, b, c, d\} = \pm\{e, f, g, h\}.$$

If they are equal, we are done. Assume therefore

$$(3.2) \quad \{a, b, c, d\} = -\{e, f, g, h\}.$$

Now we can use base change to E . In our group \bar{G} any element raised to the 5th power lies inside E_{24} (see Lemma 3.3 below). Thus any element of G raised to the 5th power lies inside $\mathrm{Gal}(L/E)$, the pre-image of E_{24} . In particular $Fr_v^5 \in \mathcal{O}_{E_w}$, where w is a prime of E above v . Then $\rho_{v,E} \leftrightarrow \pi_{v,E}$ implies $\{a^5, b^5, c^5, d^5\} = \{e^5, f^5, g^5, h^5\}$. By our assumption we have,

$$(3.3) \quad \{a^5, b^5, c^5, d^5\} = \{-a^5, -b^5, -c^5, -d^5\}.$$

Without loss of generality, assume $a^5 = -b^5$ and $c^5 = -d^5$. Then either $b = -\zeta_5 a$ or $d = -\zeta_5 c$, for otherwise $a = -b, c = -d$ which would imply $\{a, b, c, d\} = \{e, f, g, h\}$. Let us say

$$(3.4) \quad b = -\zeta_5 a.$$

Then

$$(3.5) \quad \rho(Fr_v) \sim \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -\zeta_5 a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

so

$$(3.6) \quad \bar{\rho}(Fr_v) \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\zeta_5 & 0 & 0 \\ 0 & 0 & c/a & 0 \\ 0 & 0 & 0 & d/a \end{pmatrix},$$

is an element of order divisible by 10 in $\bar{G} = \mathrm{Im}(\bar{\rho}) \subseteq \mathrm{PSp}_4(\mathbb{C})$. But \bar{G} has no elements of order 10 by Lemma 3.3 below, so ρ is modular by contradiction.

Lemma 3.3. *Every element $g \in E_{24} \rtimes C_5$ has order 5 except for the elements in the normal subgroup E_{24} , which have order 1 or 2.*

Proof. Let $g \in E_{2^4} \rtimes C_5$ such that $g \notin E_{2^4}$. We can write $g = az$ where $a \in E_{2^4}$ and $z \in C_5$, $z \neq 1$. We claim g^5 commutes with z . Write

$$\begin{aligned} (3.7) \quad g^5 &= (az)(az)(az)(az)(az) \\ (3.8) \quad &= a(zaz^{-1})(z^2az^{-2})(z^3az^{-3})(z^4az^{-4}). \end{aligned}$$

We also have

$$\begin{aligned} (3.9) \quad zg^5z^{-1} &= (za)(za)(za)(za)(za) \\ (3.10) \quad &= (zaz^{-1})(z^2az^{-2})(z^3az^{-3})(z^4az^{-4})a. \end{aligned}$$

Each $z^j az^{-j}$ lies in the normal abelian subgroup E_{2^4} and therefore the $z^j az^{-j}$'s commute. Thus we can rearrange the terms in (3.10) to get (3.8) and we have $zg^5z^{-1} = g^5$.

Now, since each term in (3.8) lies in E_{2^4} , then $g^5 \in E_{2^4}$ also. But the action of C_5 on E_{2^4} fixes only the identity. Thus $g^5 = 1$. \square

4. Transfer to GSp_4

As before, consider a Galois group G of an extension of number fields L/F and an irreducible 4-dimensional representation ρ of G into $\mathrm{GSp}_4(\mathbb{C})$. Suppose that ρ is modular, i.e. ρ corresponds to some cuspidal representation π of $\mathrm{GL}_4(\mathbb{A}_F)$. The fact that $\mathrm{Im}(\rho) \subseteq \mathrm{GSp}_4(\mathbb{C})$ implies that $L(s, \Lambda^2(\rho) \otimes \nu^{-1})$ has a simple pole at $s = 1$ for a suitable 1-dimensional representation ν of G (the ‘‘polarization’’).

This implies that the corresponding automorphic L -function $L(s, \pi; \Lambda^2 \otimes \nu^{-1})$ admits a pole at $s = 1$. An unpublished theorem of Jacquet, Piatetski-Shapiro and Shalika says that, because of this pole, π transfers to a generic irreducible cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$ with central character ν such that,

$$L^S(s, \Pi) = L^S(s, \pi),$$

for any finite set of primes S outside of which π is unramified. Here the L -function on the left is the degree 4 L -function of Π studied in [PS]; and if $L(s) = \prod_v L_v(s)$ is an Euler product, then $L^S(s)$ denotes the incomplete L -function $\prod_{v \notin S} L_v(s)$. Thus ρ in fact corresponds to the cuspidal representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$, i.e. ρ is *modular of symplectic type* as predicted by Langlands.

However we are not stressing this here because this theorem of Jacquet, Piatetski-Shapiro and Shalika remains unpublished. We hope to go into more detail in the thesis. The key point is that GL_4 maps into the connected component of GO_6 , and π gives rise to a cuspidal automorphic representation π' of $\mathrm{GO}_6(\mathbb{A}_F)^0$. The desired Π is obtained by the theta correspondence. The obstruction to this transfer is the residue of the pole of $L(s, \pi; \Lambda^2 \otimes \nu^{-1})$ at $s = 1$ (see [JS2]). Finally the ongoing work of J. Arthur will give another proof, using the trace formula, of the existence of Π and other members of its packet (see [Ar] for his program).

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