MASS FORMULAS AND EISENSTEIN CONGRUENCES IN HIGHER RANK

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Abstract. We use mass formulas to construct minimal parabolic Eisenstein congruences for algebraic modular forms on reductive groups compact at infinity, and study when these yield congruences between cusp forms and Eisenstein series on the quasi-split inner form. This extends recent work of the first author on weight 2 Eisenstein congruences for GL(2) to higher rank.

Two issues in higher rank are that the transfer to the quasi-split form is not always cuspidal and sometimes the congruences come from lower rank (e.g., are "endoscopic"). We show our construction yields Eisenstein congruences with non-endoscopic cuspidal automorphic forms on quasi-split unitary groups by using certain unitary groups over division algebras. On the other hand, when using unitary groups over fields, or other groups of Lie type, these Eisenstein congruences typically appear to be endoscopic. This suggests a new way to see higher weight Eisenstein congruences for GL(2), and leads to various conjectures about GL(2) Eisenstein congruences.

In supplementary sections, we also generalize previous weight 2 Eisenstein congruences for Hilbert modular forms, and prove some special congruence mod p results between cusp forms on $\mathrm{U}(p)$.

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1. Introduction

In [Mar17], we gave a construction for mod p congruences of weight 2 cusp forms with Eisenstein series on $\mathrm{PGL}(2)$ using the Eichler mass formula for a definite quaternion algebra and the Jacquet–Langlands correspondence. This approach has certain advantages over previous approaches to Eisenstein congruences for elliptic modular forms (e.g., [Maz77], [Yoo19]): one can treat more general levels and primes p, as well as Hilbert modular forms, without much difficulty.

In this paper, we extend this approach to groups of higher rank.

Suppose π, π' are irreducible automorphic representations of a reductive group G over a number field F, and that outside of a finite set of places S there is a hyperspecial maximal compact subgroup $K_v \subset G(F_v)$ such that π_v and π'_v are both K_v-spherical. Then we say π and π' are Hecke congruent mod p (away from S) if there exists a prime $\mathfrak{p}$ above p in a sufficiently large number field such that, for $v \notin S$, the spherical Hecke eigenvalues for $\pi_v^{K_v}$ and those for $(\pi'_v)^{K_v}$ are congruent mod $\mathfrak{p}$.

Here is a sample result for certain unitary groups over $\mathbb{Q}$.

Theorem A*. *(Example 5.10) Let $n = 2m+1$ be an odd prime, χ the idele class character for $\mathbb{Q}$ associated to the quadratic extension $E = \mathbb{Q}(i)$, and $G = \mathrm{U}(n)$ the quasi-split unitary group associated to $E/\mathbb{Q}$. Let $\mathbb{1}_G$ denote the trivial representation of G. Fix a prime $\ell \equiv 1 \bmod 4$. Suppose $p > n$ is a prime such that either $p|(\ell^r - 1)$ for some $1 \le r \le n-1$ or that p divides the numerator of the product $\prod_{r=1}^m B_{2r} \cdot \prod_{r=1}^m B_{2r+1,\chi}$ of generalized Bernoulli numbers.*

Then there exists a holomorphic weight n cuspidal representation π of $G(\mathbb{A})$ such that (i) π_v is unramified at each finite odd $v \ne \ell$, (ii) π_2 is spherical, (iii) π_ℓ is an unramified twist of the Steinberg representation, (iv) the base change π_E of π to $\mathrm{GL}_n(\mathbb{A}_E)$ is cuspidal, and (v) π is Hecke congruent to $\mathbb{1}_G$ mod p.

The asterisk in the theorem refers to an underlying assumption of the endoscopic classification for unitary groups when $n > 3$, to be discussed below.

We can regard this as an Eisenstein congruence as follows. Suppose G/F is semisimple with a Borel subgroup B. For a character χ of the Levi of B, consider the principal series representation $I(\chi)$ induced from χ. Choosing standard sections of $I(\chi)$ yields Eisenstein series, which are not in general L^2. In particular, if $\chi = \delta_G^{-1/2}$ where δ_G denotes the modulus character, then $I(\chi)$ contains $\mathbb{1}_G$ as a subrepresentation, and $\mathbb{1}_G$ contributes to the residual part of the discrete L^2 spectrum. Note one can reformulate the weight 2 Eisenstein series congruence for elliptic modular forms from [Maz77], [Mar17] as congruences with $\mathbb{1}_G$ for $G = \mathrm{PGL}(2)$.

The Hecke eigenvalues for $\mathbb{1}_G$ are relatively simple to describe, being simply the degrees of the corresponding Hecke operators. For instance, if $G = \mathrm{U}(2)$ or $\mathrm{U}(3)$, the local spherical Hecke algebra is generated by a single Hecke operator T_q. For $G = \mathrm{U}(2)$, the spherical eigenvalue of $\mathbb{1}_G$ for T_q is $q+1$ if q is split in

$E/\mathbb{Q}$ and q^2+q if q is inert in $E/\mathbb{Q}$. For $G = \mathrm{U}(3)$, the spherical eigenvalue of $\mathbb{1}_G$ for T_q is q^2+q+1 if q is split in $E/\mathbb{Q}$ and q^4+q if q is inert in $E/\mathbb{Q}$. In general, there are many local Hecke operators at q. See (3.9) and (3.13) for a general description of the unramified Hecke eigenvalues for $\mathbb{1}_G$.

As a specific example of the Bernoulli number divisibility condition, for any $\ell \equiv 1 \bmod 4$, we may take $p = 61$ if $n = 7$ or $p \in \{19, 61, 277, 691\}$ if $n = 13$.

We also remark that the condition $p > n$ is not needed in our general result.

To our knowledge, these are the first general Eisenstein congruence results in higher rank for Eisenstein series attached to minimal parabolic subgroups. Some instances of Eisenstein congruences in higher rank are known for Eisenstein series attached to maximal parabolic subgroups, e.g., see [BD16].

There are two main steps to the proof: (1) construct appropriate congruences on certain compact inner forms, and (2) transfer these congruences to the quasi-split forms via functoriality.

We begin by explaining (1) in a quite general setting. Let F be a totally real number field and G/F be a reductive group which is compact at infinity. Gross [Gro99] defined a notion of algebraic modular forms on G. Let $K = \prod K_v$ be a suitably nice compact open subgroup of $G(\mathbb{A})$. In particular, we assume K_v is a hyperspecial maximal compact subgroup almost everywhere and $K_v = G_v$ for $v|\infty$. Let $\mathcal{A}(G, K)$ denote the space of algebraic modular forms with level K and trivial weight. We may view $\mathcal{A}(G, K)$ as the space of $\mathbb{C}$-valued functions on the finite set $\mathrm{Cl}(K) = G(F)\backslash G(\mathbb{A})/K$. Let $x_1, \ldots, x_h \in G(\mathbb{A})$ be a set of representatives for $\mathrm{Cl}(K)$ and put $w_i = |G(F) \cap x_i K x_i^{-1}|$. On $\mathcal{A}(G, K)$, we consider the inner product $(\phi, \phi') = \sum \frac{1}{w_i}\phi(x_i)\overline{\phi'(x_i)}$. This space has a basis of orthogonal eigenforms for the unramified Hecke algebra. The constant function $\mathbb{1}$ is an eigenform, which we think of as a compact analogue of an Eisenstein series associated to the minimal parabolic of the quasi-split form. Let $\mathcal{A}_0(G, K)$ be the orthogonal complement on $\mathbb{1}$ in $\mathcal{A}(G, K)$. The mass of K is defined to be

$$m(K) = (\mathbb{1}, \mathbb{1}) = \frac{1}{w_1} + \cdots + \frac{1}{w_h}.$$

We say two eigenforms are congruent mod p if their automorphic representations are.

Theorem B. *(Proposition 2.1) If $p|m(K)$, then there exists an eigenform $\phi \in \mathcal{A}_0(G, K)$ which is Hecke congruent to $\mathbb{1}$ mod p.*

Explicit mass formulas have been computed in a wide variety of settings—e.g., see [Shi06] and [GHY01]. We explicate these mass formulas in a number of cases below. For the relevant inner forms and compact open subgroups for Theorem A*, the mass is $(2^n n!)^{-1} \prod_{r=1}^{n-1}(\ell^r - 1)$ times the product of Bernoulli numbers that appears in the theorem.

Now we briefly explain step (2). Theorem B gives us Eisenstein congruences on definite unitary groups. To obtain Theorem A*, we work with an inner form

G of $\mathrm{U}(n)$ which is compact at infinity and compact mod center at ℓ, i.e., G is unitary group over a division algebra. By comparing the endoscopic classification of discrete L^2 automorphic representations of G with those of the quasi-split form $\mathrm{U}(n)$, one gets a transfer of automorphic representations of G to those of $\mathrm{U}(n)$. Since G is compact mod center at ℓ and n is prime, if π is a non-abelian (not 1-dimensional) automorphic representation of G, the transfer to G' must be non-endoscopic and have cuspidal base change to $\mathrm{GL}_n(\mathbb{A}_E)$. In this case, there are no abelian automorphic representations occurring in $\mathcal{A}_0(G, K)$, which gives Theorem A*. For definite unitary groups associated to a general CM extension E/F, one can refine Theorem B to obtain a non-abelian ϕ under the condition that $p|\frac{m(K)}{n|\mathrm{Cl}(\mathrm{U}_1(E/F))|}$. See Theorem 5.7 for a precise statement.

The endoscopic classification results that we use were obtained (conditional on stabilization of trace formulas) in [Mok15] for $\mathrm{U}(n)$ and were announced in [KMSW] for inner forms. However, the proof for the case of inner forms, while known in many situations, is still work in progress, and we assume this classification in Theorem A*. For $n = 3$, the endoscopic classification was completed for all inner forms in [Rog90], and thus our results are unconditional at least for $n = 3$.

On the other hand, if one carries out this procedure for unitary groups over fields (so not compact at a finite place for $n > 2$), in all cases we have checked it appears that all Eisenstein congruences coming from the mass formula can also be explained as endoscopic lifts of $\mathrm{GL}(2)$ Eisenstein congruences in higher weight. In particular, we suggest that one *only* sees weight $\leq k$ $\mathrm{GL}(2)$ Eisenstein congruences in mass formulas for suitable $\mathrm{U}(k)$ ($k \geq 2$). We use this to formulate precise conjectures about weight k Eisenstein congruences on $\mathrm{GL}(2)$ which generalize existing results. See Conjecture 5.14 and Remark 5.15. However, at least for small p, one does not see all weight k $\mathrm{GL}(2)$ Eisenstein congruences in mass formulas for $\mathrm{U}(k)$ (see Example 5.16).

We also consider Eisenstein congruences coming from mass formulas for groups not of type A_n: specifically symplectic and odd orthogonal groups (with a focus on $\mathrm{SO}(5)$) and G_2. Again, in these, it appears that such congruences are also explained by endoscopic lifts. (These groups have no inner forms which are compact at a finite place.) For instance, if $G' = \mathrm{SO}(5) \simeq \mathrm{PGSp}(4)$, the Eisenstein congruences we obtain correspond to scalar weight 3 Siegel modular forms, and they appear to simply be Saito–Kurokawa lifts of Eisenstein congruences of weight 4 modular forms on $\mathrm{GL}(2)$. Again, this suggests some refinements of known results on Eisenstein congruences for $\mathrm{GL}(2)$. See for instance Conjecture 6.6.

Now we briefly outline the contents of our paper.

In Section 2, we give a general treatment of (trivial weight) algebraic modular forms on reductive groups compact at infinity and congruences via mass formulas. In Section 3, we discuss local Hecke algebras and the local versions of Eisenstein congruences for the groups we will consider. In Section 4, we explain how, under certain conditions, these local results let us obtain Eisenstein congruences on unitary groups via functorial liftings (endoscopic and symmetric powers) of Eisenstein congruences of smaller groups.

In Section 5, we first explain how to obtain Eisenstein congruences on $\mathrm{U}(n)$ for n prime which are not endoscopic. Then we investigate the case of definite unitary groups over fields with some examples leading to Conjecture 5.14. In Section 6 we discuss orthogonal and symplectic groups, with a focus on $\mathrm{SO}(5)$, and then consider G_2 in Section 7. In Section 8 we use the techniques here to slightly refine our results on weight 2 Eisenstein congruences for $\mathrm{GL}(2)$ from [Mar17].

Finally, in Section 9 we show that if π is a cuspidal representation of $\mathrm{U}(p)$ with trivial central character such that π_v is an unramified twist of Steinberg at some finite v, there exists a cuspidal π' on $\mathrm{U}(p)$ with the same level structure as π which is Hecke congruent to π mod p and π'_v is Steinberg at v. This is a higher rank analogue of a mod 2 congruence result on $\mathrm{GL}(2)$ from [Mar17].

Notation. Excluding the local section Section 3, F will denote a number field, $\mathfrak{o} = \mathfrak{o}_F$ its ring of integers, $\mathbb{A} = \mathbb{A}_F$ its adele ring, and v a place of F. We also denote the finite adeles by $\mathbb{A}_f$ and put $\hat{\mathfrak{o}} = \prod_{v<\infty} \mathfrak{o}_v$. At a finite place v, we denote by $\mathfrak{p}_v$ the prime ideal and q_v the size of the residue field.

For a group G, we denote its center by $Z(G)$, or just by Z if G is understood. For an algebraic group G over F, we often write G_v for $G(F_v)$. By an automorphic representation, by default we mean an irreducible L^2-discrete automorphic representation.

Finally p will typically denote our congruence prime. To denote other primes, we generally use v to denote other primes, or ℓ or q when $F = \mathbb{Q}$.

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2. Congruences from mass formulas

Let F be a totally real number field. Let G be a connected reductive linear algebraic group over F such that G_∞ is compact. Let $K = \prod K_v$ be an open compact subgroup of $G(\mathbb{A})$ such that $K_v = G_v$ for $v|\infty$. For later use of the theory of Hecke operators, we will also assume K_v is a hyperspecial maximal compact subgroup for all v outside of a finite set of places S, which contains all infinite places.

Fix a nonzero Haar measure dg on $G(\mathbb{A})$ which is a product of local Haar measures dg_v. The mass of K is defined to be

$$m(K) = \frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A}), dg)}{\mathrm{vol}(K, dg)}. \tag{2.1}$$

(As usual, we give the discrete subgroup $G(F)$ the counting measure and the volume of the quotient $G(F)\backslash G(\mathbb{A})$ really means with respect to the quotient measure.) This is nonzero, finite, and independent of the choice of dg. Note that if $K' \subset K$ is also a compact open subgroup, then $m(K') = [K : K']m(K)$.

Consider the classes $\mathrm{Cl}(K) = G(F)\backslash G(\mathbb{A})/K$. We identify $\mathrm{Cl}(K)$ with a set of representatives $\{x_1, \dots, x_h\}$, where $x_i \in G(\mathbb{A})$. Note $\mathrm{vol}(G(F)\backslash G(F)x_iK, dg) = \frac{1}{w_i}\mathrm{vol}(K, dg)$ where $w_i = |G(F) \cap x_iKx_i^{-1}|$. Thus we can also express the mass as

$$m(K) = \frac{1}{w_1} + \cdots + \frac{1}{w_h}. \tag{2.2}$$

Consequently, $m(K) \in \mathbb{Q}$.

If G is a unitary, symplectic or orthogonal group, and K is the stabilizer of a lattice Λ, then this mass corresponds to the classical mass of Λ. Mass formulas have been calculated in a considerable amount of generality in many works, e.g. see [GHY01] or [Shi06]. We will explicate these in some cases below.

2.1. Algebraic modular forms.

The basic theory of algebraic modular forms was developed in [Gro99]. Below, we will review aspects necessary for our applications.

We define the space of algebraic modular forms on $G(\mathbb{A})$ with level K and trivial weight to be

$$\mathcal{A}(G, K) = \{\phi : \mathrm{Cl}(K) \to \mathbb{C}\}. \tag{2.3}$$

As $\mathcal{A}(G, K) \subset L^2(G(F)\backslash G(\mathbb{A}))$, we can decompose this space as

$$\mathcal{A}(G, K) = \bigoplus \pi^K, \tag{2.4}$$

where π runs over irreducible automorphic representations of $G(\mathbb{A})$ with trivial infinity type. If $\pi^K \neq 0$, we will say π occurs in $\mathcal{A}(G, K)$. Since $G(F)\backslash G(\mathbb{A})$ is compact, $L^2(G(F)\backslash G(\mathbb{A}))$ decomposes discretely and each π above is finite dimensional. The usual inner product on $L^2(G(F)\backslash G(\mathbb{A}))$ restricts to an inner product on $\mathcal{A}(G,K)$, which after suitable normalization we can take to be

$$(\phi,\phi') = \sum \frac{1}{w_i}\phi(x_i)\overline{\phi'(x_i)}.$$

Let Z denote the center of G and $K_Z = K \cap Z(\mathbb{A})$. Note that $\mathrm{Cl}(K_Z) = Z(F)\backslash Z(\mathbb{A})/K_Z$ acts on elements of $\mathcal{A}(G,K)$ by (left or right) multiplication. Let $\omega : \mathrm{Cl}(K_Z) \to \mathbb{C}$ be a "class character".[1] Define the space of algebraic modular forms with central character ω, level K and trivial weight to be

$$\mathcal{A}(G,K;\omega) = \{\phi \in \mathcal{A}(G,K) : \phi(zg) = \omega(z)\phi(g) \text{ for } z \in Z(\mathbb{A}),\ g \in G(\mathbb{A})\}.$$

By decomposing $\mathcal{A}(G,K)$ with respect to the action of $\mathrm{Cl}(K_Z)$, we obtain a decomposition

$$\mathcal{A}(G,K) = \bigoplus_{\omega} \mathcal{A}(G,K;\omega),$$

where ω runs over characters of $\mathrm{Cl}(K_Z)$. We also have decompositions of the form (2.4) for each $\mathcal{A}(G,K;\omega)$, where now one runs over π with central character ω.

If $\chi : G(\mathbb{A}) \to \mathbb{C}$ is a 1-dimensional representation and $\ker\chi \supset G(F)K$, then we may view χ as an element of $\mathcal{A}(G,K)$. In particular, the space for trivial representation is the span of the constant function $\mathbb{1} \in \mathcal{A}(G,K)$. Define the codimension 1 subspace

$$\mathcal{A}_0(G,K) = \{\phi \in \mathcal{A}(G,K) : (\phi,\mathbb{1}) = 0\},$$

and put $\mathcal{A}_0(G,K;\omega) = \mathcal{A}(G,K,\omega) \cap \mathcal{A}_0(G,K)$. We say $\phi \in \mathcal{A}(G,K)$ is non-abelian if it is not a linear combination of 1-dimensional representations of $G(\mathbb{A})$.

In the special case $G = B^\times$, where B is a definite quaternion algebra over $F = \mathbb{Q}$ and K is the multiplicative group of an Eichler order of level N, then $\mathbb{1} \in \mathcal{A}(G,K)$ corresponds to a weight 2 Eisenstein series on $\mathrm{GL}(2)$ and the Jacquet–Langlands correspondence gives a Hecke isomorphism of $\mathcal{A}_0(G,K)$ with the subspace of $S_2(N)$ which are p-new for p ramified in B. However, in general $\mathcal{A}_0(G,K)$ may contain many abelian forms, as well as many non-abelian forms ϕ (even in π^K for some π occurring in $\mathcal{A}_0(G,K)$) which do not correspond to cusp forms via a generalized Jacquet–Langlands correspondence. For higher rank G, it is a difficult problem to describe the set of ϕ which correspond to cusp forms on the quasi-split form of G.

Finally, for a subring $\mathcal{O}$ of $\mathbb{C}$, and a space of algebraic modular forms (e.g., $\mathcal{A}(G,K)$) we denote with a superscript $\mathcal{O}$ (e.g., $\mathcal{A}^{\mathcal{O}}(G,K)$) the subring of $\mathcal{O}$-valued algebraic modular forms.

[1] If we relax our compact at infinity condition to compact mod center at infinity, and suppose $Z = \mathrm{GL}(1)$ and $K_Z = \hat{\mathfrak{o}}_F^\times \times F_\infty^\times$, this is just an ideal class character of F.

2.2. **Hecke operators.** For $g \in G(\mathbb{A})$ and $\phi \in \mathcal{A}(G,K)$, we define the Hecke operator

$$(T_g\phi)(x) = \sum \phi(xg_i), \quad KgK = \coprod g_iK. \tag{2.5}$$

By right K-invariance of ϕ, this is independent of the choice of representatives g_i in the coset decomposition $KgK = \coprod g_iK$, and $T_{g'} = T_g$ if $g' \in KgK$. Clearly each π^K is stable under T_g for each π occurring in $\mathcal{A}(G,K)$. In particular, T_g acts on the subspaces $\mathcal{A}_0(G,K)$ and $\mathcal{A}_0(G,K,\omega)$.

We also note that each T_g is integral in the sense that, viewing each ϕ as a column vector $(\phi(x_i)) \in \mathbb{C}^h$, the action is given by left multiplication by an integral matrix in $M_h(\mathbb{Z})$. Consequently, for any subring $\mathcal{O} \subset \mathbb{C}$, T_g restricts to an operator on $\mathcal{A}^{\mathcal{O}}(G,K)$ (and similarly, $\mathcal{A}_0^{\mathcal{O}}(G,K)$, etc.). Moreover, all eigenvalues for T_g are algebraic integers.

Consider a representation $\pi = \bigotimes' \pi_v$ occurring in $\mathcal{A}(G,K)$. Take $v \notin S$. Then π_v is K_v-spherical, and $\dim \pi_v^{K_v} = 1$. Viewing $g_v \in G(F_v)$ as an element of $G(\mathbb{A})$ which is g_v at v and 1 at all other places, we can consider the (global) Hecke operator T_{g_v}. Then T_{g_v} acts by a scalar on $\pi_v^{K_v}$, and hence is diagonalizable on $\mathcal{A}(G,K)$.

In fact, the T_{g_v}'s are simultaneously diagonalizable for all $v \notin S$ and all $g_v \in G(F_v)$. Specifically, let us call any nonzero $\phi \in \mathcal{A}(G,K)$ such that $\phi \in \pi^K$ for some π an eigenform. Such a ϕ is a simultaneous eigenform for all T_{g_v}'s with $v \notin S$. We denote the corresponding eigenvalue by $\lambda_{g_v}(\phi)$. Then any basis of $\mathcal{A}(G,K)$ of eigenforms simultaneously diagonalizes the T_{g_v}'s ($v \notin S$).

Note that $\mathbb{1}$ is always an eigenform, and $\lambda_{g_v}(\mathbb{1})$ is the degree of T_{g_v}, i.e., the number of g_i's occurring in the decomposition $Kg_vK = \coprod g_iK$, which equals $\mathrm{vol}(K_vg_vK_v)/\mathrm{vol}(K_v)$.

2.3. **Congruences.** Let $\phi, \phi' \in \mathcal{A}(G,K)$ be eigenforms. We say ϕ and ϕ' are Hecke congruent mod p (away from S) if, for all $v \notin S$ and all $g_v \in G(F_v)$, $\lambda_{g_v}(\phi) \equiv \lambda_{g_v}(\phi') \bmod \mathfrak{p}$, where $\mathfrak{p}$ is a prime of some finite extension of $\mathbb{Q}$.

For a subring $\mathcal{O} \subset \mathbb{C}$, ideal $\mathfrak{n}$ in $\mathcal{O}$ and $\phi_1, \phi_2 \in \mathcal{A}^{\mathcal{O}}(G,K)$, we write $\phi_1 \equiv \phi_2 \bmod \mathfrak{n}$ if $\phi_1(x_i) \equiv \phi_2(x_i) \bmod \mathfrak{p}$ for all $x_i \in \mathrm{Cl}(K)$. Note if ϕ_1 and ϕ_2 are eigenforms (or just mod $\mathfrak{p}$ eigenforms), then $\phi_1 \equiv \phi_2 \bmod \mathfrak{p}$ implies ϕ_1 and ϕ_2 are Hecke congruent mod $\mathfrak{p}$.

In addition, if $\alpha \in \mathbb{Q}$, by $p|\alpha$ we mean p divides the numerator of α.

Proposition 2.1. *Suppose $p|m(K)$. Then there exists an eigenform $\phi \in \mathcal{A}_0(G,K)$ which is Hecke congruent to $\mathbb{1}$ mod p.*

Proof. One can use the same arguments as those given for GL(2) in [Mar17] and [Mar18b]. In fact we give a slightly more refined argument than what we need for this proposition in order to use it later in Section 5.3.

Let $r = v_p(m(K)) \geq 1$. The first step is note that there exists a $\mathbb{Z}$-valued $\phi' \in \mathcal{A}_0^{\mathbb{Z}}(G,K)$ such that $\phi' \equiv \mathbb{1} \bmod p^r$, i.e., $\phi'(x_i) \equiv 1 \bmod p^r$ for $i = 1, \dots, h$. To see this, consider $\phi' \in \mathcal{A}^{\mathbb{Z}}(G,K)$ such that each $\phi'(x_i) = 1 + p^r a_i$ for some $a_i \in \mathbb{Z}$. We claim we can choose the a_i's so that $(\phi', \mathbb{1}) = 0$, i.e., $p^r \sum \frac{a_i}{w_i} = -\sum \frac{1}{w_i} = -m(K)$. Let $w = \prod w_i$ and $w_i^* = \frac{w}{w_i}$. Then we want $a_i \in \mathbb{Z}$ such that $\sum a_i w_i^* = -w\frac{m(K)}{p^r}$. Note that $p^j | w_i^*$ for some i implies $p^j | w$ and thus $p^{j+r} | wm(K)$. Thus $\gcd(w_1^*, \dots, w_h^*) | w\frac{m(K)}{p^r}$, and we may choose the a_i's as claimed.

Take such a ϕ', which is a mod p eigenform. Now we want to pass from ϕ' to an eigenform ϕ which is Hecke congruent to ϕ' mod p. For this, one can either use the Deligne–Serre lifting lemma as in the proof of [Mar18b, Theorem 5.1] or the reduction argument as in proof of [Mar17, Theorem 2.1]. Specifically, the subsequent Lemma 2.2 is a slight refinement of the latter, and applying it with $\mathcal{O} = \mathbb{Z}$, $\phi_1 = \mathbb{1}$, $\phi_2 = \phi'$ and $W = \mathcal{A}_0(G,K)$ gives the desired ϕ. ☐

Lemma 2.2. *Let $\mathcal{O}$ be the ring of integers of a number field L, and $\mathfrak{p}$ a prime of $\mathcal{O}$ above a rational prime p. Let $\phi_1 \in \mathcal{A}^{\mathcal{O}}(G,K)$ be an eigenform. Let W be a Hecke-stable subspace of $\mathcal{A}(G,K)$. Suppose there exists $\phi_2 \in \mathcal{A}^{\mathcal{O}}(G,K)$ such that $\phi_2 \equiv \phi_1 \bmod \mathfrak{p}$ and ϕ_2 has nonzero projection to W. Then there exists an eigenform $\phi \in W$ such that ϕ is Hecke congruent to $\phi_1 \bmod p$ for all Hecke operators T_g.*

Proof. Enlarge L if necessary to assume that $\mathcal{A}^{\mathcal{O}}(G,K)$ contains a basis of eigenforms $\psi_1, \dots, \psi_h$. Let Φ denote the collection of $\phi \in \mathcal{A}^{\mathcal{O}}(G,K)$ such that ϕ is congruent to a nonzero multiple of $\phi_1 \bmod \mathfrak{p}$ and ϕ has nonzero projection to W. The hypothesis on ϕ_2 means $\Phi \neq \emptyset$. Let m be minimal such that, after a possible reordering of $\psi_1, \dots, \psi_h$, there exists $\phi = c_1\psi_1 + \cdots + c_m\psi_m \in \Phi$ with each $c_i \in L^\times$ and $\psi_1 \in W$. Take such a ϕ.

Fix any Hecke operator T_g, and put $\phi' = [T_g - \lambda_g(\psi_1)]\phi$. Then note that

$$\phi' \equiv (\lambda_g(\phi_1) - \lambda_g(\psi_1))\phi \mod \mathfrak{p}.$$

Hence $\phi' \in \Phi$ unless $\lambda_g(\psi_1) \equiv \lambda_g(\phi_1) \bmod \mathfrak{p}$. But ϕ' is of the form $c_2'\psi_2 + \cdots + c_m'\psi_m$ for some $c_i' \in L^\times$. Thus $\phi' \notin \Phi$ by minimality of m. Consequently, $\lambda_g(\psi_1) \equiv \lambda_g(\phi_1) \bmod \mathfrak{p}$ for all g, and we may take ψ_1 for our desired ϕ. ☐

Remark 2.3. Let $\mathfrak{p}$ be the prime above p in a sufficiently large extension of $\mathbb{Q}_p$, with ramification index e. The work [BKK14] considers the notion of *depth* of congruences, which is $\frac{1}{e}$ times the number of Hecke eigensystems satisfying a congruence mod $\mathfrak{p}$ counted with multiplicity (a congruence mod $\mathfrak{p}^r$ means multiplicity r). Combining this theorem with Proposition 4.3 of *op. cit.* gives a lower bound on the depth of congruences of $v_p(m(K))$.

We can also guarantee the existence of such a ϕ with trivial central character.

Corollary 2.4. *Set $\bar{G} = G/Z$. Suppose that $\bar{G}(k) = G(k)/Z(k)$ holds for any field k of characteristic zero, and $p | \frac{m(K)}{m(K_Z)}$. Then there exists an eigenform $\phi \in \mathcal{A}_0(G,K;1)$ which is Hecke congruent to $\mathbb{1} \bmod p$.*

Proof. Let $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Then $\mathcal{A}_0(G, K; 1)$ may be identified with $\mathcal{A}_0(\bar{G}, \bar{K})$. Now note that $m(\bar{K}) = \frac{m(K)}{m(K_Z)}$, and apply the proposition to $\mathcal{A}(\bar{G}, \bar{K})$. $\square$

The assumption for $\bar{G}$ in Corollary 2.4 is satisfied when G is a unitary group of odd degree.

Ideally, we would like to be able to say when there is such a ϕ which is non-abelian, or more generally non-endoscopic. This topic is the focus of the remainder of the paper.

3. Local congruences

3.1. **Algebraic groups, L-parameters and Satake parameters.** We recall well-known facts and results for L-parameters of discrete series of real groups, and Stake transforms and parameters of spherical representations of p-adic groups. For details of L-parameters over the real field, we refer to [Bor79], [Kot90, Section 7], [LP02], and [CR15]. For details of spherical representations over p-adic fields, we refer to [Bor79, Chapter II], [Car79, Chapter III], [Gro98], [Mín11] and [Sat63].

3.1.1. *Some quasi-split reductive groups.* Let F be a field of characteristic zero. Assume that E is either $F \times F$ or a quadratic extension of F. When E is a quadratic extension of F, we write σ for the non-trivial element in $\mathrm{Gal}(E/F)$. When $E = F \times F$, we define $\sigma(x, y) = (y, x)$ for $x, y \in F$. Set $\overline{z} = \sigma(z)$ $(z \in E)$ and let Φ_n denote an $n \times n$ matrix with alternating ± 1's on the anti-diagonal and zeros elsewhere. A quasi-split unitary group $\mathrm{U}(n) = \mathrm{U}_{E/F}(n)$ of degree n over F is defined by

$$\mathrm{U}(n) = \mathrm{U}_{E/F}(n) := \{g \in \mathrm{Res}_{E/F}\,\mathrm{GL}(n) \mid g\Phi_n {}^t\overline{g} = \Phi_n\}. \tag{3.1}$$

Note that $\mathrm{U}(n, E) \simeq \mathrm{GL}(n, E)$. If $E = F \times F$, then $\mathrm{U}(n)$ is isomorphic to $\mathrm{GL}(n)$ over F.

A split symplectic similitude group $\mathrm{GSp}(4)$ over F is defined by

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) \mid \exists \mu(g) \in \mathbb{G}_m \text{ s.t. } gJ{}^tg = \mu(g)J\}, \qquad J := \begin{pmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{pmatrix}.$$

An exceptional split simple algebraic group G_2 over F is defined by an automorphism group of a split octonion over F, see [SV00]. It is known that G_2 is realized as a closed subgroup in a split special orthogonal group $\mathrm{SO}(Q)$, where Q is a symmetric matrix of degree seven. We give such a realization which is a slight modification of that in [CNP98, Section 4.2]. Let $Q := E_{71} + E_{62} + E_{53} + 2E_{44} + E_{35} + E_{26} + E_{17}$ where E_{ij} denotes the matrix unit of (i, j) in M_7. Consider the vector space $\mathfrak{t}$ (a Cartan subalgebra) given by

$$\mathfrak{t} := \langle H_1 := \mathrm{diag}(0, -1, 1, 0, -1, 1, 0), \quad H_2 := \mathrm{diag}(-1, -1, 0, 0, 0, 1, 1)\rangle.$$

We may assume that the fundamental roots α and β satisfy $\alpha(H_1) = 1$, $\alpha(H_2) = 0$, $\beta(H_1) = -2$, $\beta(H_2) = -1$ (α is short, β is long). Then $\mathrm{Lie}(G_2)$ is generated by H_1, H_2, $X_\alpha := E_{12} - E_{34} + 2E_{45} - E_{67}$, $X_\beta := -E_{23} + E_{56}$, and $X_{-3\alpha-2\beta} :=$

$E_{61}-E_{71}$. In particular, these give a Chevalley basis that we can use to construct G_2 as a closed subgroup of $\mathrm{SO}(Q)$. Let T denote the split torus of G_2 over F such that $\mathfrak{t}=\mathrm{Lie}(T)$. We choose an isomorphism $T\cong\mathbb{G}_m\times\mathbb{G}_m$ over F corresponding to basis elements H_1 and H_2, namely $\alpha(a,b)=a$, $\beta(a,b)=a^{-2}b^{-1}$, where (a,b) is identified with $\mathrm{diag}(b^{-1},a^{-1}b^{-1},a,1,a^{-1},ab,b)$.

3.1.2. *Discrete series of* $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(n,\mathbb{R})$. The group $W_{\mathbb{R}}$ is given by $W_{\mathbb{R}}=\mathbb{C}^\times\sqcup\mathbb{C}^\times j$, $j^2=-1$, $zj=j\overline{z}$ $(z\in\mathbb{C}^\times)$. Let $\psi:W_{\mathbb{R}}\to\mathrm{GL}(n,\mathbb{C})\rtimes W_{\mathbb{R}}$ denote a tempered discrete relevant L-parameter of $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(n,\mathbb{R})$ which is of the form

(3.2)
$$\psi(z)=\mathrm{diag}((z/\overline{z})^{l_1+\frac{n+1}{2}-1},(z/\overline{z})^{l_2+\frac{n+1}{2}-2},\dots,(z/\overline{z})^{l_n+\frac{n+1}{2}-n})\rtimes z,\quad \psi(j)=\Phi_n\rtimes j$$

where $l_1,\dots,l_n\in\mathbb{Z}$ and $l_1\geq l_2\geq\cdots\geq l_n$. Indeed, ψ corresponds to a L-packet of discrete series of $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(n,\mathbb{R})$. When $l_1+\cdots+l_n=0$, the image of ψ is contained in $\mathrm{SL}(n,\mathbb{C})$, namely the central character of the corresponding discrete series is trivial. In particular, if $(l_1,\dots,l_n)=(0,\dots,0)$, then the L-parameter ψ corresponds to the L-packet of discrete series of $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(n,\mathbb{R})$ with the same infinitesimal character as that of the trivial representation.

3.1.3. *Discrete series of* $\mathrm{GL}(2,\mathbb{R})$. For each $l\in\mathbb{Z}$, we write $\psi_l:W_{\mathbb{R}}\to\mathrm{GL}(2,\mathbb{C})$ for a tempered discrete relevant L-parameter of $\mathrm{GL}(2,\mathbb{R})$ which is of the form

$$\psi_l(z)=\begin{pmatrix}(z/\overline{z})^{l/2}&0\\0&(z/\overline{z})^{-l/2}\end{pmatrix},\quad \psi_l(j)=\begin{pmatrix}0&(-1)^l\\1&0\end{pmatrix}. \tag{3.3}$$

Note that $\psi_l\cong\psi_{-l}$ up to conjugation. If $l\neq0$, then ψ corresponds to the discrete series of $\mathrm{GL}(2,\mathbb{R})$ including $[e^{i\theta}\mapsto e^{i(l+1)\theta}]$ as a minimal $\mathrm{SO}(2)$-type.

3.1.4. *Discrete series of* $\mathrm{GSp}(4,\mathbb{R})$. Choose an element $(a,b)\in\mathbb{Z}^{\oplus2}$ such that

$$a\equiv b\mod 2.$$

We denote by $\psi_{a,b}:W_{\mathbb{R}}\to\mathrm{GSp}(4,\mathbb{C})$ a tempered discrete relevant L-parameter of $\mathrm{GSp}(4,\mathbb{R})$ which is of the form

(3.4)
$$\psi_{a,b}(z)=\begin{pmatrix}(z/\overline{z})^{a/2}&0&0&0\\0&(z/\overline{z})^{b/2}&0&0\\0&0&(z/\overline{z})^{-a/2}&0\\0&0&0&(z/\overline{z})^{-b/2}\end{pmatrix},\quad \psi_{a,b}(j)=\begin{pmatrix}0&0&(-1)^a&0\\0&0&0&(-1)^b\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Note that $\psi_{a,b}\cong\psi_{b,a}$ and $\psi_{a,b}\cong\psi_{\pm a,\pm b}$ up to conjugation. If $a>b>0$, then the L-parameter $\psi_{a,b}$ corresponds to a holomorphic discrete series of $\mathrm{GSp}(4,\mathbb{R})$ with minimal $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(2,\mathbb{R})$-type $\det^{\frac{a-b}{2}+2}\otimes\mathrm{Sym}_{b-1}$. In particular, $\psi_{3,1}$ corresponds to the trivial representation of the compact real form of $\mathrm{GSp}(4)$. The elliptic endoscopic group $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\{(x,x^{-1})\mid x\in\mathbb{G}_m\}$ comes from the centralizer of $\mathrm{diag}(1,-1,1,-1)$ in $\mathrm{GSp}(4,\mathbb{C})$.

3.1.5. *Discrete series of* $G_2(\mathbb{R})$. Recall the notations Q, α, β, and T for G_2 given in Section 3.1.1. Choose an element $(a,b) \in (2\mathbb{Z})^{\oplus 2}$. We denote by $\tilde{\psi}_{a,b} : W_{\mathbb{R}} \to G_2(\mathbb{C})$ a tempered discrete relevant L-parameter of $\mathrm{GSp}(4,\mathbb{R})$ which is of the form

$$\tilde{\psi}_{a,b}(z) = ((z/\overline{z})^{a/2}, (z/\overline{z})^{b/2}) \in T(\mathbb{C}) \subset G_2(\mathbb{C}) \subset \mathrm{SO}(Q,\mathbb{C})$$

and $\tilde{\psi}_{a,b}(j) = Q - E_{44}$. Note that we have $\tilde{\psi}_{a,b} \cong \tilde{\psi}_{-a,a+b}$ and $\tilde{\psi}_{a,b} \cong \tilde{\psi}_{-a-b,b}$ up to conjugation. When $a > b > 0$, the L-parameter $\tilde{\psi}_{a,b}$ corresponds to an L-packet of a discrete series for $G_2(\mathbb{R})$. In particular, $\tilde{\psi}_{4,2}$ corresponds to the trivial representation of the compact real form of G_2, and also corresponds to the quaternionic discrete series of $G_2(\mathbb{R})$ whose minimal $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\{\pm 1\}$-type is $\mathrm{Sym}_4 \otimes 1$. The elliptic endoscopic group $\mathrm{SO}(2,2)$ of G_2 arises from the centralizer of $(1,-1)$ in $G_2(\mathbb{C})$. For odd integers c and d, the L-embedding of $\psi_c \oplus \psi_d$ for $\mathrm{SO}(4,\mathbb{C}) = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\{\pm 1\}$ is $\tilde{\psi}_{c+d,c-d} (\cong \tilde{\psi}_{d-c,2c} \cong \tilde{\psi}_{2c,d-c} \cong \cdots)$.

3.1.6. *Satake parameters.* Now we review Satake transforms and parameters of spherical representations. Let F be a nonarchimedean local field of characteristic 0, and G a connected reductive group over F. For simplicity, we write G for the group $G(F)$ of F-rational points in G. Suppose that G is unramified. Then G has a hyperspecial maximal compact subgroup K and a Borel subgroup B containing a maximal F-torus T. We have an Iwasawa decomposition $G = BK$ and a Cartan decomposition $G = KTK$. Also, $T \cap K$ is the maximal compact subgroup of T. Denote by $X^{\mathrm{un}}(T)$ the group of unramified characters of T, i.e., characters trivial on $T \cap K$. We have $T/(T \cap K) \cong \mathbb{Z}^n$, where n is the rank of G. The Weyl group $W := N_G(T)/T$ acts on $X^{\mathrm{un}}(T)$ via $(w\chi)(t) = \chi(w^{-1}tw)$ for $t \in T$, $w \in W$. As usual, $N_G(T)$ denotes the normalizer of T in G.

We write $\mathcal{H}(G,K)$ for the spherical Hecke algebra of G and K, that is,

$$\mathcal{H}(G,K) := \{ f \in C_c^\infty(G) \mid f(hxh') = f(x) \quad \text{for } x \in G, \quad h, h' \in K \}.$$

The convolution $(f_1 * f_2)(x) := \int_G f_1(xy^{-1}) f_2(y)\,\mathrm{d}y$ defines a product on $\mathcal{H}(G,K)$, where $\mathrm{d}y$ is the Haar measure on G normalized by $\int_K \mathrm{d}y = 1$. The Satake transform $\mathscr{S} : \mathcal{H}(G,K) \to \mathcal{H}(T, T \cap K)^W$ is a $\mathbb{C}$-algebra isomorphism defined by

$$\mathscr{S}f(t) := \delta_B(t)^{1/2} \int_N f(tu)\,\mathrm{d}u = \delta_B(t)^{-1/2} \int_N f(ut)\,\mathrm{d}u \qquad (f \in \mathcal{H}(G,K))$$

where δ_B is the modulus function of B, N is the unipotent radical of B, and $\mathrm{d}u$ denotes the Haar measure on N satisfying $\int_{N\cap K} \mathrm{d}u = 1$. Let $\omega_\chi : \mathcal{H}(G,K) \to \mathbb{C}$ be the linear map defined by

$$\omega_\chi(f) := \int_T \mathscr{S}f(t)\,\chi(t)\,\mathrm{d}t \qquad (f \in \mathcal{H}(G,K))$$

where $\mathrm{d}t$ is a Haar measure on T such that $\int_{T\cap K} \mathrm{d}t = 1$. It is known that, for any nonzero $\mathbb{C}$-algebra homomorphism $\lambda : \mathcal{H}(G,K) \to \mathbb{C}$, there is a unique character $\chi \in X^{\mathrm{un}}(T)/W$ such that $\lambda = \omega_\chi$. We denote by $\mathrm{Irr}^K(G)$ the set of equivalence classes of K-spherical irreducible representations of G. Each $\pi \in \mathrm{Irr}^K(G)$ defines a $\mathbb{C}$-algebra homomorphism $\lambda_\pi : \mathcal{H}(G,K) \to \mathbb{C}$ by $\mathrm{Tr}\,\pi(f)$, so we may associate a character $\chi \in X^{\mathrm{un}}(T)/W$ to π via $\omega_\chi = \lambda_\pi$. This gives a bijection from $\mathrm{Irr}^K(G)$ to $X^{\mathrm{un}}(T)/W$.

There is a maximal F-split torus A in T. The inclusion mapping $A \to T$ induces isomorphisms $A/(A\cap K) \cong T/(T\cap K)$ and $X^{\mathrm{un}}(A) \cong X^{\mathrm{un}}(T)$. Choose a basis $(a_1, a_2, \dots, a_n)$ in $A/A\cap K$. For each $\chi \in X^{\mathrm{un}}(A)$, one has an n-tuple

$$\alpha = (\chi(a_1), \chi(a_2), \dots, \chi(a_n)).$$

Write $\omega_\alpha = \omega_\chi$. The n-tuple α determines a unique spherical representation corresponding to ω_α as above, and α is called the Satake parameter. By abuse of notation, we also denote the spherical representation by ω_α. Note that in the case of $\alpha_0 = (\delta_B(a_1)^{1/2}, \dots, \delta_B(a_n)^{1/2})$, ω_{α_0} corresponds to the trivial representation, that is,

$$\omega_{\alpha_0}(f) = \int_G f(g)\,\mathrm{d}g. \tag{3.5}$$

Throughout this section, to consider mod p congruences between Hecke eigenvalues we always assume that, for each double coset KaK ($a \in A$), the Hecke eigenvalue $\omega_\alpha(f_{KaK})$ is an algebraic integer for the characteristic functions f_{KaK} of KaK. Here, mod p is used in the same sense as above (mod $\mathfrak{p}$ for a suitable prime $\mathfrak{p}$ above p).

3.2. **Local congruence lifts.** Now we consider local lifts of Eisenstein congruences. Let F denote a nonarchimedean field of characteristic 0, and $|\ |$ the normalized valuation of F. The integer ring of F is given by $\mathfrak{o} := \{x \in F \mid |x| \leq 1\}$. We choose a prime element ϖ of $\mathfrak{o}$ and put $q := |\mathfrak{o}/\varpi\mathfrak{o}|$, that is, $|\varpi| = q^{-1}$.

3.2.1. *Symmetric functions.* We shall review some symmetric function theory. For details, we refer to [Mac95].

A partition means any sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_j \in \mathbb{Z}_{\geq 0}$, $\lambda_j \geq \lambda_{j+1}$, and $l(\lambda) := |\{j \in \mathbb{Z}_{>0} \mid \lambda_j > 0\}|$ is finite. We call $l(\lambda)$ the length of λ. We write $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j$ holds for all j, where $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$. This relation is a partial ordering on partitions. For each partition λ, we denote by $\lambda^c = (\lambda_1^c, \lambda_2^c, \dots)$ the conjugate of λ, i.e., $\lambda_j^c = |\{k \in \mathbb{Z}_{>0} \mid \lambda_k \geq j\}|$. Set $|\lambda| := \lambda_1 + \lambda_2 + \cdots$. Notice that $\lambda \geq \mu$ does not hold if $|\lambda| < |\mu|$. If a partition λ satisfies $l(\lambda) \leq n$, then we write $\lambda = (\lambda_1, \dots, \lambda_n)$ as a n-tuple.

Let S_n denote the symmetric group of degree n. We write m_λ for the monomial symmetric function of λ, i.e.,

$$m_\lambda(x_1, \dots, x_n) := \sum_{w \in S_n/S_n^\lambda} x_{w(1)}^{\lambda_1} \cdots x_{w(n)}^{\lambda_n},$$

where S_n^λ denotes the stabilizer of λ in S_n. We also write e_r for the r-th elementary symmetric function, i.e.,

$$e_r(x_1,\dots,x_n) := \sum_{1\le m_1<m_2<\cdots<m_r\le n} x_{m_1}x_{m_2}\cdots x_{m_r}.$$

Obviously one has the relation $e_r = m_{(1^r,0^{n-r})}$.

Let $\Lambda_n := \mathbb{Z}[x_1,\dots,x_n]^{S_n}$. It is obvious that m_λ (for $l(\lambda)\le n$) form a $\mathbb{Z}$-basis of Λ_n. By [Mac95, (2.3) on p.20], for a partition $\lambda = (\lambda_1,\dots,\lambda_n)$, there exist non-negative integers $a_{\lambda\mu}$ such that

$$e_{\lambda_1^c}\cdots e_{\lambda_r^c} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu}m_\mu \tag{3.6}$$

where $r=\lambda_1$ and $|\mu|=|\lambda|$. Hence, we also have $\Lambda_n = \mathbb{Z}[e_1,\dots,e_n]$.

3.2.2. $\mathrm{GL}(n)$. Set

$$G := \mathrm{GL}(n,F),\quad G^+ := G\cap M(n,\mathfrak{o}),\quad \text{and}\quad K := \mathrm{GL}(n,\mathfrak{o}).$$

Let $T=T_n$ denote the usual maximal split torus in G consiting of diagonal matrices, and $B=TN$ the standard Borel subgroup consisting of upper triangular matrices with unipotent radical N. We denote by X_j the characteristic function of $\{\mathrm{diag}(x_1,\dots,x_n)\in T \mid x_k\in\mathfrak{o}^\times\ (\forall k\ne j),\ x_j\in\varpi\mathfrak{o}^\times\}$. For $u\in\mathbb{Z}$, X_j^u denotes the characteristic function of $\{\mathrm{diag}(x_1,\dots,x_n)\in T \mid x_k\in\mathfrak{o}^\times\ (\forall k\ne j),\ x_j\in\varpi^u\mathfrak{o}^\times\}$. The Satake transform gives an isomorphism between $\mathcal{H}(G^+,K)$ and $\mathbb{C}[X_1,\dots,X_n]^{S_n}$.

For each partition λ with $l(\lambda)\le n$, we set

$$\varpi^\lambda := \mathrm{diag}(\varpi^{\lambda_1},\varpi^{\lambda_2},\dots,\varpi^{\lambda_n}).$$

The set $\{\varpi^\lambda \mid \lambda \text{ is a partition with } l(\lambda)\le n\}$ is a system of representative elements of $K\backslash G^+/K$. Let c_λ denote the characteristic function of $K\varpi^\lambda K$. The functions c_λ ($l(\lambda)\le n$) form a basis of $\mathcal{H}(G^+,K)$ as a vector space over $\mathbb{C}$. We set

$$\rho=\rho_n := \frac{1}{2}(n-1,n-3,\dots,1-n), \tag{3.7}$$

$$\langle (t_1,\dots,t_n),(t_1',\dots,t_n')\rangle := t_1t_1'+\cdots+t_nt_n'.$$

We write $\widehat{c}_\lambda$ for the Satake transform $\mathscr{S}(c_\lambda)$ of c_λ, that is, $\widehat{c}_\lambda(X)$ is in $\mathbb{C}[X]^{S_n}$ where $X=(X_1,X_2,\dots,X_n)$. It is known that $\widehat{c}_\lambda$ is described by the Hall–Littlewood symmetric function $P_\lambda(x_1,\dots,x_n;t)$, namely $\widehat{c}_\lambda(X) = q^{\langle\lambda,\rho\rangle}P_\lambda(X;q^{-1})$ (see [Mac95, (3.3) on p.299]). The following fact is well known.

Lemma 3.1. *For any partition $\lambda=(\lambda_1,\dots,\lambda_n)$ and any $g\in G$, the element g belongs to $K\varpi^\lambda K$ if and only if $d_r(g) = q^{-\lambda_n-\lambda_{n-1}-\cdots-\lambda_{n-r+1}}$ for all $1\le r\le n$, where $d_r(g)$ denotes the maximum of the valuations of all $r\times r$ minors of g.*

This lemma leads to the equality

$$\int_N c_\lambda(u\varpi^\mu)\,\mathrm{d}u = \begin{cases} 0 & \text{if } |\mu| \neq |\lambda|, \text{ or if } \mu \not\leq \lambda \text{ and } |\mu| = |\lambda|, \\ 1 & \text{if } \mu = \lambda. \end{cases} \tag{3.8}$$

Lemma 3.2. *For any partition λ, $l(\lambda) \leq n$, the symmetric function $\widehat{c}_\lambda$ belongs to the subalgebra $\mathbb{Z}[Z_1, \dots, Z_n]$, where $Z_r := \widehat{c}_{(1^r, 0^{n-r})} = q^{\frac{(n-r)r}{2}} e_r$ $(1 \leq r \leq n)$.*

Proof. It follows from (3.8) and the symmetry of $\widehat{c}_\lambda$ that, for some $a_{1,\lambda\mu} \in \mathbb{Z}_{\geq 0}$, we have

$$\widehat{c}_\lambda = q^{\langle \lambda, \rho \rangle} m_\lambda + \sum_{\mu < \lambda} a_{1,\lambda\mu}\, q^{\langle \mu, \rho \rangle} m_\mu$$

where $|\mu| = |\lambda|$. Note that Lemma 3.1 ensures that, if $\mu > \lambda$, then such m_μ does not appear in the above equality. We have $\langle \lambda - \mu, \rho \rangle \in \mathbb{Z}_{>0}$ for any μ satisfying $\mu < \lambda$ and $|\mu| = |\lambda|$, and $\langle \lambda, \rho \rangle = \sum_{j=1}^{\lambda_1} \langle (1^{\lambda_j^c}, 0^{n-\lambda_j^c}), \rho \rangle$. Hence, this assertion follows from (3.6). $\square$

Take the basis of $T/(T \cap K)$ given by

$$\mathrm{diag}(\varpi, 1, \dots, 1), \ \ \mathrm{diag}(1, \varpi, \dots, 1), \ \ \dots, \ \ \mathrm{diag}(1, 1, \dots, \varpi).$$

As above, write ω_α for the K-spherical representation of G with the Satake parameter $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then, one has

$$\widehat{c}_\lambda(\alpha) = \omega_\alpha(c_\lambda).$$

Set

$$\mathbb{L} = \mathbb{L}_n := \rho + \{(l_1, l_2, \dots, l_n) \in \mathbb{Z}^{\oplus n} \mid l_1 \geq l_2 \geq \cdots \geq l_n\}.$$

Choose an n-tuple $k = (k_1, k_2, \dots, k_n) \in \mathbb{L}$. Here k corresponds to the L-parameter $\mathbb{C}^\times \ni z \mapsto \mathrm{diag}((z/\overline{z})^{k_1}, (z/\overline{z})^{k_2}, \dots, (z/\overline{z})^{k_n}) \in \mathrm{GL}(n, \mathbb{C})$ for the discrete series of $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(n, \mathbb{R})$. We set

$$q^k := (q^{k_1}, q^{k_2}, \dots, q^{k_n}).$$

Since such an L-parameter is associated to the local component of a global representation induced from the Borel, we may think of $\widehat{c}_\lambda(q^k)$ as a local Eisenstein Hecke eigenvalue. One can show $\widehat{c}_\lambda(q^k)$ is in $\mathbb{Z}[q^{-1}]$ using Lemma 3.2. Since ω_{q^ρ} is the trivial representation, one gets

$$\widehat{c}_\lambda(q^\rho) = \mathrm{vol}(K\varpi^\lambda K). \tag{3.9}$$

If there exists a prime number p such that $(p, q) = 1$ and

$$\widehat{c}_\lambda(\alpha) \equiv \widehat{c}_\lambda(q^k) \mod p \tag{3.10}$$

holds for any partition λ, then we say that the spherical representation ω_α satisfies the Eisenstein congruence mod p for the L-parameter (associated to) k. Here, as in the global case, what we mean by this congruence notation is that both quantities lie in some ring of integers $\mathcal{O}$ and they are congruent modulo a prime ideal of $\mathcal{O}$ above p. It follows from (3.9) that ω_α satisfies the Eisenstein congruence mod p for the L-parameter ρ if and only if

$$\widehat{c}_\lambda(\alpha) \equiv \mathrm{vol}(K\varpi^\lambda K) \mod p$$

holds for any partition λ.

Proposition 3.3. *Let ω_α (resp. $\omega_{\alpha'}$) be the spherical representation of $\mathrm{GL}(n,F)$ (resp. $\mathrm{GL}(n',F)$) with the Satake parameter $\alpha = (\alpha_1, \dots, \alpha_n)$ (resp. $\alpha' = (\alpha'_1, \dots, \alpha'_{n'})$). Assume that ω_α (resp. $\omega_{\alpha'}$) satisfies the Eisenstein congruence mod p for $k = (k_1, \dots, k_n)$ (resp. $k' = (k'_1, \dots, k'_{n'})$). For $m = n, n'$, choose $u_m \in \mathbb{Z}$, set*

$$\nu_m := \begin{cases} u_m & \text{if } m \text{ is even,} \\ 1/2 + u_m & \text{if } m \text{ is odd,} \end{cases}$$

and let $\beta_m \in \mathbb{C}^\times$ such that $q^{-\nu_m}\beta_m$ is an algebraic integer with $q^{-\nu_m}\beta_m \equiv 1 \bmod p$. We also set $\alpha\beta_{n'} = (\alpha_1\beta_{n'}, \dots, \alpha_n\beta_{n'})$, $\alpha'\beta_n = (\alpha'_1\beta_n, \dots, \alpha'_{n'}\beta_n)$, $k + \nu_{n'} = (k_1 + \nu_{n'}, \dots, k_n + \nu_{n'})$, and $k' + \nu_n = (k'_1 + \nu_n, \dots, k'_{n'} + \nu_n)$. Suppose that $(k + \nu_{n'}, k' + \nu_n)$ is in $\mathbb{L}_{n+n'}$ by changing its ordering. Then the spherical representation $\omega_{(\alpha\beta_{n'}, \alpha'\beta_n)}$ of $\mathrm{GL}(n+n',F)$ satisfies the Eisenstein congruence mod p for L-parameter $(k + \nu_{n'}, k' + \nu_n)$.

This proposition is relevant for local endoscopic lifts of unitary groups at a split finite place.

Proof. By using Lemma 3.2, it suffices to prove the Eisenstein congruence of $\widehat{c}_{(1^r, 0^{n+n'-r})}(\alpha\beta_{n'}, \alpha'\beta_n)$ for $1 \le r \le n+n'$. It is obvious that there exist integers $a_{2,j,j'}$ such that

$$q^{\frac{(n+n'-r)r}{2}} e_r(\alpha\beta_{n'}, \alpha'\beta_n) =$$
$$\sum_{j+j'=r} a_{2,j,j'} q^{\frac{(n-j)j' + (n'-j')j}{2} + \nu_{n'} j + \nu_n j'} (q^{-\nu_{n'}}\beta_{n'})^j (q^{-\nu_n}\beta_n)^{j'} \, q^{\frac{(n-j)j}{2}} e_j(\alpha) \, q^{\frac{(n'-j')j'}{2}} e_{j'}(\alpha').$$

Hence, the statement follows from the facts that $\frac{(n-j)j' + (n'-j')j}{2} + \nu_{n'} j + \nu_n j' \in \mathbb{Z}$, $q^{\frac{(n-j)j}{2}} e_j(\alpha) \equiv q^{\frac{(n-j)j}{2}} e_r(q^k) \bmod p$, and $q^{\frac{(n'-j')j'}{2}} e_{j'}(\alpha') \equiv q^{\frac{(n'-j')j'}{2}} e_r(q^{k'}) \bmod p$. $\square$

Proposition 3.4. *Let ω_α be the spherical representation of $\mathrm{GL}(2,F)$ with the Satake parameter $\alpha = (\alpha_1, \alpha_2)$. Assume that ω_α satisfies the Eisenstein congruence mod p for $k = (k_1, k_2)$. The spherical representation of $\mathrm{GL}(m+1,F)$ with the Satake parameter $(\alpha_1^m, \alpha_1^{m-1}\alpha_2, \dots, \alpha_2^m)$ satisfies the Eisenstein congruence mod p for L-parameter $(mk_1, (m-1)k_1 + k_2, \dots, mk_2)$. Furthermore, when $k_1 - k_2 \ge m$, the spherical representation of $\mathrm{GL}(2m,F)$ with the Satake parameter $(\alpha_1 q^{\rho_m}, \alpha_2 q^{\rho_m})$ satisfies the Eisenstein congruence mod p for L-parameter $((k_1, \dots, k_1) + \rho_m, (k_2, \dots, k_2) + \rho_m)$.*

Proof. The assertions follow from Lemma 3.2, since it is easy to prove the congruences for Z_r. $\square$

3.2.3. $\mathrm{U}_{E/F}(N)$. Let E be an unramified quadratic extension of F. Then $G = \mathrm{U}_{E/F}(N, F)$ is unramified. Assume that K is a hyperspecial maximal compact subgroup of G. Let T denote the maximal F-torus consisting of diagonal matrices. The torus T includes a maximal F-split torus A. We set $n := [N/2]$, so $N = 2n$ or $2n+1$. Then A is isomorphic to $\mathbb{G}_m^n$ over F. Write X_j for the characteristic function of $\{\mathrm{diag}(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) \in A$ (resp. $\mathrm{diag}(x_1, \dots, x_n, 1, x_n^{-1}, \dots, x_1^{-1}) \in A) \mid x_k \in \mathfrak{o}^\times\ (\forall k \neq j),\ x_j \in \varpi\mathfrak{o}^\times\}$ if $N = 2n$ (resp. $N = 2n+1$). The Satake transform gives an isomorphism between $\mathcal{H}(G, K)$ and $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}]^{W_n}$. Here $W_n = S_n \ltimes \{\pm 1\}^n$ where the -1 in the j-th component acts on X_j by $X_j \mapsto X_j^{-1}$. Hence we obtain the following isomorphism

$$\mathcal{H}(G, K) \cong \mathbb{C}[Y_1, Y_2, \dots, Y_n]^{S_n}$$

where $Y_j = X_j + X_j^{-1}$. Here we have used

$$X_j^k + X_j^{-k} = (X_j + X_j^{-1})^k - \sum_{u=1}^{k-1} \binom{k}{u} X_j^{k-2u}. \tag{3.11}$$

For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $l(\lambda) \leq n$, we set

$$\varpi^\lambda := \begin{cases} \mathrm{diag}(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \dots, \varpi^{\lambda_n}, \varpi^{-\lambda_n}, \dots, \varpi^{-\lambda_1}) & \text{if } N = 2n, \\ \mathrm{diag}(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \dots, \varpi^{\lambda_n}, 1, \varpi^{-\lambda_n}, \dots, \varpi^{-\lambda_1}) & \text{if } N = 2n+1. \end{cases}$$

The set $\{\varpi^\lambda \mid l(\lambda) \leq n\}$ is a system of representative elements of $K \backslash G / K$. Let c_λ denote the characteristic function of $K\varpi^\lambda K$. The functions c_λ with $l(\lambda) \leq n$ form a basis of $\mathcal{H}(G, K)$. We set

$$\widehat{c}_\lambda := \mathscr{S}(c_\lambda), \quad \varrho = \varrho_N := \begin{cases} (2n-1, 2n-3, \dots, 1) & \text{if } N = 2n, \\ (2n, 2n-2, \dots, 2) & \text{if } N = 2n+1. \end{cases}$$

Lemma 3.5. *Set $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ ($Y_j = X_j + X_j^{-1}$). For any partition λ, $l(\lambda) \leq n$, the function $\widehat{c}_\lambda$ is in $\mathbb{Z}[\tilde{Z}_1, \dots, \tilde{Z}_n]$, where*

$$\tilde{Z}_r(X) := q^{Nr - r^2}\, e_r(Y) \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}]^{W_n} \quad (1 \leq r \leq n).$$

We note $Nr - r^2 = \langle (1^r, 0^{n-r}), \varrho \rangle$.

Proof. For each $1 \leq r \leq n$, one can prove

$$\widehat{c}_{(1^r, 0^{n-r})}(X) \in \tilde{Z}_r(X) + \mathbb{Z}[\tilde{Z}_1, \dots, \tilde{Z}_{r-1}] \tag{3.12}$$

by Lemma 3.1, $\mathscr{S}(c_\lambda) \in \mathbb{C}[Y_1, Y_2, \dots, Y_n]^{S_n}$ and a direct calculation for representative elements of left cosets of $K\varpi^\lambda K$. Furthermore, for any partition λ,

$l(\lambda) \leq n$, using Lemma 3.1 and (3.11), one gets

$$\widehat{c}_\lambda(X) = q^{\langle \lambda, \varrho \rangle} m_\lambda(Y) + \sum_{\mu < \lambda,\, |\mu| = |\lambda|} a_{3,\lambda\mu} q^{\langle \mu, \varrho \rangle} m_\mu(Y) + \sum_{\mu < \lambda,\, |\mu| \leq |\lambda| - 2} a_{4,\lambda\mu} q^{\langle \mu, \varrho \rangle} m_\mu(Y)$$

for some $a_{3,\lambda\mu} \in \mathbb{Z}_{\geq 0}$ and $a_{4,\lambda\mu} \in \mathbb{Z}$. Hence the proof is completed by (3.6) and (3.12). □

Set $a_j := \mathrm{diag}(\overbrace{1, \ldots, 1}^{j-1}, \varpi, 1, \ldots, 1, \varpi^{-1}, \overbrace{1, \ldots, 1}^{j-1}) \in A$. Then $a_1, a_2, \ldots, a_n$ form a basis of $A/A \cap K$. For the Satake parameter $\alpha = (\alpha_1, \ldots, \alpha_n)$, let ω_α denote the associated irreducible spherical representation of G. We have $\widehat{c}_\lambda(\alpha) = \omega_\alpha(c_\lambda)$. We shall use the same notations $\rho = \rho_N$, $\mathbb{L} = \mathbb{L}_N$, and q^k as in the previous subsection. Choose an N-tuple $k = (k_1, k_2, \ldots, k_N) \in \mathbb{L}$ and a prime number p such that $(p, q) = 1$. We set

$$\tilde{k} := (k_1 - k_N, k_2 - k_{N-1}, \ldots, k_n - k_{N-n+1}),$$
$$k' := (k_1 + k_N, k_2 + k_{N-1}, \ldots, k_n + k_{N-n+1}).$$

For $t = (t_1, \ldots, t_n)$ and $s = (s_1, \ldots, s_n)$, we set

$$(-1)^t q^s := ((-1)^{t_1} q^{s_1}, \ldots, (-1)^{t_n} q^{s_n}).$$

Then, defining $\tilde{\rho}$ and ρ' in the same manner, one has $\varrho = \tilde{\rho}$, $\rho' = (0, \ldots, 0)$, and

$$\widehat{c}_\lambda((-1)^{\rho'} q^{\tilde{\rho}}) = \mathrm{vol}(K \varpi^\lambda K). \tag{3.13}$$

We say that the spherical representation ω_α satisfies the Eisenstein congruence mod p for the L-parameter (associated to) k if

$$\widehat{c}_\lambda(\alpha) \equiv \widehat{c}_\lambda((-1)^{k'} q^{\tilde{k}}) \mod p \tag{3.14}$$

holds for any partition λ. In particular, ω_α satisfies the Eisenstein congruence mod p for the parameter ρ if and only if

$$\widehat{c}_\lambda(\alpha) \equiv \mathrm{vol}(K \varpi^\lambda K) \mod p$$

holds for any partition λ. Notice that this local Eisenstein congruence was defined from the viewpoint of the L-parameter of ω_α over F (cf. [Mín11, Section 3.2.6]).

Proposition 3.6. *Let ω_α (resp. $\omega_{\alpha'}$) be the spherical representation of $\mathrm{U}_{E/F}(N, F)$ (resp. $\mathrm{U}_{E/F}(N', F)$) with the Satake parameter $\alpha = (\alpha_1, \ldots, \alpha_n)$ (resp. $\alpha' = (\alpha'_1, \ldots, \alpha'_{n'})$), where $n = [N/2]$ (resp. $n' = [N'/2]$). Assume that ω_α (resp. $\omega_{\alpha'}$) satisfies the Eisenstein congruence mod p for $k = (k_1, \ldots, k_N)$ (resp. $k' = (k'_1, \ldots, k'_{N'})$). For a natural number $m = N$, N', choose an integer u_m, and set*

$$\mu_m := \begin{cases} -1 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even,} \end{cases} \qquad \nu_m := \begin{cases} 1/2 + u_m & \text{if } m \text{ is odd,} \\ u_m & \text{if } m \text{ is even.} \end{cases}$$

Further, we put

$$\mu_m(\alpha_1, \ldots, \alpha_n) := (\mu_m \alpha_1, \ldots, \mu_m \alpha_n), \quad (k_1, \ldots, k_n) + \nu_m := (k_1 + \nu_m, \ldots, k_n + \nu_m).$$

Case 1. *Suppose N and N' are not both odd. Suppose that there exists a new sequence $r = (r_1, r_2, \dots, r_{N+N'}) \in \mathbb{L}_{N+N'}$ such that $(r_1, r_2, \dots, r_{N+N'})$ is obtained from $(k + \nu_{N'}, k' + \nu_N)$ by changing the ordering, for any $1 \le j \le [(N+N')/2]$ we have $(r_j, r_{N+N'-j+1}) = (k_u + \nu_{N'}, k_{N-u+1} + \nu_{N'})$ for some $1 \le u \le [N/2]$ or $(r_j, r_{N+N'-j+1}) = (k'_{u'} + \nu_N, k'_{N'-u'+1} + \nu_N)$ for some $1 \le u' \le [N'/2]$. Under these assumptions, the spherical representation $\omega_{(\mu_{N'}\alpha, \mu_N\alpha')}$ of $\mathrm{U}_{E/F}(N+N', F)$ satisfies the Eisenstein congruence mod p for L-parameter r.*

Case 2. *Suppose N and N' are both odd. Suppose that there exists a new sequence $r = (r_1, r_2, \dots, r_{N+N'}) \in \mathbb{L}_{N+N'}$ such that $(r_1, r_2, \dots, r_{N+N'})$ is obtained from $(k + \nu_{N'}, k' + \nu_N)$ by changing the ordering, for any $1 \le j \le (N+N')/2 - 1$ we have $(r_j, r_{N+N'-j+1}) = (k_u + \nu_{N'}, k_{N-u+1} + \nu_{N'})$ for some $1 \le u \le [N/2]$ or $(r_j, r_{N+N'-j+1}) = (k'_{u'} + \nu_N, k'_{N'-u'+1} + \nu_N)$ for some $1 \le u' \le [N'/2]$, and we have $(r_{(N+N')/2}, r_{(N+N')/2+1}) = (k_{(N+1)/2} + \nu_{N'}, k_{(N'+1)/2} + \nu_N)$. Furthermore, we suppose*

$$\widehat{c}_{(1)}(-1) \equiv \widehat{c}_{(1)}((-1)^{f'} q^{\tilde{f}}) \mod p \quad \textit{where } f = (k_{(N+1)/2} + \nu_{N'}, k_{(N'+1)/2} + \nu_N).$$

Under these assumptions, the spherical representation $\omega_{(\mu_{N'}\alpha, \mu_N\alpha', -1)}$ of $\mathrm{U}_{E/F}(N+N', F)$ satisfies the Eisenstein congruence mod p for L-parameter r.

Proof. By Lemma 3.5, it is sufficient to prove the Eisenstein congruence of $\tilde{Z}_r(\mu_{N'}\alpha, \mu_N\alpha')$ $(1 \le \forall r \le n + n')$ and $\tilde{Z}_r(\mu_{N'}\alpha, \mu_N\alpha', -1)$ $(1 \le \forall r \le n + n' + 1)$. Hence, this assertion follows from the same argument as in the proof of Proposition 3.3. Notice that $(-1)^{2\nu_m} = \mu_m$ holds. □

Proposition 3.7. *Let ω_α be the spherical representation of $\mathrm{U}_{E/F}(2, F)$ with the Satake parameter $\alpha = (\alpha_1)$. Assume that ω_α satisfies the Eisenstein congruence mod p for $k = (k_1, k_2)$. Choose a natural number m, and we put $\delta_m := 2$ if m is even, and $\delta_m := 1$ if m is odd. The spherical representation of $\mathrm{U}_{E/F}(m+1, F)$ with the Satake parameter $(\alpha_1^m, \alpha_1^{m-2}, \dots, \alpha_1^{\delta_m})$ satisfies the Eisenstein congruence mod p for L-parameter $(mk_1, (m-1)k_1 + k_2, \dots, mk_2)$. Further, when $k_1 - k_2 \ge m$, the spherical representation of $\mathrm{U}_{E/F}(2m, F)$ with the Satake parameter $(\alpha_1 q^{\rho_m})$ satisfies the Eisenstein congruence mod p for L-parameter $((k_1, \dots, k_1) + \rho_m, (k_2, \dots, k_2) + \rho_m)$.*

Proof. If we put $k = (mk_1, (m-1)k_1 + k_2, \dots, mk_2)$ (resp. $(k_1 + \rho_m, k_2 + \rho_m)$), then $k' = (k_1 + k_2)(m, m, \dots, m)$ and $\tilde{k} = (k_1 - k_2)(m, m-2, \dots, \delta_m)$ (resp. $k' = (k_1 + k_2)(1, 1, \dots, 1)$ and $\tilde{k} = (k_1 - k_2 - m)(1, 1, \dots, 1) + \varrho_{2m}$). Since we have $x^l + x^{-l} = (x + x^{-1})^l + c_1(x + x^{-1})^{l-2} + \cdots$ for some $c_1, \dots \in \mathbb{Z}$, we obtain the assertions. □

3.2.4. $\mathrm{GL}(2)$ *and* $\mathrm{PGL}(2)$. Let $G := \mathrm{GL}(2, F)$, $K := G(\mathfrak{o})$, and let T denote the maximal split torus consisting of diagonal matrices in G. We choose $\mathrm{diag}(\varpi, 1)$ and $\mathrm{diag}(1, \varpi)$ as a basis of the group $T/T \cap K$. For each spherical representation

ω_α with the Satake parameter $\alpha = (\alpha_1, \alpha_2)$, we say that ω_α satisfies the Eisenstein congruence mod p for weight k and character (χ_1, χ_2) if

$$q^{(k-1)/2}(\alpha_1 + \alpha_2) \equiv \chi_1(\varpi)q^{k-1} + \chi_2(\varpi) \mod p, \quad \alpha_1\alpha_2 \equiv \chi_1(\varpi)\chi_2(\varpi) \mod p \tag{3.15}$$

hold, where χ_1 and χ_2 are unramified characters on $F^\times$ and weight $k \in \mathbb{Z}_{\geq 1}$ means the L-parameter $\psi_{(k-1)/2}$ of $\mathrm{GL}(2, \mathbb{R})$ cf. (3.3). Here, we assume that $q^{(k-1)/2}(\alpha_1 + \alpha_2)$ and $\alpha_1\alpha_2$ are algebraic as always. Note that (3.15) is equivalent to (3.10), $n = 2$, $(k_1, k_2) = ((k-1)/2, -(k-1)/2)$. In particular, if $k = 2$ and χ_1, χ_2 are trivial, then (3.15) means

$$q^{1/2}(\alpha_1 + \alpha_2) \equiv \mathrm{vol}(K \,\mathrm{diag}(1, \varpi)K) \mod p, \quad \alpha_1\alpha_2 \equiv \mathrm{vol}(K) \mod p. \tag{3.16}$$

For the group $\mathrm{PGL}(2, F)$, Eisenstein congruences can be defined by putting $\alpha_2 = \alpha_1^{-1}$ and supposing that weight k is even in the situation of G. We choose $\mathrm{diag}(\varpi, 1)$ as a generator on $T/(T \cap K)Z$, where Z denotes the center of G. For the spherical representation ω_α of $\mathrm{PGL}(2, F)$ with the Satake parameter α, we say that ω_α satisfies the Eisenstein congruence mod p for weight k if

$$q^{(k-1)/2}(\alpha + \alpha^{-1}) \equiv q^{k-1} + 1 \mod p. \tag{3.17}$$

3.2.5. $\mathrm{GSp}(4)$. Set $G := \mathrm{GSp}(4, F)$, $K := G(\mathfrak{o})$, and $G^+ := G \cap M(4, \mathfrak{o})$. We write T for the maximal split torus consisting of diagonal matrices in G. Choose

$$\mathrm{diag}(1, 1, \varpi, \varpi), \quad \mathrm{diag}(\varpi, 1, \varpi^{-1}, 1), \quad \mathrm{diag}(1, \varpi, 1, \varpi^{-1}),$$

as a basis of the group $T/T \cap K$. The Hecke algebra $\mathcal{H}(G^+, K)$ is generated by c_1 and c_2, where c_1 (resp. c_2) denotes the characteristic function of $K \,\mathrm{diag}(1, 1, \varpi, \varpi)K$ (resp. $K \,\mathrm{diag}(1, \varpi, \varpi^2, \varpi)K$). We write $\widehat{c}_1$ and $\widehat{c}_2$ for the Satake transform of c_1 and c_2 respectively.

Consider the spherical representation ω_α with the Satake parameter $\alpha = (\alpha_0, \alpha_1, \alpha_2)$. Then, it is known that the values $\widehat{c}_1(\alpha) = \omega_\alpha(c_1)$ and $\widehat{c}_2(\alpha) = \omega_\alpha(c_2)$ are given by

$$\widehat{c}_1(\alpha) = q^{3/2}(\alpha_0 + \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_0\alpha_1\alpha_2), \tag{3.18}$$

$$\widehat{c}_2(\alpha) = q^2(\alpha_0^2\alpha_1 + \alpha_0^2\alpha_2 + \alpha_0^2\alpha_1^2\alpha_2 + \alpha_0^2\alpha_1\alpha_2^2) + (q^2 - 1)\alpha_0^2\alpha_1\alpha_2, \tag{3.19}$$

see, e.g., [Gro98, (3.15)], [RS07, Section 6.1]. Furthermore, it is easy to prove that $\omega_\alpha(c)$ is in $\mathbb{Z}[\widehat{c}_1(\alpha), \widehat{c}_2(\alpha)]$ for each characteristic functions c of double cosets $K \,\mathrm{diag}(\varpi^{c_1}, \varpi^{c_2}, \varpi^{-c_1+c_3}, \varpi^{-c_2+c_3})K$, see, e.g., [Gro98, (2.9)]. Therefore, the Eisenstein congruences of $\omega_\alpha(c)$ are determined by those of $\widehat{c}_1(\alpha)$ and $\widehat{c}_2(\alpha)$, and so it is enough to consider congruences only for $\widehat{c}_1(\alpha)$ and $\widehat{c}_2(\alpha)$.

The parameter $(k_1, k_2) \in \mathbb{Z}^{\oplus 2}$ ($k_1 + 1 \geq k_2 \geq 2$) corresponds to the L-parameter $\psi_{k_1+k_2-3, k_1-k_2+1}$, cf. (3.4). If $k_1 \geq k_2 \geq 3$, then (k_1, k_2) means the minimal $\mathrm{U}(2)$-type $\det^{k_2} \otimes \mathrm{Sym}_{k_1-k_2}$ of the holomorphic discrete series for $\psi_{k_1+k_2-3, k_1-k_2+1}$. We say that ω_α satisfies the Eisenstein congruence mod p for weight (k_1, k_2) if

$$q^{\frac{k_1+k_2}{2}-3}\,\widehat{c}_1(\alpha) \equiv q^{k_1+k_2-3} + q^{k_1-1} + q^{k_2-2} + 1 \mod p, \tag{3.20}$$

$$q^{k_1-3}\widehat{c}_2(\alpha) \equiv q^{2k_1-2} + q^{k_1+k_2-3} + q^{k_1-k_2+1} + q^{k_1-1} - q^{k_1-3} + 1 \mod p \tag{3.21}$$

hold. When $(k_1, k_2) = (3, 3)$, the congruences (3.20) and (3.21) are respectively equivalent to

$$\widehat{c}_1(\alpha) \equiv \mathrm{vol}(K \operatorname{diag}(1, 1, \varpi, \varpi)K) \mod p, \tag{3.22}$$

$$\widehat{c}_2(\alpha) \equiv \mathrm{vol}(K \operatorname{diag}(1, \varpi, \varpi^2, \varpi)K) \mod p. \tag{3.23}$$

The following propositions follow from a direct calculation for (3.15), (3.18), (3.19), (3.20), and (3.21).

Proposition 3.8. *Let $k \geq k' \geq 1$ and ω_α (resp. $\omega_{\alpha'}$) be the spherical representation of $\mathrm{GL}(2, F)$ with the Satake parameter $\alpha = (\alpha_1, \alpha_2)$ (resp. $\alpha' = (\alpha'_1, \alpha'_2)$). Suppose that $k + k'$ is even, $\alpha_1\alpha_2 = \alpha'_1\alpha'_2$ holds, and ω_α (resp. $\omega_{\alpha'}$) satisfies the Eisenstein congruence mod p for weight k (resp. k'). Then, the spherical representation ω_β of G with the Satake parameter $\beta = (\alpha_1, \frac{\alpha'_1}{\alpha_1}, \frac{\alpha'_2}{\alpha_1})$ satisfies the Eisenstein congruence mod p for weight $(\frac{k+k'}{2}, \frac{k-k'}{2} + 2)$.*

In this proposition, ω_β is the local Yoshida lift of ω_α and $\omega_{\alpha'}$. If one chooses $k' = 2$ and $\alpha' = (q^{1/2}, q^{-1/2})$, then ω_β is the local Saito-Kurokawa lift.

Proposition 3.9. *Let $k \geq 1$ and ω_α be the spherical representation of $\mathrm{GL}(2, F)$ with $\alpha = (\alpha_1, \alpha_2)$. Suppose that ω_α satisfies the Eisenstein congruence mod p for weight k. Then, the spherical representation ω_β of G with $\beta = (\alpha_1^3, \alpha_2/\alpha_1, (\alpha_2/\alpha_1)^2)$ satisfies the Eisenstein congruence mod p for weight $(k - \frac{1}{2}, \frac{k+1}{2})$.*

In this proposition, ω_β is the symmetric cube lift of ω_α.

3.2.6. G_2. Recall the notations in Section 3.1.5 for G_2. Set $G = G_2(F)$ and $K := G_2(\mathfrak{o})$. According to [Gro98, Section 5] and [GGS02, Section 13], we write λ_1 and λ_2 for the fundamental co-weights, and then they satisfy $\lambda_1(t) = (t^3, t^{-2})$ and $\lambda_2(t) = (t^4, t^{-3})$ $(t \in \mathbb{G}_m)$. The Hecke algebra $\mathcal{H}(G, K)$ is generated by c_1 and c_2, which denote the characteristic functions of $K\lambda_1(\varpi)K$ and $K\lambda_2(\varpi)K$ respectively. We choose $\lambda_1(\varpi)$ and $\lambda_2(\varpi)$ as a basis of the group $T/T \cap K$. Let ω_α be the spherical representation of G with the Satake parameter $\alpha = (\alpha_1, \alpha_2)$. By [Gro98, (5.7)], the eigenvalues $\widehat{c}_1(\alpha) = \omega_\alpha(c_1)$ and $\widehat{c}_2(\alpha) = \omega_\alpha(c_2)$ satisfy

$$\widehat{c}_1(\alpha) = q^3 - 1 + q^3(\alpha_1 + \alpha_1^{-1} + \alpha_1^2\alpha_2^{-1} + \alpha_1^{-2}\alpha_2 + \alpha_1^3\alpha_2^{-1} + \alpha_1^{-3}\alpha_2), \tag{3.24}$$

$$\begin{aligned}\widehat{c}_2(\alpha) = 2q^5 - q^4 - 1 - \widehat{c}_1(\alpha) + q^5(&\alpha_1^2\alpha_2^{-1} + \alpha_1^{-2}\alpha_2 + \alpha_1^3\alpha_2^{-2} + \alpha_1^{-3}\alpha_2^2 \\ &+ \alpha_1\alpha_2^{-1} + \alpha_1^{-1}\alpha_2 + \alpha_1 + \alpha_1^{-1} + \alpha_1^3\alpha_2^{-1} + \alpha_1^{-3}\alpha_2 + \alpha_2 + \alpha_2^{-1}).\end{aligned} \tag{3.25}$$

The Eisenstein congruences mod p of $\widehat{c}_1(\alpha)$ and $\widehat{c}_2(\alpha)$ provide those of any characteristic functions of double cosets $K\alpha K$.

The parameter $(a,b) \in (2\mathbb{Z})^{\oplus 2}$ corresponds to the L-parameter $\tilde{\psi}_{a,b}$ over $\mathbb{R}$. We say that ω_α satisfies the Eisenstein congruence mod p for weight (a,b) if

$$\widehat{c}_1(\alpha) \equiv q^3 - 1 + q^3(q^{a/2} + q^{-a/2} + q^{b/2} + q^{-b/2} + q^{(a+b)/2} + q^{-(a+b)/2}) \mod p, \tag{3.26}$$

$$\begin{aligned}\widehat{c}_2(\alpha) \equiv\ & 2q^5 - q^4 - 1 - \widehat{c}_1(\alpha) + q^5(q^{\frac{a}{2}} + q^{-\frac{a}{2}} + q^{\frac{b}{2}} + q^{-\frac{b}{2}} \\ & + q^{\frac{a+b}{2}} + q^{-\frac{a+b}{2}} + q^{\frac{a-b}{2}} + q^{-\frac{a-b}{2}} + q^{a+\frac{b}{2}} + q^{-a-\frac{b}{2}} + q^{\frac{a}{2}+b} + q^{-\frac{a}{2}-b} \mod p.\end{aligned} \tag{3.27}$$

When $(a,b) = (4,2)$, the congruences (3.26) and (3.27) are equivalent to

$$\widehat{c}_1(\alpha) \equiv q^6 + q^5 + q^4 + q^3 + q^2 + q = \mathrm{vol}(K\lambda_1(\varpi)K) \mod p, \tag{3.28}$$

$$\widehat{c}_2(\alpha) \equiv q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 = \mathrm{vol}(K\lambda_2(\varpi)K) \mod p \tag{3.29}$$

cf. [Gro98, Section 7].

Proposition 3.10. *Let c and d be odd natural numbers, and ω_{β_1} (resp. ω_{β_2}) be the spherical representation of $\mathrm{PGL}(2,F)$ with the Satake parameter β_1 (resp. β_2). Suppose that ω_{β_1} (resp. ω_{β_2}) satisfies the Eisenstein congruence mod p for weight $c+1$ (resp. $d+1$). Then, the spherical representation ω_α of G with the Satake parameter $\alpha = (\beta_1\beta_2, \beta_2^{-2})$ satisfies the Eisenstein congruence mod p for weight $(c+d, c-d)$.*

Proof. This follows from a direct calculation, see [LP02, Section 4.3] $\square$

If we choose $(c,d) = (3,1)$ or $(1,5)$ in Proposition 3.10, then one gets the Eisenstein congruence mod p of ω_α for weight $(4,2)$. (Note $\tilde{\psi}_{6,-4} \cong \tilde{\psi}_{4,2}$.)

4. Lifting congruences

In this section, we explain how the results of the previous section imply that various functorial lifts of unitary groups preserve Eisenstein congruences. For simplicity we will restrict to the case $F = \mathbb{Q}$, and sometimes assume E has class number 1, but this is not crucial. We discuss lifts to $\mathrm{SO}(5)$ and G_2 in Sections 6.2 and 7.2.

4.1. **Eisenstein series and congruences for GL(2).** Set $F = \mathbb{Q}$, and choose a natural number N, an integer $k \geq 2$, and a Dirichlet character χ on $\mathbb{Z}/N\mathbb{Z}$ such that $\chi(-1) = (-1)^k$. We also denote by $\chi = \otimes_v \chi_v$ its lift on $F^\times \mathbb{R}_{>0} \backslash \mathbb{A}^\times$. Let $S_k(N,\chi)$ denote the space of holomorphic cusp forms of weight k and level N (i.e., $\Gamma_0(N)$) with nebentypus χ. For simplicity, we put $S_k(N) = S_k(N,\chi)$ if χ is trivial. It is well known that each cusp form f in $S_k(N,\chi)$ lifts to a function on $\mathrm{GL}(2,\mathbb{Q}) \backslash \mathrm{GL}(2,\mathbb{A}_F)$, and if f is a Hecke eigenform, it generates an irreducible automorphic representation $\pi_f^{\mathrm{GL}(2)} = \otimes_v \pi_{f,v}^{\mathrm{GL}(2)}$ in $L^2_{\mathrm{cusp}}(\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}_F), \chi)$. The L-parameter of $\pi_{f,\infty}^{\mathrm{GL}(2)}$ is $\psi_{\frac{k-1}{2}}$, cf. (3.3). Let $N = N_1N_2$ with $(N_1,N_2) = 1$, and choose Dirichlet characters χ_1 on $\mathbb{Z}/N_1\mathbb{Z}$ and χ_2 on $\mathbb{Z}/N_2\mathbb{Z}$ such that $\chi =$

$\chi_1\chi_2$ on $\mathbb{Z}/N\mathbb{Z}$. For $k > 2$, we can define an Eisenstein series E_{k,χ_1,χ_2} of weight k and character (χ_1, χ_2) by

$$E_{k,\chi_1,\chi_2}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \chi_1(m)\chi_2(n)\,(mz+n)^{-k}.$$

If $N \neq 1$, a regularized expression similarly gives a weight 2 Eisenstein series E_{2,χ_1,χ_2}. For $k \geq 2$, E_{2,χ_1,χ_2} is a Hecke eigenform with T_p- eigenvalue $\chi_1(p)p^{k-1} + \chi_2(p)$ for $p \nmid N$. An eigenform $f \in S_k(N, \chi)$ is Hecke congruent to E_{k,χ_1,χ_2} mod p away from a set of places S if and only if $\pi_{f,v}^{\mathrm{GL}(2)}$ satisfies the local Eisenstein congruence mod p in (3.15) for weight k and character $(\chi_{1,v}, \chi_{2,v})$ for all $v \notin S$. (Here we are fixing the a ring of integers and a prime ideal above p for these congruences—we of course are not allowing the use of different prime ideals above p as we vary v in the local congruence.) In particular, for weight 2 and trivial characters $f \in S_2(N)$ being Hecke congruent to $E_{2,1,1}$ mod p corresponds to $\pi_f^{\mathrm{GL}(2)}$ is Hecke congruent to $\mathbb{1}$ mod p.

4.2. **The Kudla lift and symmetric power lifts.** Choose a fundamental discriminant $D < 0$, and set $F = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt{D})$. Suppose that the class number of E is one. We write χ_D for the quadratic Dirichlet character on $\mathbb{Z}/D\mathbb{Z}$ corresponding to E/F.

We will consider the following two cases:

(I) $k \geq 2$, k is even, N is arbitrary, and χ is trivial.
(II) $k \geq 3$, k is odd, N is divisible by D, and $\chi = \chi_D$.

Choose a character $\omega = \otimes_v \omega_v$ on $\mathrm{U}(1, F)\setminus \mathrm{U}(1, \mathbb{A}_F)$ as follows. In case (I), take ω to be trivial. In case (II), let $\Omega = \otimes_v \Omega_v$ denote a character on $E^\times\backslash\mathbb{A}_E^\times$ such that $\Omega|_{\mathbb{A}_F^\times} = \chi_D$ and $\Omega_\infty(z) = |z|/z$ ($\forall z \in E_\infty^\times = \mathbb{C}^\times$), and we define $\omega = \otimes_v \omega_v$ to be the restriction of Ω to $\mathrm{U}(1, \mathbb{A})$.

The mapping $\mathrm{U}(1) \times \mathrm{SL}(2) \ni (z, h) \mapsto zh \in G = \mathrm{U}(2)$ gives rise to the isomorphism $(\mathbb{C}^1 \times \mathrm{SL}(2,\mathbb{R}))/\{\pm 1\} \cong \mathrm{U}(2,\mathbb{R})$. In addition, one has $G(\mathbb{Q}) \cap \prod_{v<\infty} G(\mathbb{Z}_v) = (\mathfrak{o}_E^\times \times \mathrm{SL}(2,\mathbb{Z}))/\{\pm 1\}$, where $G(\mathbb{Z}_v) := G(\mathbb{Q}_v) \cap M(2, \mathfrak{o}_E \otimes_{\mathbb{Z}} \mathbb{Z}_v)$. Hence, each cusp form f in $S_k(N, \chi)$ is identified with a smooth function $\varphi_{f,\mathbb{R}}$ on $\mathrm{U}(2,\mathbb{R})$ satisfying

$$\varphi_{f,\mathbb{R}}(\gamma z g k) = \omega_\infty(z)\,\chi(\gamma)\,e^{ik\theta}\,\varphi_{f,\mathbb{R}}(g)$$

for any $\gamma \in \Gamma_0(N)$, $z \in \mathbb{C}^1$, $g \in \mathrm{SL}(2,\mathbb{R})$, $k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$. Furthermore, $\varphi_{f,\mathbb{R}}$ can be lifted as a smooth function $\varphi_{f,\mathbb{A}}$ in $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F), \omega)$ by the assumption that E has class number 1. Let π_f be an irreducible component of the automorphic representation of $G(\mathbb{A}_F)$ generated by $\phi_{f,\mathbb{A}}$. In case (I) (resp. (II)), the L-parameter ψ_∞ of $\pi_{f,\infty}$ is given by $\psi_\infty(z) = \mathrm{diag}((z/\overline{z})^{(k-1)/2}, (z/\overline{z})^{(-k+1)/2})$ (resp. $\psi_\infty(z) = \mathrm{diag}((z/\overline{z})^{(k-2)/2}, (z/\overline{z})^{-k/2})$).

Lemma 4.1. *Let v be a finite place of F which is non-split in E, and suppose that $G(F_v)$ is unramified, and $\pi_{f,v}$ and $\pi_{f,v}^{\mathrm{GL}(2)}$ are spherical. Suppose that*

- *$(k_1, k_2) = ((k-1)/2, (-k+1)/2)$ and χ_v is trivial in case (I);*
- *$(k_1, k_2) = ((k-2)/2, -k/2)$ and χ_v is the unramified quadratic character on $F_v^\times$ in case (II).*

Then, $\pi_{f,v}$ satisfies the Eisenstein congruence mod p in (3.14) for (k_1, k_2) if $\pi_{f,v}^{\mathrm{GL}(2)}$ satisfies the Eisenstein congruence mod p in (3.15) for weight k and character $(\chi_v, 1)$ or $(1, \chi_v)$.

Proof. Let q denote the prime number corresponding to v. The assertion can be proved by considering the action of the Hecke operator of $\Gamma_0(N)\operatorname{diag}(q^{-1}, q)\Gamma_0(N)$ on f, and its lifts to $G(\mathbb{A}_F)$ and $\mathrm{GL}(2, \mathbb{A}_F)$. □

For split places, we only consider local Eisenstein congruences (3.15) for trivial character.

Lemma 4.2. *Let v be a finite place of F split in E, and suppose that $\pi_{f,v}$ and $\pi_{f,v}^{\mathrm{GL}(2)}$ are spherical. In case (I), $\pi_{f,v}$ satisfies the Eisenstein congruence mod p in (3.10) for $((k-1)/2, (-k+1)/2)$ if and only if $\pi_{f,v}^{\mathrm{GL}(2)}$ satisfies the Eisenstein congruence mod p in (3.15) for weight k. In case (II), $\pi_{f,v} \boxtimes \omega_v^{-1} \circ \det$ satisfies the Eisenstein congruence for weight k in (3.15) (not (3.10)) if and only if $\pi_{f,v}^{\mathrm{GL}(2)}$ satisfies the Eisenstein congruence mod p in (3.15) for weight k.*

Proof. Let q denote the prime number corresponding to v. Since we are now supposing the class number of E is one, for each a place w of E dividing v, there exists a prime element $\varpi_w \in \mathfrak{o} = \mathfrak{o}_E$ of $\mathfrak{o}_w$ such that $q = \varpi_w \overline{\varpi_w}$. Then, from the viewpoint of the action on f, the Hecke operator of $\Gamma_0(N)\operatorname{diag}(\varpi_w^{-1}, \varpi_w^{-1} q)\Gamma_0(N)$ on G is identified with that of $\Gamma_0(N)\operatorname{diag}(1, q)\Gamma_0(N)$ on $\mathrm{GL}(2)$. Hence, the assertion can be proved by considering their lifts and the central character ω. □

The following is a global congruence lift by the Kudla lift.

Theorem 4.3. *Suppose that the class number of E is one. Let $N = N_1 N_2$ where N_1 and N_2 be natural numbers such that $(N_1, N_2) = 1$. Choose real-valued Dirichlet characters χ_1 on $\mathbb{Z}/N_1\mathbb{Z}$ and χ_2 on $\mathbb{Z}/N_2\mathbb{Z}$ such that $\chi_D = \chi_1 \chi_2$ on $\mathbb{Z}/N\mathbb{Z}$, and a finite set S of places of F such that $\infty \in S$ and G is unramified for $v \notin S$. Assume that there exists a Hecke eigen cusp form f in $S_3(N, \chi_D)$ such that f is Hecke congruent to $E_{3,\chi_1,\chi_2}(z) \bmod p$. Then, there exists an automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{U}(3)$ such that π is Hecke congruent to $\mathbb{1} \bmod p$ and the L-parameter of π_∞ is associated with $(1, 0, -1)$.*

Remark 4.4. There is also an endoscopic lift from $S_2(N)$ to an automorphic representation of $\mathrm{U}(3)$ with the same real component, but it does not preserve the Eisenstein congruences at places v which are non-split over E in general, see Section 3.2.3.

Proof. By the Kudla lift, one has a lifting of f to an automorphic form F_f in $L^2_{\mathrm{cusp}}(\mathrm{U}(3,F)\backslash \mathrm{U}(3,\mathbb{A}_F))$, see [Kud79] and [MS07]. The automorphic representation of F_f corresponds to the L-parameter of π_f of $\mathrm{U}(2)$ and the trivial representation of $\mathrm{U}(1)$ via the L-embedding ${}^L\mathrm{U}(2) \times {}^L\mathrm{U}(1) \to {}^L\mathrm{U}(3)$ with Ω^{-1}, cf. [Rog90, Section 4.8]. Therefore, the assertion follows from Proposition 3.6, Lemmas 4.1 and 4.2, and the proof of Proposition 3.3. Notice that the condition for $\pi_{f,v}$ in Lemma 4.2 is different from Proposition 3.3, but the local congruence lift can be proved by the same manner as in the proof of Proposition 3.3. □

The following tells us that symmetric power lifts, when they exist, preserve Eisenstein congruences. For non-CM forms cusp forms f, the m-th symmetric power lift to $\mathrm{GL}(m+1)$ is known for $m \leq 8$ (see [CT17] and references therein), which can be transferred to $\mathrm{U}(m+1)$ using [Mok15]. (In fact [CT17] first constructs the representation on a definite $\mathrm{U}(m+1)$.)

Theorem 4.5. *Suppose that the class number of E is one. Assume $f \in S_2(N)$ is an eigenform which is Hecke congruent to $E_{2,1,1}$ mod p. If there exists an m-th symmetric power lift π of π_f from $\mathrm{U}(2)$ to $\mathrm{U}(m+1)$, then π is Hecke congruent to $\mathbb{1}$ mod p.*

Proof. This follows from Propositions 3.4 and 3.7 and Lemmas 4.1 and 4.2. □

4.3. **Congruences on $\mathrm{U}(1)$.** Since endoscopic lifts for $\mathrm{U}(n)$ involve characters of $\mathrm{U}(1)$, to understand when these lifts preserve Eisenstein congruences, we need to know about congruences of characters with different infinity types. Let E be an imaginary quadratic field over $F = \mathbb{Q}$ of class number 1.

First we briefly discuss characters of even infinity type. For any $u \in \mathbb{Z}$, one can define a Hecke character $\Omega = \otimes_v \Omega_v$ on $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ by

$$\Omega_\infty(z) = (z/\overline{z})^{-u}.$$

Suppose that $\Omega_\infty(\varepsilon) = 1$ for any $\varepsilon \in \mathfrak{o}_E^\times$, i.e., suppose that $\frac{|\mathfrak{o}_E^\times|}{2} | u$. Then Ω is identified with a character on $\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A})$ by the isomorphism $\mathbb{G}_m \backslash R_{E/F}(\mathbb{G}_m) \ni z \mapsto z/\overline{z} \in \mathrm{U}(1)$ over F. Note the corresponding character on $\mathbb{C}^1 = \mathrm{U}(1,\mathbb{R})$ is $e^{i\theta} \mapsto e^{-iu\theta}$. When a prime number q is not split in E, $\mathrm{U}(1,F_q)$ is compact, and so the spherical representation Ω_q is trivial. When a prime number q of F is split in E, we choose a place $w|q$ and we take a prime element $\varpi_w \in \mathfrak{o}_E$ of $\mathfrak{o}_w$. Then one has

$$\Omega_q(\varpi_w) = \Omega_\infty(\varpi_w^{-1}) = (\varpi_w/\overline{\varpi_w})^u = q^{-u} \times \varpi_w^{2u}.$$

Therefore, if $\varpi_w^{2u} \equiv 1 \bmod p$ holds for the place q, then Ω satisfies the local Eisenstein congruence mod p of L-parameter $-u$ for the split place q, that is, $q^u \Omega_w(\varpi_w) \equiv 1 \bmod p$. This local congruence holds for all (split) q whenever $(p-1)|2u$ (in particular for any u if $p = 2, 3$), in which case we get that Ω satisfies a global Eisenstein mod p congruence with archimedean parameter $-u$.

Next we give an example of a congruence for odd infinity type. Let $E = \mathbb{Q}(\sqrt{-7})$, χ_{-7} denote the quadratic Dirichlet character on $\mathbb{Z}/7\mathbb{Z}$ corresponding to E/F. A character $\Omega = \otimes_v \Omega_v$ on $E^\times \backslash \mathbb{A}_E^\times$ is defined by

$$\Omega_\infty(z) = (z/\overline{z})^{-u}\,|z|/z \ \ (z \in \mathbb{C}^\times), \quad \Omega_7(x + y\sqrt{-7}) = \chi_{-7}(x) \ \ (x \in \mathbb{Z}_7^\times, \ \ y \in \mathbb{Z}_7).$$

Note that $\Omega|_{\mathbb{A}_F} = \chi_{-7}$ holds. When a prime number q_1 of F is non-split in E and not 7, we have $\Omega_{q_1}(q_1) = -1$. Further, for each prime q split in E, one has

$$\Omega_q(\varpi_w) = q^{-u-\frac{1}{2}} \times \Omega_7(\varpi_w) \times \varpi_w^{2u+1}.$$

Since $a^3 \equiv \chi_{-7}(a) \bmod 7$ ($a \in (\mathbb{Z}/7\mathbb{Z})^\times$), Ω_q satisfies the local Eisenstein congruence mod 7 of L-parameter $-u - \frac{1}{2}$ if $2u + 1 \equiv 3 \bmod 6$.

4.4. **Endoscopic lifts for** $\mathrm{U}(n)$**.** Here we briefly discuss elliptic endoscopic lifts of congruences for $\mathrm{U}(n)$. Let E be an imaginary quadratic field over $F = \mathbb{Q}$, and n_1, n_2 be nutural numbers. Suppose that n_2 is even. We set $n = n_1 + n_2$ and consider endoscopic lifts from $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$ to $\mathrm{U}(n)$ for E/F. Let $\pi_1 = \otimes_v \pi_{1,v}$ be an automorphic representation of $\mathrm{U}(n_1)$ such that $\pi_{1,\infty}$ corresponds to the L-parameter $k_1 = \rho_{n_1}$ (see (3.7) and Section 3.1.2 for $\rho_{n_1} \in \mathbb{L}_{n_1}$). Choose an integer u if n_1 is odd. Let $\pi_2 = \otimes_v \pi_{2,v}$ be an automorphic representation of $\mathrm{U}(n_2)$ such that $\pi_{2,\infty}$ corresponds to the L-parameter

$$k_2 = \begin{cases} \frac{1}{2}(n - 2 - 2u, \ldots, n_1 - 2u, -n_1 - 2 - 2u, \ldots, -n - 2u) & \text{if } n_1 \text{ is odd}, \\ \frac{1}{2}(n - 1, \ldots, n_1 + 1, -n_1 - 1, \ldots, -n + 1) & \text{if } n_1 \text{ is even}. \end{cases}$$

Assume that π_1 and π_2 are associated with global generic parameters in the sense of [Mok15, Section 2.3]. When n_1 is odd, choose a character $\chi = \otimes_v \chi_v$ on $E^\times \backslash \mathbb{A}_E^\times$ such that $\chi_\infty(z) = (z/\overline{z})^u z/|z|$ and $\chi|_{\mathbb{A}_F^\times}$ is the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ corresponding to E/F. When n_1 is even, we set $\chi = 1$. We also choose a finite set S of places of F containing ∞ such that $\pi_{1,v}$, $\pi_{2,v}$, and χ_v are unramified for any $v \notin S$. In the case that n_1 is odd, we suppose that $\chi_v(\varpi_v) q_v^{-u-\frac{1}{2}} \equiv 1 \bmod p$ holds for each place $v \notin S$ when v is split in E, and $\chi_v(\varpi_v) \equiv -1 \bmod p$ holds for each place $v \notin S$ when v is non-split in E, see Section 4.3 for an example. Assume that $\pi_{1,v}$ (resp. $\pi_{2,v}$) satisfies the Eisenstein congruences mod p of L-parameter k_1 (resp. k_2) for any $v \notin S$. Then, there exists an automorphic representation π of $\mathrm{U}(n)$, which is an endoscopic lift of π_1 and π_2 by the L-embedding of χ (see [Mok15]), and it follows from Propositions 3.3 and 3.6 that π is Hecke congruent to $\mathbb{1} \bmod p$.

One can also consider global congruence lifts as above when both n_1 and n_2 are odd. However it is more complicated due to character considerations (cf. Proposition 3.6) and we do not discuss it here.

On the other hand, it is easier to lift congruences from from $\mathrm{U}(n-2) \times \mathrm{GL}(2)$, generalizing Theorem 4.3 and the above argument with $n_1 = n - 2$ and $n_2 = 2$.

Theorem 4.6. *Let ρ_m denote the L-parameter ψ of $\mathrm{U}_{\mathbb{C}/\mathbb{R}}(m, \mathbb{R})$ for $l_1 = \cdots = l_m = 0$ in (3.2). Let $n \geq 2$, and π_1 be an automorphic representation of $\mathrm{U}(n-2)$ which has the L-parameter ρ_{n-2} at ∞, is associated with a global generic parameter, and is Hecke congruent to $\mathbb{1}$ mod p. Let $f \in S_n(N, \chi_D^n)$ be an eigenform which is Hecke congruent to $E_{n,1,\chi_D^n}(z) \bmod p$. Then, there is an automorphic representation π of $\mathrm{U}(n)$ such that π is an endoscopic lift of π_1 and π_f, π has L-parameter ρ_n at ∞, and π is Hecke congruent to $\mathbb{1}$ mod p.*

Theorem 5.7 implies the existence of π_1 for $n-2$ prime in the above theorem. The assumption of a global generic parameter ensures that the endoscopic lift has the L-parameter ρ_n at ∞.

4.5. **The Ikeda lift.** Finally, we discuss congruence lifts obtained from the Ikeda lifts to $\mathrm{U}(2m)$, cf. Ikeda [Ike08] and Yamana [Yam]. The global A-parameters of the Ikeda lifts are of the form $\mu \boxtimes \nu$, where μ is the base change lift of a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ whose component at ∞ is a holomorphic discrete series, and ν is the m-dimensional irreducible representation of $\mathrm{SU}(2)$. Hence, almost all local components of the Ikeda lifts are non-tempered.

Theorem 4.7. *As in Section 4.2, assume $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{D})$, and the class number of E is one. Choose natural numbers N and m. Let f be a Hecke eigen cusp form in $S_{2m}(N)$ which is Hecke congruent to $E_{2m,1,1} \bmod p$. Then, there exists a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{U}(4m-2)$ such that π is Hecke congruent to $\mathbb{1} \bmod p$ and π_∞ is the holomorphic discrete series whose the L-parameter is associated with ρ_{4m-2}.*

Proof. By [Yam, Theorem 1.1], we get an automorphic representation π_1 of $\mathrm{GU}(4m-2)$ which contains a nonzero Hermitian cusp form in the lift of f. We can choose a nonzero irreducible constitutent π of the restriction of π_1 to $\mathrm{U}(4m-2)$, whose component is the holomorphic discrete series. Hence, this theorem follows from Propositions 3.4 and 3.7 and Lemmas 4.1 and 4.2. $\square$

5. Unitary groups

Let E/F be a CM extension of number fields, and $G' = \mathrm{U}(n)$ be the associated quasi-split unitary group over F in n variables defined in (3.1).

Let G be an inner form of G'. We can realize G as follows. There exist (i) a central simple algebra A/E of degree n, i.e., $\dim_E A = n^2$, and (ii) an involution $\alpha \mapsto \alpha^*$ of A of the second kind with $\alpha^* = \bar{\alpha}$ for $\alpha \in E$, such that

$$G = \{g \in A^\times : g^* g = 1\}.$$

We remark that G is the automorphism group of the Hermitian form $\langle \alpha, \beta \rangle = \alpha^* \beta$ on A. The center of G is $E^\times \cap G$ (viewing $E^\times$ as the algebraic group $\mathrm{Res}_{E/F} \mathbb{G}_m$), which we may identify with $\mathrm{U}(1) = E^1 = \{a \in E^\times : a\bar{a} = 1\}$.

To specify A and/or $*$ below, we will also denote $G = \mathrm{U}_A(n) = \mathrm{U}_{A,*}(n)$. (The isomorphism class depends on both A and $*$, but as we will typically only be concerned about specifying A we often just write $\mathrm{U}_A(n)$.) Landherr's theorem on the classification of involutions of the second kind tells us that if v is inert or ramified in E/F, then A_v is split. Moreover, if v splits in E/F as $v = ww'$, then $*$ interchanges the factors of A_w and $A_{w'}$, giving an isomorphism $A_w \simeq A_{w'}^{\mathrm{opp}}$ and $G_v = \mathrm{U}_A(n, F_v) \simeq A_w^\times \simeq A_{w'}^\times$.

We will now assume $G = \mathrm{U}_{A,*}(n)$ is a definite unitary group, i.e., the associated Hermitian form is totally definite. This means G_v is compact for all $v|\infty$. Note that one can make a definite involution on A from any involution by conjugation (see [Sch85, Remark 10.6.11]).

Let det denote the reduced norm on A. By restriction to $G = \mathrm{U}_A(n)$, we may view det as a homomorphism of algebraic groups $\det : G \to \mathrm{U}(1)$. The derived subgroup $\mathrm{SU}_A(n)$ of G is the kernel of det, so any 1-dimensional automorphic representation of $G(\mathbb{A})$ factors through det.

Lemma 5.1. *The map* $\det : G(k) \to \mathrm{U}(1, k)$ *is a surjective map of rational points for any* $k = F_v$ *and for* $k = F$.

Proof. By the Hasse principle for the norm map of unitary groups ([PR94, Theorem 6.28]), the result for F follows from the result for each F_v. If v is split in E_v/F_v, the local result follows from surjectivity of reduced norm for central simple algebras over p-adic fields. Otherwise, $G(F_v)$ is an honest unitary group, and it is clear det restricted to the diagonal torus surjects onto $\mathrm{U}(1, k)$. $\square$

5.1. **Endoscopic classification.** Here we briefly explain certain aspects of the endoscopic classification for unitary groups as asserted in [KMSW, Theorem* 1.7.1], and refer the reader to *op. cit.* and [Mok15] for more precise details.

The endoscopic classification was treated by Rogawski [Rog90] for U(3) and its inner forms (as well as quasi-split U(2)), by Mok [Mok15] for quasi-split $\mathrm{U}(n)$, and by Kaletha–Minguez–Shin–White [KMSW] for inner forms of $\mathrm{U}(n)$ under some hypotheses. (See [Mok15, Section 2.6] for a summary of some intermediary results.) These latter results rely on the stabilization of the twisted trace formula which was established in [MW17], and also require the general weighted fundamental lemma which is expected to be finished by Chaudouard and Laumon. Work in progress of Kaletha–Minguez–Shin is expected to complete the proof of [KMSW, Theorem* 1.7.1], and we will assume this in our subsequent congruence results.

In fact the cases that we need are in some sense easier than cases already established in the literature (e.g., [HT01], [Lab11], [Shi11], [Mok15]), as the only non-quasi-split forms we consider are certain compact forms, where the trace formula analysis is simpler and one does not have endoscopic contributions. However, to our knowledge the cases we use (definite unitary groups over division algebras) have not been explicitly dealt with in the literature.

To describe the classification, in this section we let G be an arbitrary inner form of $G' = \mathrm{U}(n)$. In particular, we allow $G = G'$.

As in [Mok15], the set of formal global parameters for G' is the set $\Psi(G')$ consisting of formal sums (up to equivalence) $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ of formal tensors $\psi_i = \mu_i \boxtimes \nu_i$, where μ_i is a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ and ν_i is the r_i-dimensional irreducible representation of $\mathrm{SU}(2)$, such that $\sum n_i r_i = n$ and the parameter ψ is conjugate self-dual. If $m = 1$, we call ψ simple. If each $\nu_i = 1$, we call ψ generic. Set $\dim \psi_i = n_i r_i$.

According to the Moeglin–Waldspurger classification, $\mu_i \boxtimes \nu_i$ corresponds to a discrete automorphic representation σ_{ψ_i} of $\mathrm{GL}_{n_i r_i}(\mathbb{A}_E)$, which is cuspidal if $r_i = 1$. Thus by Langlands theory of Eisenstein series, ψ corresponds to an automorphic representation σ_ψ of $\mathrm{GL}_n(\mathbb{A}_E)$. Let $\Psi_2(G')$ denote the subset of square-integrable parameters, which are of the form $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ where the ψ_i's are all distinct and each ψ_i is conjugate self-dual. Let $\Psi_2(G', \mathrm{std})$ be the subset of $\Psi_2(G')$ which "factor through" the standard L-embedding $\mathrm{std} : {}^L G' \to {}^L \mathrm{Res}_{E/F}(G')$ (this set is denoted $\Psi_2(G', \xi_1)$ in [Mok15, Definition 2.4.5]).

Let $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m \in \Psi_2(G')$. One associates to ψ a component group $\mathcal{S}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^{m'}$ (denoted $\bar{\mathcal{S}}_\psi$ in [KMSW]), and a canonical sign character ϵ_ψ of $\mathcal{S}_\psi$. Here $0 \leq m' \leq m$—see [Mok15, (2.4.14)] for a precise description of m'. We note $\epsilon_\psi = 1$ if ψ is generic. Then there is a global packet $\Pi_\psi(G) = \Pi_\psi(G, \xi, \epsilon_\psi)$ of representations attached to an inner twist (G, ξ) that is a certain subset of a restricted product of local packets consisting of elements which are globally compatible with ϵ_ψ. (Here ξ is an $\bar{F}$-isomorphism from G to G' exhibiting G as an inner form of G'.) The role of ϵ_ψ is to give a parity condition for a product of members of local packets to lie in the global packet.

The packet $\Pi_\psi(G)$ is necessarily empty if ψ is not locally relevant everywhere for G. Specifically, if v is split in E/F, and $G(F_v) \simeq M_{r_v}(D_v)$ where D_v is a central F_v-division algebra of degree d_v, then for $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ to be relevant it is necessary that $d_v | \dim \psi_i$ for each i.

For $\psi \in \Psi_2(G', \mathrm{std})$ and $\pi \in \Pi_\psi(G)$, we call the associated automorphic representation $\pi_E := \sigma_\psi$ of $\mathrm{GL}_n(\mathbb{A}_E)$ the (standard) base change of π. Note that π_E is cuspidal if and only if $\psi = \pi_E \boxtimes 1$, i.e., if and only if ψ is simple generic. If π_E is cuspidal and $v = ww'$ is a split place for E/F, then $\pi_v \simeq \pi_{E,w}$ when A_v is split, and more generally π_v corresponds to $\pi_{E,w}$ via the Jacquet–Langlands correspondence for $\mathrm{GL}(n)/E$.

Then the $\kappa = 1$ and $\chi_\kappa = 1$ case of [KMSW, Theorem* 1.7.1] states that we have a $G(\mathbb{A})$-module isomorphism:

$$\text{(EC-U)} \qquad L^2_{\mathrm{disc}}(G(F) \backslash G(\mathbb{A})) \simeq \bigoplus_{\psi \in \Psi_2(G', \mathrm{std})} \bigoplus_{\pi \in \Pi_\psi(G)} \pi.$$

A consequence of this is a generalized Jacquet–Langlands correspondence for unitary groups. Namely, fix an inner form G of G', so $G(F_v) \simeq G'(F_v)$ for almost all v. For simplicity, assume $\psi \in \Psi_2(G', \mathrm{std})$ is simple generic, so we may view ψ as a conjugate self-dual cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then the packet $\Pi_\psi(G')$ is non-empty—in fact it contains a cuspidal generic representation of G' [Mok15, Corollary 9.2.4]. If $\pi \in \Pi_\psi(G)$ we write $\mathrm{JL}(\pi) = \Pi_\psi(G')$ for the Jacquet–Langlands correspondent to the packet $\Pi_\psi(G)$. For v split in E/F and $\pi' \in \mathrm{JL}(\pi)$, π_v and π'_v correspond via the local Jacquet–Langlands correspondence for $\mathrm{GL}_n(F_v)$, and necessarily $\pi'_v \simeq \pi_v$ if $G(F_v) \simeq G'(F_v) \simeq \mathrm{GL}_n(F_v)$.

It is expected that generic packets are tempered. If ψ is cohomological, then Shin [Shi11] (together with [CH13] when n is even and ψ_∞ is not Shin-regular) guarantees that ψ_v is tempered at all finite v. Now let us also assume ψ is cohomological.

For $\pi' \in \Pi_\psi(G')$, the local packets Π_{ψ_v} for π and π' are the same at almost all places. But, by definition, elements of the global packets correspond locally to the trivial character of the component group (and thus unramified local parameters ψ_v) almost everywhere. Since ψ_v is generic and bounded (tempered), the local packet $\Pi_{\psi_v}(G'(F_v))$ is in bijection with the dual of the component group at nonarchimedean v ([Mok15, Theorem 2.5.1(b)]). Consequently, $\pi_v \simeq \pi'_v$ for almost all v.

In fact we can say more. Since ψ is simple generic, we have $|\mathcal{S}_\psi| = 1$ ([Mok15, (2.4.14)]). This means there is no parity condition associated to ϵ_ψ required for a product $\pi' = \otimes \pi'_v$ of local components of packets to lie in the global packet $\Pi_\psi(G')$. Hence, given π, we may always choose $\pi' \in \mathrm{JL}(\pi)$ such that $\pi'_v \simeq \pi_v$ whenever $G(F_v) \simeq G'(F_v)$. Moreover, at all other v, we can choose π'_v freely within the local packet $\Pi_{\psi_v}(G'(F_v))$.

5.2. **A cuspidality criterion for base change.** For the remainder of this section, we return to our assumption that $G = \mathrm{U}_{A,*}(n)$ is a definite unitary group.

Proposition 5.2. *Assume* (EC-U). *Suppose n is prime and A_w is a division algebra for some finite prime w of E. If π occurs in $\mathcal{A}(G, K)$, and π is not 1-dimensional, then π_E is cuspidal.*

Proof. Necessarily, there is a finite prime v of F which splits as $v = ww'$ for some w'. Then $G(F_v) \simeq A_w^\times$ is the multiplicative group of a degree n division algebra. Let $\psi \in \Psi_2(G', \mathrm{std})$ be the parameter associated to π. Then for ψ to be relevant, we need ψ to be simple, i.e., $\psi = \mu \boxtimes \nu$ for some cuspidal automorphic representation μ of $\mathrm{GL}_m(\mathbb{A}_E)$ and ν of dimension $r = \frac{n}{m}$.

Since n is prime, either $m = 1$ or $m = n$. If $m = n$, we are done. Otherwise, the proposition follows from the following lemma, which was kindly explained to us by Sug Woo Shin. ☐

Lemma 5.3. *Assume* (EC-U). *Suppose π is an automorphic representation of G associated to a simple parameter $\psi = \mu \boxtimes \nu$ where μ is a representation of $\mathrm{GL}_1(\mathbb{A}_E)$. Then π is 1-dimensional.*

Proof. Suppose $v = ww'$ is split in E. The local base change $\pi_{E,w}$ is a 1-dimensional representation of $\mathrm{GL}_n(E_w)$. Then $\pi_v \simeq \pi_{E,w}$, so π_v is 1-dimensional. Since the strong approximation property with respect to v is satisfied by $G^1 = \{g \in G : \det g = 1\}$ (see [PR94, Theorem 7.12]), π_v trivial on $G^1(F_v)$ implies π is trivial on $G^1(\mathbb{A})$. Thus π is 1-dimensional. □

5.3. **Eisenstein congruences.** Let $K = \prod K_v \subset G(\mathbb{A})$ be a compact open subgroup which is hyperspecial and maximal at almost all v. We assume that $K_v = G_v$ for $v|\infty$, and place the following assumptions on K_v for $v < \infty$.

First suppose v splits in E/F. Then we can write $G_v = \mathrm{GL}_{r_v}(D_v)$ for some division algebra D_v of degree d_v with $d_v r_v = n$. Let $\mathcal{O}_v$ be an order of D_v containing the unramified field extension of F_v of degree d_v (e.g., $\mathcal{O}_v$ is the maximal order in D_v). We assume the diagonal subgroup $(\mathcal{O}_v^\times)^{r_v} \subset K_v$. This holds, for instance, when K_v is the stabilizer of a lattice of the form $\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_{r_v} \subset D_v^{r_v}$ where each $\mathcal{I}_i$ is left $\mathcal{O}_v$-ideal on D_v.

Next suppose v is ramified or inert in E/F, so A_v is split. Assume G_v has a maximal torus $T_v \simeq (E_v^\times)^r \times (E_v^1)^s$ for some r, s with $2r + s = n$, such that the integral points of T_v are contained in K_v, i.e., $(\mathfrak{o}_{E_v}^\times)^r \times (E_v^1)^s \subset K_v$. This holds for instance if K_v is the stabilizer of a lattice of the form $\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_n \subset E_v^n$ where each $\mathcal{I}_i$ is a $\mathfrak{o}_{E_v}$-ideal (in which case $s = n$).

The above assumptions guarantee that for all $v < \infty$, (i) $K_v \cap Z(G_v) = \mathrm{U}(1, \mathfrak{o}_v) = \{a \in \mathfrak{o}_{E,v}^\times : a\bar{a} = 1\}$, and (ii) $\det K_v = \mathrm{U}(1, \mathfrak{o}_v)$. Note for $v < \infty$, if E_v/F_v is a field then $\mathrm{U}(1, \mathfrak{o}_v) = \mathrm{U}(1, F_v) = E_v^1$, whereas if E_v/F_v is split then $\mathrm{U}(1, \mathfrak{o}_v) \simeq \mathfrak{o}_v^\times$.

Consequently, if π occurs in $\mathcal{A}(G, K, \omega)$, then ω is a character of $\mathrm{U}(1, \mathbb{A})$ which is invariant under $\mathrm{U}(1, F)$ and $K \cap Z(\mathbb{A}) = \mathrm{U}(1, \hat{\mathfrak{o}})\, \mathrm{U}(1, F_\infty)$. Thus the relevant central characters for us will be characters ω of the class group $\mathrm{Cl}(\mathrm{U}(1)) = \mathrm{U}(1, F) \backslash \mathrm{U}(1, \mathbb{A}_f) / \mathrm{U}(1, \hat{\mathfrak{o}})$.

Any 1-dimensional representation π occurring in $\mathcal{A}(G, K)$ is of the form $\pi = \chi \circ \det$, where χ is a character of $\mathrm{U}(1, \mathbb{A})$. From Lemma 5.1 and our assumptions on K, we in fact see that χ must be a character of $\mathrm{Cl}(\mathrm{U}(1))$.

We can apply Proposition 2.1 or Corollary 2.4 to construct congruences on $\mathcal{A}(G, K)$. However, since $\mathcal{A}(G, K)$ admits many 1-dimensional representations in general, even with trivial central character, we need more to guarantee we get congruences with non-abelian forms.

5.3.1. *Congruence modules.* Fix a finite abelian group H and let L be a number field which contains all character values for H. Let $X(R)$ be the set of R-valued class functions for $R = \mathbb{Z}$ or $R = L$. Endow $X(L)$ with the usual inner product $(\cdot, \cdot)$. Decompose $X(L) = X_{\mathbb{1}}(L) \oplus X_0(L)$ where $\mathbb{1}$ is the trivial character of H and $X_{\mathbb{1}}(L) = L\mathbb{1}$. Let $X_{\mathbb{1}}(\mathbb{Z}) = X_{\mathbb{1}}(L) \cap X(\mathbb{Z}) = \mathbb{Z}\mathbb{1}$ and $X_0(\mathbb{Z}) = X_0(L) \cap X(\mathbb{Z})$. Also, let $X^{\mathbb{1}}(\mathbb{Z})$ (resp. $X^0(\mathbb{Z})$) be the image of the orthogonal projection $X(\mathbb{Z}) \to X_{\mathbb{1}}(L)$ (resp. $X(\mathbb{Z}) \to X_0(L)$). Then $X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z}) \subset X(\mathbb{Z}) \subset X^{\mathbb{1}}(\mathbb{Z}) \oplus X^0(\mathbb{Z})$.

We consider the congruence module $C_0(H) = X(\mathbb{Z})/(X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z}))$. One readily sees that the projection $X(L) \to X_{\mathbb{1}}(L)$ induces an isomorphism $C_0(H) \simeq X(\mathbb{Z})/(X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z})) \simeq X^{\mathbb{1}}(\mathbb{Z})/X_{\mathbb{1}}(\mathbb{Z})$. One similarly has an isomorphism with $X^0(\mathbb{Z})/X_0(\mathbb{Z})$.

Lemma 5.4. *For a positive integer n, there exists $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod n$ if and only if $C_0(H)$ contains an element of order n.*

Proof. First note if $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod n$, then the projection of $\frac{1}{n}(\phi - \mathbb{1})$ to $X_{\mathbb{1}}(L)$ is the element $-\frac{1}{n}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$, and thus gives an element of order n in $C_0(H)$. Conversely, suppose $\psi \in X(\mathbb{Z})$ is an element of order n in $C_0(H)$. Then we can write $\psi = \frac{a}{n}\mathbb{1} - \frac{1}{n}\phi$ where $a \in \mathbb{Z}$ and $\phi \in X_0(\mathbb{Z})$. Since projection gives the isomorphism $C_0(H) \simeq X^{\mathbb{1}}(\mathbb{Z})/\mathbb{Z}\mathbb{1}$, $\frac{a}{n}$ has order n mod $\mathbb{Z}$. Thus after scaling ψ (and correspondingly ϕ) we may assume $a \equiv 1 \mod n$. Then $\phi \equiv \mathbb{1} \mod n$. $\square$

Lemma 5.5. *As $\mathbb{Z}$-modules, $C_0(H) \simeq H$.*

Proof. First suppose that $H = H_1 \times H_2$. For $i = 1, 2$, write $X(R; H_i)$, $X_0(R; H_i)$, etc. for the corresponding objects for the group H_i. It is not hard to see that $X(\mathbb{Z}) = X(\mathbb{Z}; H) = \{\phi_1 \otimes \phi_2 : \phi_i \in X(\mathbb{Z}; H_i)\}$. Thus we may identify $X(\mathbb{Z}; H) = X(\mathbb{Z}; H_1) \oplus X(\mathbb{Z}; H_2)$. This identifies the $\mathbb{Z}$-submodule $X_{\mathbb{1}}(\mathbb{Z}; H) \oplus X_0(\mathbb{Z}; H)$ with $X_{\mathbb{1}}(\mathbb{Z}; H_1) \oplus X_0(\mathbb{Z}; H_1) \oplus X_{\mathbb{1}}(\mathbb{Z}; H_2) \oplus X_0(\mathbb{Z}; H_2)$. Hence $C_0(H) \simeq C_0(H_1) \oplus C_0(H_2)$. This reduces the proof to the case that $H = \langle g \rangle$ is cyclic of order n, which we assume now.

If $\chi_1, \dots, \chi_n$ are the irreducible characters of H, then $\frac{1}{n}(\chi_1 + \cdots + \chi_n) \in X(\mathbb{Z})$. Hence $\frac{1}{n}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$. Conversely, suppose $\frac{1}{m}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$. Then there exists $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod m$. Let $a_j = \phi(g^j)$ for $1 \le j \le n$. Then $n \cdot (\chi, \mathbb{1}) = \sum a_j = 0$ but also $\sum a_j \equiv n \mod m$, hence $m|n$. Therefore $C_0(H) \simeq X^1(\mathbb{Z})/\mathbb{Z}\mathbb{1} \simeq H$. $\square$

The relevant consequence for us is the following. Let $e(H)$ denote the exponent of a finite group H: if $p^r \nmid e(H)$ then there is no congruence mod p^r between the trivial character of H and any $\mathbb{Z}$-valued linear combination of the non-trivial characters of H.

Proposition 5.6. *Let $h_E^1 = |\mathrm{Cl}(\mathrm{U}(1))|$ and e_E^1 be the exponent of $\mathrm{Cl}(\mathrm{U}(1))$. Suppose $p|\frac{m(K)}{\gcd(n, e_E^1)h_E^1}$ and n is odd. Then there is a non-abelian eigenform $\phi \in \mathcal{A}(G, K, 1)$ such that ϕ is Hecke congruent to $\mathbb{1} \bmod p$.*

Proof. By our assumptions on K, we have $K_Z = \mathrm{U}(1, \hat{\mathfrak{o}})U(1, F_\infty)$, so $m(K_Z) = |\mathrm{Cl}(\mathrm{U}(1))|$. Thus Corollary 2.4 says there exists an eigenform $\phi \in \mathcal{A}_0(G, K, 1)$ which is Hecke congruent to $\mathbb{1} \bmod p$. We want to show we can take ϕ to be non-abelian.

Let $\bar{G} = G/Z$ and $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Note the abelian elements of $\mathcal{A}(G, K, 1) = \mathcal{A}(\bar{G}, \bar{K})$ are generated by the characters $\chi \circ \det$ where χ is a character of $\mathrm{Cl}(\mathrm{U}(1))$

of order dividing n. We may view such χ as factoring through the largest quotient H of $\mathrm{Cl}(\mathrm{U}(1))$ of exponent dividing n.

Recall that the existence of such a ϕ arose from an integral element $\phi' \in \mathcal{A}_0^{\mathbb{Z}}(\bar{G}, \bar{K})$ such that $\phi' \equiv \mathbb{1} \bmod p^r$ where $r = v_p(m(\bar{K})) = v_p(m(K)) - v_p(h_E^1)$. For a suitable rationality field L, decompose $\mathcal{A}_0^L(\bar{G}, \bar{K}) = X_1(L) \oplus X_2(L)$ where $X_1(L)$ consists of the abelian forms orthogonal to $\mathbb{1}$ and $X_2(L)$ is spanned by the non-abelian eigenforms.

We claim $\phi' \notin X_1(L)$. Since $\det : G(\mathbb{A}) \to \mathrm{U}(1, \mathbb{A})$ is surjective, our assumptions on K imply that det induces a surjective map $\Delta : \mathrm{Cl}(\bar{K}) \to H$. Thus if we had $\phi' \in X_1(L)$, composing it with Δ gives $\mathbb{Z}$-valued class function ψ on H such that $\psi \equiv \mathbb{1} \bmod p^r$. But this is impossible by the above lemmas as $v_p(\frac{p^r}{\gcd(n, e_E^1)}) > 0$ implies p^r does not divide the exponent of H.

Hence ϕ' has nonzero projection to $X_2(L)$. Therefore applying the lifting lemma, Lemma 2.2, with $W = X_2(L)$, we obtain an eigenform $\phi \in X_2(L)$ which is Hecke congruent to $\mathbb{1} \bmod p$. $\square$

5.3.2. *Non-endoscopic congruences.* We now define the notion of congruences on the quasi-split form G'. For convenience, we talk about congruences of representations. Suppose $K' = \prod K'_v$ is an open compact subgroup of G' which is hyperspecial at all $v \notin S$, and π and π' are automorphic representations of $G'(\mathbb{A})$ which are K'_v-unramified at all $v \notin S$. For $\alpha_v \in G'_v$, we let $\lambda_{\alpha_v}(\pi)$ be the eigenvalue of the local Hecke operator $K'_v \alpha_v K'_v$ on $\pi_v^{K'_v}$. We say π and π' are Hecke congruent (away from S) mod p if $\lambda_{\alpha_v}(\pi) \equiv \lambda_{\alpha_v}(\pi') \bmod \mathfrak{p}$ for some prime $\mathfrak{p}$ of $\bar{\mathbb{Q}}$ above p and all $v \notin S$, $\alpha_v \in G'_v$.

Consider the simple parameter $\psi_0 = 1 \boxtimes \nu(n) \in \Psi_2(G', \mathrm{std})$, where $\nu(n)$ is the irreducible n-dimensional representation of $\mathrm{SU}(2)$. This is the parameter of the trivial representations $\mathbb{1}_G$ and $\mathbb{1}_{G'}$ of G and G'. The base change of $\mathbb{1}_{G'}$ to $\mathrm{GL}_n(\mathbb{A}_E)$ is the residual contribution of the Eisenstein series induced from the $\delta_{\mathrm{GL}(n)}^{-1/2}$ of the Borel. The unramified Hecke eigenvalues for $\mathbb{1}_{G'}$ are given by (3.9) and (3.13).

Theorem 5.7. *Suppose n is an odd prime and assume (EC-U) for n. Let A/E be a degree n central simple algebra which is division at a non-empty set $\mathrm{Ram}_0(A)$ of finite places of F which split in E/F, and let $S_0 \subset \mathrm{Ram}_0(A)$. Consider a definite unitary group $G = \mathrm{U}_A(n)$ over A as above. Let $K = \prod K_v \subset G(\mathbb{A})$ be a compact open subgroup satisfying the assumptions at the beginning of this section, and also assume that $K_v = G^1(F_v)$ for $v \in S_0$.*

Suppose that $p | \frac{m(K)}{\gcd(n, e_E^1) h_E^1}$. Then there exists a cuspidal automorphic representation π of $G'(\mathbb{A})$ with trivial central character such that: (i) the base change π_E is cuspidal, (ii) π_{v_0} is an unramified twist of Steinberg for $v_0 \in S_0$; (iii) π_v has a nonzero K_v-fixed vector when $G(F_v) \simeq G'(F_v)$; (iv) π_v is a holomorphic weight n discrete series for $v | \infty$; and (v) π is Hecke congruent to $\mathbb{1}_{G'} \bmod p$.

Note that by the classification of central simple algebras over number fields and Landherr's theorem, given E/F and any non-empty finite set Σ of finite places of F split in E/F, there exists $G = \mathrm{U}_A(n)$ as in Theorem 5.7 with $\mathrm{Ram}_0(A) = \Sigma$.

We make a few remarks on such a π as in the theorem. First, it cannot arise as an endoscopic lift from smaller unitary groups, so this congruence is "native" to $\mathrm{U}(n)$. Second, by the central character condition, (ii) means π_{v_0} is a twist of Steinberg by an unramified character of order dividing n. Also, (iii) implies π_v will be unramified whenever K_v is hyperspecial. Moreover, if every finite place $v \notin S_0$ satisfies $G(F_v) \simeq G(F'_v)$, and if K_v is good special maximal compact subgroup at all of these places, then we have strong control over π at all places: (iv) describes π_∞ completely; (ii) says π_v is an unramified twist of Steinberg for $v \in S_0$, and (iii) says π_v is K_v-spherical at all remaining v.

Proof. First Proposition 5.6 tells us there exists a non-abelian eigenform $\phi \in \mathcal{A}(G, K, 1)$ which is Hecke congruent to $\mathbb{1} \bmod p$. Let σ be the associated automorphic representation of $G(\mathbb{A})$. By Proposition 5.2, we know σ_E is cuspidal. We may take $\pi \in \mathrm{JL}(\sigma)$ such that (iv) holds and $\pi_v \simeq \sigma_v$ when $G(F_v) \simeq G'(F_v)$. For $v \in S_0$, since $K_{v_0} = G^1_{v_0}$ we must have that $\sigma_{v_0} = \chi_{v_0} \circ \det$, where χ_{v_0} is an unramified character of $F^\times_{v_0}$, so the local Jacquet–Langlands correspondent π_{v_0} is Steinberg twisted by χ_{v_0}. Finally, π satisfies (v) because $\mathbb{1}_{G'}$ has the same Hecke eigenvalues as $\mathbb{1} \in \mathcal{A}(G, K, 1)$ at almost all places. □

We now describe $m(K)$ for nice maximal compact subgroups K using [GHY01]. For simplicity we restrict to odd n. If desired, one can obtain masses for smaller compact subgroups $K' \subset K$ by recalling that $m(K') = [K : K']m(K)$. Let $\chi_{E/F}$ be the quadratic idele class character of F associated to E/F.

Proposition 5.8. *Let $G = \mathrm{U}_A(n)$ be a definite unitary group over A where n is odd. Let $\mathrm{Ram}_f(E)$ (resp. $\mathrm{Ram}_f(A)$) denote the set of finite primes of F above which E (resp. A) is ramified. Assume A_w is division for each w above $v \in \mathrm{Ram}_f(A)$. Let $S = \mathrm{Ram}_f(E) \cup \mathrm{Ram}_f(A)$. Take $K = \prod K_v$ such that K_v is maximal hyperspecial for finite $v \notin S$, $K_v = G^1(F_v)$ for $v \in \mathrm{Ram}_f(A)$, K_v is the stabilizer of a maximal lattice for $v \in \mathrm{Ram}_f(E)$, and $K_v = G(F_v)$ for $v|\infty$. Then*

$$m(K) = 2^{1-nd-|\mathrm{Ram}_f(E)|} \times \prod_{r=1}^{n} L(1-r, \chi^r_{E/F}) \times \prod_{v \in \mathrm{Ram}_f(A)} \left(\prod_{r=1}^{n-1} (q_v^r - 1) \right), \tag{5.1}$$

where $d = [F : \mathbb{Q}]$.

Proof. A general mass formula is given in [GHY01, Proposition 2.13], which is explicated for definite odd unitary groups over fields in Proposition 4.4 of *op. cit.* From those calculations, it follows that

$$m(K) = 2^{1-nd} \times \prod_{r=1}^{n} L(1-r, \chi^r_{E/F}) \times \prod_{v \in S} \lambda_v,$$

where λ_v is as follows. For a finite place v, let $\underline{H}'_v$ be Gross's canonical integral model of $H_v := G'_v$. Let $\underline{G}_v$ be the smooth integral model associated to a parahoric such that $K_v = \underline{G}_v(\mathfrak{o}_v)$. By our hypotheses, S is the set of finite places such that $\underline{G}_v \not\simeq \underline{H}^0_v$. Let $\bar{G}_v$ and $\bar{H}^0_v$ be the maximal reductive quotients of the special fibers of $\underline{G}_v$ and $\underline{H}^0_v$, which are reductive groups over $k_v = \mathfrak{o}_v/\mathfrak{p}_v$, with $\bar{G}_v$ possibly being disconnected. Then for $v \in S$,

$$\lambda_v = \frac{q_v^{-N(\bar{H}^0_v)}|\bar{H}^0_v(k_v)|}{q_v^{-N(\bar{G}_v)}|\bar{G}_v(k_v)|},$$

where $N(\cdot)$ denotes the number of positive roots over $\bar{k}_v$. When G_v is quasi-split, *loc. cit.* tells us $\lambda_v = \frac{1}{2}$ if E_v/F_v is ramified.

So we need only to compute λ_v for $v \in \mathrm{Ram}_f(A)$. In this case v splits in E/F so $\bar{H}^0_v \simeq \mathrm{GL}_n(k_v)$. Let $\mathcal{O}_v = A_v$ and $\mathfrak{P}_v$ the prime ideal of $\mathcal{O}_v$. Then $G_v \simeq \mathcal{O}_v^\times/(1+\mathfrak{P}_v) \simeq \mathbb{F}_{q_v^n}^\times$, which gives $\lambda_v = q_v^{-n(n-1)/2}\prod_{r=1}^{n-1}(q_v^n - q_v^r) = \prod_{r=1}^{n-1}(q_v^r - 1)$. □

Remark 5.9. By [GHY01], we can extend the formula (5.1) to include finite places v such that G_v is quasi-split and K_v is a special but not hyperspecial maximal compact. Each such place will contribute a factor of $\lambda_v = \frac{q^n+1}{q+1}$ to $m(K)$.

Consequently, Theorem 5.7 gives non-endoscopic Eisenstein congruences mod p which are Steinberg at v whenever p is a sufficiently large (depending on n and E/F) prime dividing some $q_v^r - 1$ (for $1 \le r \le n-1$).

Example 5.10. *Suppose $F = \mathbb{Q}$, $E = \mathbb{Q}(i)$. Then $|\mathrm{Cl}(\mathrm{U}(1))| = 1$. Let A/E be a central division algebra of odd prime degree $n = 2m+1$ which is ramified only at the primes of E above a fixed rational prime $\ell \equiv 1 \bmod 4$ (so necessarily division at $w|\ell$). Write $\chi = \chi_{E/F}$. It is well known that $L(1-r,\chi^r) = -\frac{1}{r}B_{r,\chi^r}$ (generalized Bernoulli number). Thus taking G and K as in Proposition 5.8, we get*

$$m(K) = \frac{1}{2^n n!}\prod_{r=1}^{m} B_{2r} \times \prod_{r=1}^{m} B_{2r+1,\chi} \times \prod_{r=1}^{n-1}(\ell^r - 1).$$

Suppose $p > n$ is a prime dividing some $\ell^r - 1$ where $1 \le r \le n-1$. Since $p > n$, the von Staudt–Clausen theorem tells us that p does not divide the denominators of any of the Bernoulli numbers $B_2, B_4, \dots, B_{2m}$. Also $B_{1,\chi}, B_{3,\chi}, \dots, B_{n,\chi}$ all have denominator 2. Hence Theorem 5.7 yields a non-endoscopic holomorphic weight n cuspidal representation π of $G'(\mathbb{A}) = \mathrm{U}(n,\mathbb{A})$ Hecke congruent to $\mathbb{1}_{G'} \bmod p$ such that π is (i) unramified at each odd finite $v \neq \ell$, (ii) spherical at $v = 2$, and (iii) an unramified twist of Steinberg at $v = \ell$. (By working with smaller compact subgroups K, one can remove the condition $p > n$.)

The same result is true for some additional values of p, independent of ℓ, coming from numerators of Bernoulli numbers. For instance, we can always take $p = 61$ for $7 \le n \le 59$ as $61|B_{7,\chi}$; we can take $p \in \{277, 2659\}$ if $11 \le n < p$

as $277 \cdot 2659 | B_{9,\chi}$*; we can take* $p = 19$ *if* $n = 13, 17$ *as* $19 | B_{11,\chi}$*; or we can take* $p \in \{43, 691, 967\}$ *if* $13 \leq n < p$ *as* $691 | B_{12}$ *and* $43 \cdot 97 | B_{13,\chi}$.

5.3.3. *Congruences from definite unitary groups over fields.* The reason to work with definite unitary groups over division algebras A is to guarantee we get a representation π with an Eisenstein congruence such that π is non-endoscopic. For instance, the congruences coming from Bernoulli numbers in Example 5.10 also occur on the definite unitary group $G = \mathrm{U}_A(3)$ where $A = M_3(\mathbb{Q}(i))$. Namely, if $p > n = 2m + 1$ and $p | B_{2r}$ or $p | B_{2r+1,\chi}$ for some $1 \leq r \leq m$, we still get an Eisenstein congruence mod p for some non-abelian π occurring in $\mathcal{A}_0(G, K, 1)$ where K_v is maximal everywhere. However, it may be that π is a lift from a smaller group, and that this congruence may be explained either as the symmetric square lift of a weight 2 Eisenstein congruence or as the Kudla lift of a weight 3 Eisenstein congruence as in Section 4.2.

First we state a sample result using definite unitary groups over fields. Unspecified notation is as in Theorem 5.7.

Proposition 5.11. *Let* E/F *be a CM extension of number fields,* n *a positive integer, and* G *the definite unitary group preserving* $\Phi = I \in M_n(E)$*. Take* $\mathfrak{l}$ *to either be* $\mathfrak{o}_E$ *or a prime ideal of* E *not ramified in in* E/F*. Let* $\ell \in \mathbb{N}$ *be the absolute norm of* $\mathfrak{l}$*. Let* $K = \prod K_v$ *where* $K_v = G_v$ *for* $v | \infty$*,* K_v *is the stabilizer of a maximal lattice for all finite* $v \neq \mathfrak{l}$ *(so all finite* v *if* $\ell = 1$*) such that* K_v *is hyperspecial at all such unramified* v*, and* $K_{\mathfrak{l}}$ *be the subgroup of* $G(\mathfrak{o}_{\mathfrak{l}})$ *fixing the (localization of the) lattice* $\Lambda = \mathfrak{o}_E \oplus \cdots \oplus \mathfrak{o}_E \oplus \mathfrak{l}\mathfrak{o}_E \subset E^n$*. If* n *is odd, put* $\lambda_v = \lambda_v(n) = \frac{1}{2}$*. If* n *is even, put* $\lambda_v = \lambda_v(n) = 1$ *or* $\frac{1}{2} \cdot \frac{q^n - 1}{q+1}$ *according to whether* $(-1)^{n/2} \in N(E_v^\times)$ *or not. Suppose* p *divides*

$$m(K) = 2^{1-nd} \times \frac{\ell^n - \chi_{E/F}(\ell)^n}{\ell - \chi_{E/F}(\ell)} \times \prod_{v \in \mathrm{Ram}_f(E)} \lambda_v \times \prod_{r=1}^{n} L(1 - r, \chi_{E/F}^r). \tag{5.2}$$

Then there exists an eigenform $\phi \in \mathcal{A}_0(G, K, 1)$ *which is Hecke congruent to* $\mathbb{1}$ mod p*. The factor* $\frac{\ell^n - 1}{\ell - 1}$ *is interpreted to be* 1 *if* $\ell = 1$*.*

Proof. The only thing to do is check the above formula for $m(K)$. This follows from the formulas in [GHY01] together with the fact that $[G(\mathfrak{o}_{\mathfrak{l}}) : K_{\mathfrak{l}}] = (\ell^n - 1)/(\ell - 1)$ (resp. $(\ell^n - (-1)^n)/(\ell + 1)$) if $\mathfrak{l}$ is split (resp. unramified inert) in E. Note $[G(\mathfrak{o}_{\mathfrak{l}}) : K_{\mathfrak{l}}]$ can be computed as the index of an $(n-1, 1)$-type maximal parabolic inside $\mathrm{GL}_n(\mathbb{F}_\ell)$ (resp. $\mathrm{U}_n(\mathbb{F}_\ell)$). □

We can numerically compute examples in Magma [BCP97] on definite U(3) (or more generally $\mathrm{U}(n)$) over a field using Markus Kirschmer and David Lorch's code for Hermitian lattices.[2] We compute unramified eigenvalues on $\mathrm{U}(n)$ at both split and inert places using the lattice method described in [GV14]. For

[2] Available at: http://www.math.rwth-aachen.de/~Markus.Kirschmer/magma/lat.html

convenience, we calculate with the standard definite form G of $\mathrm{U}(n)$ over $F = \mathbb{Q}$, taking $\Lambda = \mathfrak{o}_E \oplus \cdots \oplus \mathfrak{o}_E \oplus \mathfrak{l}\mathfrak{o}_E$, K_v to be stabilizer of Λ_v at all finite v, and $K_\infty = \mathrm{U}_3(\mathbb{R})$. In this case, denote $K(\ell) := K = \prod K_v$ where ℓ is the absolute norm of $\mathfrak{l}$. Note K_v is of the form in given the proposition except possibly at the finite $v \in \mathrm{Ram}_f(E)$ as now K_v need not be maximal. But since $K(\ell)$ is contained in some K' of the form in the proposition, we can still compute congruence examples as in the proposition on the potentially larger spaces of level $K(\ell)$.

Our calculations suggest that the Eisenstein congruences from the $n = 3$ case of Proposition 5.11 can also be explained by Theorem 4.3 as Kudla lifts of weight 3 Eisenstein congruences for $\mathrm{GL}(2)$. We numerically check if a form ϕ on $\mathrm{U}(3)$ is a Kudla lift of a weight 3 elliptic form f by comparing Hecke eigenvalues. Specifically, from computing local Hecke operators in Section 3, it follows that the p-th Hecke eigenvalue of ϕ is $a_p(f)+p$ at split primes p and $a_p(f)^2+2p^2+p-1$ at unramified inert places p.

Example 5.12. *Taking $n = 3$, $E/F = \mathbb{Q}(i)/\mathbb{Q}$, and $K = K(\ell) = K(5)$ we get $m(K) = \frac{31}{384}$, which gives an eigenform $\phi \in \mathcal{A}_0(G, K, 1)$ with an Eisenstein congruence mod 31. (Here $K(5)$ is of the form in the proposition.) We compute that $\mathcal{A}_0(G, K, 1)$ is spanned by 2 forms, say ϕ and $\bar{\phi}$, which are Galois conjugate over $\mathbb{Q}(\sqrt{5})$. We may choose ϕ so that the first few eigenvalues at split primes v for $\mathbb{1}$ and ϕ are given by:*

v	13	17	29	37
$\mathbb{1}$	183	307	871	1407
ϕ	$9-2\sqrt{5}$	$11+8\sqrt{5}$	$27-4\sqrt{5}$	$41-10\sqrt{5}$

and the first few eigenvalues at unramified inert primes v are given by

v	3	7	11	19
$\mathbb{1}$	84	2408	14652	130340
ϕ	$10+2\sqrt{5}$	$50+10\sqrt{5}$	$52-88\sqrt{5}$	$580-32\sqrt{5}$

These eigenvalues for $\mathbb{1}$ and ϕ are congruent mod $\mathfrak{p} = (\frac{1+5\sqrt{5}}{2})$. Note this congruence comes from the factor $\frac{\ell^3-1}{\ell-1}$ in the mass formula.

Numerically, ϕ agrees the Kudla lift of a newform $f \in S_3(20, \chi_{-4})$. The rationality field of f is a CM extension of $\mathbb{Q}(\sqrt{5})$ and f has a mod 31 congruence with $E_{3,1,\chi_{-4}} \in M_3(20, \chi_{-4})$.

Example 5.13. *Taking $n = 3$, $E/F = \mathbb{Q}(\sqrt{-19})/\mathbb{Q}$ and $K = K(1)$, we get $m(K) = \frac{11}{48}$. Here $\mathcal{A}_0(G, K, 1)$ is spanned by 2 rational eigenforms. One of these forms, which we call ϕ, has a congruence mod 11 with $\mathbb{1}$. Note this congruence comes from the numerator of the Bernoulli number $B_{3,\chi_{-19}}$.*

The space $S_3(19, \chi_{-19})$ has dimension 3, spanned by newforms $f, \bar{f}, g$, where f has rationality field $\mathbb{Q}(\sqrt{-13})$. Numerically, ϕ agrees with the Kudla lift of f (and of $\bar{f}$), and f has mod 11 congruences with the Eisenstein series $E_{3,1,\chi_{-19}}$ and $E_{3,\chi_{-19},1}$.

(In this example $K(1)$ is not of the form in the proposition, as the global lattice $\mathfrak{o}_E^{\oplus 3}$ is not maximal, but it has the same mass as a maximal lattice Λ'. However, taking K' to be the stabilizer of $\prod_q \Lambda'_q$, one only sees the Kudla lift of g in $\mathcal{A}_0(G, K')$ and not that of $f, \bar{f}$.)

Additional calculations together with known results and expectations about GL(2) Eisenstein congruences suggest that the congruences in Proposition 5.11 always arise as endoscopic lifts. Specifically, we expect the following GL(2) Eisenstein congruences. For simplicity, we only explicate this $F = \mathbb{Q}$.

Conjecture 5.14. *Let $k \geq 2$ be an integer, E be an imaginary quadratic field of discriminant D, and $\ell = 1$ or $\ell \nmid D$ a prime. Define $\lambda_q = \lambda_q(k)$ as in Proposition 5.11. If p divides*

$$\frac{1}{2^{k-1}k!} \times \frac{\ell^k - \chi_D(\ell)^k}{\ell - \chi_D(\ell)} \times \prod_{q \in \mathrm{Ram}_f(E)} \lambda_q \times \prod_{r=1}^{k} B_{r,\chi_D^r}, \tag{5.3}$$

then there exists an eigenform $f \in S_{k'}(\ell|D|, \chi_D^{k'})$ which is Hecke congruent mod p to the Eisenstein series $E_{k',1,\chi_D^{k'}} \in M_{k'}(\ell|D|, \chi_D^{k'})$ for some $2 \leq k' \leq k$.

Moreover we may take $k = k'$ in the following situations. If $\mathrm{Ram}_f(E)$ contains a single prime and k is odd, denote this prime by p_D; otherwise let $p_D = 1$. Let N be the conductor of χ_D^k, and put $\delta = |\mathrm{Ram}_f(E)|$ if k is odd and $\delta = 0$ if k is even.

(i) If $\ell = 1$, and p divides

$$\frac{1}{2^{k-1+\delta} p_D^{(k-1)/2}} \prod_{m=1}^{\lfloor (k-1)/2 \rfloor} \Big(\prod_{(q-1)|2m} \frac{1}{q} \Big) \times \frac{B_{k,\chi_D^k}}{k},$$

then we may take $k' = k$ and $f \in S_k(N, \chi_D^k)$ to be a newform.

(ii) If $k \geq 3$, ℓ is prime, and p divides

$$\frac{1}{2^{k-1+\delta} p_D^{\lfloor (k+1)/2 \rfloor}} \prod_{m=1}^{\lfloor k/2 \rfloor} \Big(\prod_{(q-1)|2m} \frac{1}{q} \Big) \times \frac{\ell^k - \chi_D(\ell)^k}{\ell - \chi_D(\ell)},$$

then we may take $k' = k$ and $f \in S_k(\ell N, \chi_D^k)$ to be a newform.

Note if k is even, then $N = 1$ and there is no dependence on D in (i) or (ii)—e.g., the final factor in (ii) is $\frac{\ell^k - 1}{\ell - 1}$.

The first part of the conjecture corresponds to the expectation that the congruences in Proposition 5.11 can also be produced as lifts of GL(2) Eisenstein congruences of the above type. Specifically, when $k = k'$ above, the posited GL(2) congruence corresponds to a congruence in Proposition 5.11 via an endoscopic lift as in Theorem 4.6. However, there are many lifts from GL(2) to U(n) (e.g., the Ikeda lift in Theorem 4.7) and not all of the congruences in Proposition 5.11

can directly be explained by Theorem 4.6. For instance, when $D = -19$ and we have mod 11 Eisenstein congruences on both U(3) and U(4) with $\ell = 1$ in Proposition 5.11 because $11|L(-2, \chi_{-19})$. Correspondingly, there is a mod 11 congruence in $S_3(19, \chi_{-19})$ but not one in $S_4(19)$. It appears that both the U(3) and U(4) congruences arise as lifts of the weight 3 congruence (the former being the Kudla lift, and the latter not of the type considered in Theorem 4.6).

The reasoning for the second part of the conjecture is as follows. The condition in (i) means that a new factor of p arises in (5.3) when increasing the weight from $k-1$ to k. (In other words, the depth of the congruence in the sense of Remark 2.3 should increase when going from weight $k-1$ to weight k.) Here we explicated denominators (and ignored numerators) of Bernoulli numbers, using that the denominator of $\frac{B_{2n}}{2n}$ is $\prod_{(q-1)|2n} q$ (where q denotes a prime) and the denominator of $\frac{B_{2n+1,\chi_D}}{2n+1}$ is p_D. Similarly, for the ℓ prime case the divisibility condition in (ii) forces a new factor of p in (5.3) when compared with the $\ell = 1$ case.

For $k \geq 4$ even, we also applied the following reasoning: if one has a mod p congruence in $S_k(\ell D)$ for all D, almost surely there is one in $S_k(\ell)$. (Note $k = 2$ is special: one does not have congruences in $S_2(\ell)$ for $p|(\ell+1)$, but rather for $p|(\ell-1)$, and one really should look at the spaces $S_2(\ell D)$—cf. [Maz77], [Yoo19], and Section 8).

This explicit conjecture is mostly known by now, and our primary reason for stating the above conjecture was to make precise the apparent connection between mass formulas on U(k) and weight k GL(2) congruences. For instance, generalizing work of Billerey and Menares [BM16] (cf. Section 6.2), Spencer [Spe18] showed that for a Dirichlet character χ of conductor N, there is a mod p Hecke congruence with $E_{k,1,\chi}$ in $S_k(\ell N, \chi)$ if $p \nmid 6N$, $\ell \nmid N$ and p divides $(p^k - \chi(p))\frac{B_{k,\chi}}{k}$. Thus apart from the specification of the exact level of f in (i) and (ii), the above conjecture is known if $p > k+1$ and $p \nmid N$. However, at least when k is even, one can typically deduce the exact level of f in the conjecture by comparing [Spe18] with work of Ribet [Rib75] which says that (for $p > k+1$) one has a mod p Eisenstein congruence in $S_k(1)$ if and only if $p|\frac{B_k}{k}$. See also [BK, Proposition 5.2] for a similar congruence result with an argument that is valid for small primes, but with less control over the level of f. We also checked the above conjecture numerically for small choices of parameters.

Remark 5.15. Conjecture 5.14 can be extended to different ways. First, there is an obvious extension to Hilbert modular forms using (5.2). Second, one can use the same idea of checking when new factors divide the mass to extend these conjectures to newforms of higher levels (e.g., replace ℓ with a product $\ell_1 \cdots \ell_t$ by allowing more general compact subgroups in Proposition 5.11). E.g., see Conjecture 6.6 and Remark 8.2. To our knowledge, there are no general results along the lines of Conjecture 5.14 for higher levels or for Hilbert modular forms when $k > 2$.

We note one more phenomenon. While the above conjecture says that the congruences seen in the mass formula in Proposition 5.11 should also be seen on GL(2), we can find examples to see the converse of this statement is not true, at least for small p.

Example 5.16. *There is a unique CM newform $f \in S_3(11, \chi_{-11})$. It has rational coefficients and is congruent to $E_{3,1,\chi_{-11}}$ mod 3. Taking $n = 3$, $E = \mathbb{Q}(\sqrt{-11})$ and $K = K(1)$, we compute $\dim \mathcal{A}(G, K) = 2$. Say $\mathcal{A}_0(G, K) = \langle \phi \rangle$. Numerically ϕ agrees with the Kudla lift of f and is congruent to $\mathbb{1}$ mod 3. However, the mass here is $\frac{1}{16}$, so we do not see this lifted congruence in the mass formula.*

6. Symplectic and odd orthogonal groups

Here we will discuss our construction of Eisenstein congruences for groups of type B_n and C_n, i.e., odd orthogonal and symplectic groups. For simplicity we will focus on the rank 2 case where these coincide—namely $\mathrm{SO}(5) \simeq \mathrm{PGSp}(4)$—and we have a more explicit understanding of the representations. Then we briefly discuss the higher rank situation in Section 6.3.

6.1. SO(5)**.** Let F be a totally real number field of degree d and B/F a totally definite quaternion algebra. Consider the projective similitude group $G = \mathrm{PGU}(2, B)$, which is an inner form of the quasi-split $G' = \mathrm{SO}(5) \simeq \mathrm{PGSp}(4)$ that is compact at infinity. We consider a maximal compact subgroup $K = \prod K_v$ as follows. Let S be a finite set of finite places containing all finite places where B is ramified. Take $K_v = G(F_v)$ for $v|\infty$ and K_v hyperspecial maximal compact for finite $v \notin S$. Let S_1 (resp. S_2) be the set of finite places which are ramified in B such that K_v stabilizes a maximal lattice in the principal genus (resp. non-principal genus). Let S_3 be the set of places in S which are split in B, and assume K_v is not hyperspecial for $v \in S_3$, i.e., K_v is the level $\mathfrak{p}_v$ paramodular subgroup of $\mathrm{PGSp}_4(F_v)$.

Proposition 6.1. *Suppose p divides the numerator of*

$$m(K) = 2^{1-2d-|S|} |\zeta_F(-1)\zeta_F(-3)| \prod_{v \in S} \lambda_v,$$

where

$$\lambda_v = \begin{cases} (q_v + 1)(q_v^2 + 1) & \text{if } v \in S_1, \\ q_v^2 - 1 & \text{if } v \in S_2, \\ q_v^2 + 1 & \text{if } v \in S_3. \end{cases}$$

Then there exists an eigenform $\phi \in \mathcal{A}_0(G, K)$ which is Hecke congruent to $\mathbb{1}$ mod p.

Note that in all cases the local factors λ_v are factors of $q_v^4 - 1$.

Proof. By Proposition 2.1, it suffices to prove the mass formula. This mass formula follows from the mass formula for SO(5) in [GHY01]. In the case of $F = \mathbb{Q}$, the mass formula was given earlier in the context of $\mathrm{PGU}(2, B)$ by Hashimoto and Ibukiyama ([HI81], [HI83]). $\square$

As with the case of unitary groups, there should be a generalized Jacquet–Langlands correspondence associating automorphic representations of $G(\mathbb{A})$ to automorphic representations of $G'(\mathbb{A})$. This would follow from the endoscopic classification of representations of $G(\mathbb{A})$, as conjectured in [Art13, Chapter 9] (see also [Sch18] for an explicit description of this classification for SO(5)). However, even for SO(5) this has not been completed, though some instances of the Jacquet–Langlands correspondence are known by [Sor09].

We would like to know whether this method can give cuspidal, and if possible non-CAP, representations of $G'(\mathbb{A})$ with Eisenstein congruences. The local archimedean parameter corresponding to the trivial representation for $G(\mathbb{R})$ matches the local parameter ρ_3 corresponding to the packet containing the holomorphic discrete series $\mathcal{D}_3$ of scalar weight 3 for $G'(\mathbb{R})$. This packet has size 2, and the other (generic) member $\mathcal{D}_{(3,-1)}$ is the large discrete series with minimal K-type $(3, -1)$. Hence we expect the Eisenstein congruences from Proposition 6.1 to transfer to Eisenstein congruences for holomorphic or non-holomorphic Siegel modular forms of archimedean type ρ_3—specifically, discrete automorphic representations π' of $G'(\mathbb{A})$ such that either $\pi'_v \simeq \mathcal{D}_3$ for all $v|\infty$, or for all but one $v|\infty$ and the remaining archimedean component is $\mathcal{D}_{(3,-1)}$.

To be more explicit about the weight, for a global Arthur parameter ψ for $G'(\mathbb{A})$, one in general needs a parity condition to be satisfied for a global tensor product $\pi' = \bigotimes \pi'_v$ to lie in the discrete spectrum of $G'(\mathbb{A})$. This parity condition tells us whether we can take π' to be holomorphic or not. We will say π' (or ψ) is of Saito–Kurokawa type if the Arthur parameter ψ is of the form $\mu \boxplus (\chi \otimes \nu(2))$, where μ is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, χ a quadratic idele class character of $\mathrm{GL}_1(\mathbb{A})$ and $\nu(2)$ is the irreducible 2-dimensional representation of SU(2). It follows from [Sch18] that, if π' has a paramodular fixed vector for all finite v, then the parity condition means we can take π_∞ to be holomorphic if and only if ψ is not of Saito–Kurokawa type $\mu \boxplus (\chi \otimes \nu(2))$ for some μ, χ such that $\varepsilon(1/2, \mu \otimes \chi) = -1$.

While the behavior of level structure (existence of fixed vectors under explicit compact subgroups) along the conjectural Jacquet–Langlands correspondence for SO(5) is not understood even conjecturally in general, Ibukiyama ([Ibu85], [Ibu18]) has made global conjectures that suggest how the level structure behaves for maximal compact K as above. Indeed, it follows from the tables [Sch17] and [RS07] that the (principal genus stabilizer) K_v-level structure on $G(F_v)$ for $v \in S_1$ corresponds to being new of Klingen level $\mathfrak{p}_v$ on $G'(F_v)$, and the (non-principal genus stabilizer) K_v-level structure on $G(F_v)$ for $v \in S_2$ corresponds to paramodular level structure on $G'(F_v)$. We remark the paramodular case of Ibukiyama conjectures were recently proven by [vHo]. Hence Proposition 6.1 suggests the following congruence statement about Siegel modular forms.

Conjecture 6.2. *For $p|m(K)$ in the notation of Proposition 6.1, there exists a holomorphic or non-holomorphic degree 2 Siegel modular form f of archimedean type ρ_3 such that (i) f is unramified outside S; (ii) f has Klingen level $\mathfrak{p}_v$ for $v \in S_1$; (iii) f has paramodular level $\mathfrak{p}_v$ for $v \in S_2 \cup S_3$; and (iv) f is Hecke congruent mod p to the trivial representation $\mathbb{1}$ of $\mathrm{PGSp}(4)$.*

In the next section, we will suggest that, at least when $S_1 = \emptyset$, there is always a holomorphic Saito–Kurokawa form f satisfying the above congruence.

6.2. **Lifting GL(2) Eisenstein congruences.** Here we will explain how, under certain conditions, one can also construct congruences predicted in Conjecture 6.2 via Saito–Kurokawa lifts. First we state recent results on GL(2) Eisenstein congruences in higher weight.

Proposition 6.3 ([BM16], [GP18])**.** *Let $k \geq 2$, $p \geq \min\{2k-1, 5\}$, and ℓ a prime. If $p|(\ell^{2k}-1)$, then there exists a newform $f \in S_{2k}(\Gamma_0(\ell))$ which is congruent mod p to a weight $2k$ Eisenstein series $E_{k,\ell} \in M_{2k}(\Gamma_0(\ell))$. Moreover, if $p|(\ell^k \pm 1)$, then we may take f to have Atkin–Lehner sign ± 1 at p.*

In fact, these works also give congruences if $p|(\ell^{2k-2}-1)$ with additional divisibility criteria of Bernoulli numbers, and [BM16] proves an if-and-only-if statement (without the Atkin–Lehner sign condition) under the stronger hypothesis that $p \geq 2k+3$. The specification of Atkin–Lehner sign and the extension to $p \geq 2k-1$ was done in [GP18]. In [BM16], the authors also conjecture that their congruence result extends to squarefree level in a natural way. Note Conjecture 5.14(iii) suggests an extension of this result to small primes p.

We recall some facts about the Saito–Kurokawa lift. Let μ be a discrete L^2-automorphic representations of of $\mathrm{PGL}_2(\mathbb{A})$. We define the Saito–Kurokawa lift $SK(\mu)$ of μ to be a certain discrete automorphic representation of $\mathrm{SO}_5(\mathbb{A})$ associated to the formal global parameter $\mu \boxplus (\mathbb{1} \boxtimes \nu(2))$ as in [Art13]. This corresponds to the endoscopic functorial lifting of $(\mu, \mathbb{1} \boxtimes \nu(2))$ associated to the standard embedding $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ from the dual group of $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ to the dual group of $\mathrm{SO}(5)$. We have not specified $SK(\mu)$ as a unique representation in a global packet $\Pi(\mu)$. Under a parity condition, $SK(\mu)$ can be defined as in [Sch05].

For simplicity, we just specify some information about $SK(\mu)$ in the special case $F = \mathbb{Q}$ and μ corresponds to an elliptic newform $f \in S_4(\Gamma_0(N))$ with N squarefree. Then we may take $SK(f) := SK(\mu)$ to be unramified at primes $p \nmid N$ and paramodular of level p for $p|N$ [Sch07]. Let $\omega(N)$ be the number of primes dividing N. If $\varepsilon(1/2, f) = (-1)^{\omega(N)}$, then we may take $SK(f)$ to be a holomorphic Siegel cusp form of scalar weight 3 [Sch05]. If $\varepsilon(1/2, f) = 1$, then an examination of the parity condition shows we cannot take $SK(f)$ to be holomorphic, but we can take $SK(f)$ to be generic of weight $(3,-1)$. (For the product $\otimes\pi_v$ of local components to occur discretely, one needs the associated character of the component group to have order $\varepsilon(1/2, f)$. Both the non-archimedean local paramodular representations and the archimedean representation $\mathcal{D}_{(3,-1)}$ correspond to the trivial characters of the local component groups—cf. [Sch18].)

Let E_4 be the weight 4 Eisenstein series in $M_4(\Gamma_0(1))$. Then the trivial representation $\mathbb{1}$ of SO(5) locally agrees with the local Saito–Kurokawa lifts of (the local principal series representations attached to) E_4. Thus it follows from Proposition 3.8 ($k=4$, $k'=2$) that if an eigenform $f \in S_4(\Gamma_0(N))$ is Hecke congruent to E_4 mod p, then the Saito–Kurokawa lift $SK(f)$ will be Hecke congruent to $\mathbb{1}$ mod p.

Hence Proposition 6.3, together with the above discussion of Saito–Kurokawa lifts, gives us a special case of Conjecture 6.2, namely when whe $F=\mathbb{Q}$, $S = S_2 \cup S_3 = \{\ell\}$ and $p \geq 5$:

Corollary 6.4. *Let $\varepsilon = \pm 1$. Suppose ℓ is a prime and $p \geq 5$ such that $p|(\ell^k+\varepsilon)$. Then there exists a degree 2 archimedean type ρ_3 Saito–Kurokawa type Siegel modular form $SK(f)$ Hecke congruent to $\mathbb{1}$ mod p, such that f is cuspidal and holomorphic if $\varepsilon=-1$ and $SK(f)$ is non-holomorphic if $\varepsilon=+1$.*

Remark 6.5. By considering the action of ramified Hecke operators on $\mathbb{1}$, it should be possible to show that, for p odd, the ϕ from Proposition 6.1 has local epsilon sign -1 for $v \in S_2$ and sign $+1$ for $v \in S_3$. (It is not hard to prove the analogous statement for GL(2) assuming $p \neq 2$—for $p=2$ it is not true as one cannot distinguish signs mod 2.) Consequently, looking at the parity condition for the global packet, we expect that one can take the f in Conjecture 6.2 to be holomorphic when $S_1=\emptyset$ and $|S_2| \equiv [F:\mathbb{Q}] \mod 2$. Since B is ramified at an even number of places, the condition $S_1=\emptyset$ should be sufficient.

Since local conditions, such as $p|(q_v^2-1)$, are often sufficient to produce congruences, and the local factor in Proposition 6.3 for weight 4 matches with the local factors in the mass formula in Proposition 6.1, it is reasonable to expect that the congruences predicted in Conjecture 6.2 are always achieved by Saito–Kurokawa lifts. As with the case of unitary groups over fields in Section 5.3.3, this suggests that the mass formula does not give "new" Eisenstein congruences for SO(5), but does provide a way to see higher weight Eisenstein congruences on GL(2). Specifically, Conjecture 6.2 suggests the following generalization of Proposition 6.3 in weight 4.

Conjecture 6.6. *Let N_1, N_2 be coprime squarefree positive integers with $\omega(N_1)$ odd. Suppose p is an odd prime dividing $\frac{1}{45}\prod_{\ell|N_1}(\ell^2-1)\prod_{\ell|N_2}(\ell^2+1)$. Then there exists an eigenform $f \in S_4(\Gamma_0(N_1N_2))$ which is Hecke congruent to E_4 mod p. If $p \geq 7$, and $p|(\ell^2-1)$ for each $\ell|N_1$ and $p|(\ell^2+1)$ for each $\ell|N_2$, then we* *may take f to be new of level N_1N_2 such that f has Atkin–Lehner sign -1 (resp. $+1$) if $\ell|N_1$ (resp. $\ell|N_2$).*

In general, the f in the conjecture need not be new for level N_1N_2. For instance if $N_1 = 2$ and $N_2 = 5$, it predicts an Eisenstein congruence mod 13 in level 10. However there is no new congruence at level 10, but rather an old one coming from level 5. The reason to expect that we may take f to be new in the situation prescribed in the second part of the conjecture is because then each local factor contributes to the depth of the congruence as mentioned in Remark 2.3 (see also Remark 8.2). The same reasoning suggests the obvious analogue of Conjecture 6.6 for parallel weight 4 Hilbert modular forms. We also remark that by considering mass formulas for K which are non-maximal, one is led to expect weight 4 Eisenstein congruences mod p in levels divisible by p^2, analogous to what happens in weight 2 (see Theorem 8.1).

6.3. **Higher rank.** Here we briefly discuss the analogue of the mass formulas for the higher rank analogues of $\mathrm{Sp}(4)$ and $\mathrm{SO}(5) \simeq \mathrm{PGSp}(4)$.

Suppose G is a definite form of $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$ over $\mathbb{Q}$ and $K = \prod K_v$ where $K_v = G_v$ for $v|\infty$ and K_v is hyperspecial maximal compact outside of a finite set of finite places S. Then Shimura's mass formula as stated in [GHY01, Propositions 7.4 and 9.4] is of the form

$$m(K) = \frac{\tau(G)}{2^n} \prod_{r=1}^{n} \zeta(1-2r) \prod_{v \in S} \lambda_v. \tag{6.1}$$

Here $\tau(G) = 1$ or 2 according to whether $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$.

If $G(F_v)$ is quasi-split and we take K_v to be the Iwahori subgroup at some finite quasi-split v, we simply have $\lambda_v = [G(\mathfrak{o}_v) : K_v]$. Putting $q = q_v$, these indices are, respectively, $|\,\mathrm{Sp}(2n, \mathbb{F}_q)| = q^{n^2} \prod_{r=1}^n (q^{2r}-1)$ and (for q odd) $|\,\mathrm{SO}(2n+1, \mathbb{F}_q)| = q^{2n^2+3n} \prod_{r=1}^n (q^{2r}-1)$. Alternatively, if K_v is not hyperspecial but stabilizes a maximal integral lattice, then the precise form of Shimura's mass formulas tells us that the numerator of λ_v is a divisor of $q_v^{2n} - 1$ if $G = \mathrm{SO}(2n+1)$ and $\lambda_v = \prod_{r=1}^n (q_v^r + (-1)^r)$ if $G = \mathrm{Sp}(2n)$. See [GHY01] for details.

If $G = \mathrm{Sp}(2n)$ and G' is the quasi-split inner form, then the Eisenstein congruences on Proposition 2.1 should, via a generalized Jacquet–Langlands correspondence between G and G', give rise to Eisenstein congruences for scalar weight $n+1$ holomorphic degree n Siegel modular forms (at least under suitable parity conditions for the local parameters).

However, there are conjecturally many functorial lifts from $\mathrm{GL}(2)$ to $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$, and there seems to be no reason to expect that the Eisenstein congruences constructed by Proposition 2.1 will not arise from lifts of $\mathrm{GL}(2)$ congruences. For instance, when n is odd, one has an Ikeda lift from weight 2 forms on $\mathrm{PGL}(2)$ to scalar weight $n+1$ Siegel modular forms on $\mathrm{Sp}(2n)$. Thus we expect congruences coming from the factors $q_v \pm 1$ in the mass formula for

$\mathrm{Sp}(2n)$ to already be explained by Ikeda lifts of $\mathrm{GL}(2)$ congruences in weight 2. This suggests suitable generalizations of Conjecture 6.6 to higher weight for both elliptic and Hilbert modular forms.

7. G_2

7.1. **Eisenstein congruences for G_2.** Let F be a totally real number field of degree d and G/F be the form of G_2 which is compact at each infinite place. Necessarily G_v is quasi-split at each finite place. Let $K = \prod K_v$ be a maximal compact subgroup of $G(\mathbb{A})$ which is hyperspecial at almost all places. At a finite place v, by Bruhat–Tits theory, the conjugacy classes maximal compact subgroups correspond to (complements of) vertices in the extended Dynkin diagram for G_2. There are 2 types of non-hyperspecial maximal compact subgroups: (i) those corresponding to Dynkin diagram $A_1 \times A_1$, and (ii) those corresponding to Dynkin diagram A_2. Let S_1 (resp. S_2) be the set of finite places v where K_v is of type (i) (resp. (ii)).

Theorem 7.1. *Suppose p divides*

$$m(K) = \frac{1}{2^d}\zeta_F(-1)\zeta_F(-5)\prod_{v\in S_1}(q_v^4+q_v^2+1)\times\prod_{v\in S_2}(q_v^3+1).$$

Then there exists an eigenform $\phi \in \mathcal{A}_0(G,K)$ such that ϕ is Hecke congruent to $\mathbb{1}$ mod p.

In particular, if $F=\mathbb{Q}$, we have $m(K) = \frac{1}{2^5\cdot 3^3\cdot 7}\prod_{\ell\in S_1}(\ell^4+\ell^2+1)\cdot\prod_{\ell\in S_2}(\ell^3+1)$. Note that each factor for $v \in S_1 \cup S_2$ is a factor of $\frac{q_v^6-1}{q_v-1}$, which would be the local factor if we took K_v to be Iwahori.

Proof. Again it suffices, to prove the mass formula for $m(K)$. This is proved in [CNP98, Theorem 3.7 and (2)]. ☐

7.2. **Lifts from GL(2).** Conjecturally, there are three functorial lifts from $\mathrm{PGL}(2)$ to G, corresponding to three classes of embeddings of dual groups $\iota : \mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$, and are described in [LP02]. (There are 4 classes of embeddings from $\mathrm{SL}_2(\mathbb{C})$ to $G_2(\mathbb{C})$, but one is not relevant for compact G_2.) We can label these by ι_s, ι_l and ι_r, which respectively send a unipotent element of $\mathrm{SL}_2(\mathbb{C})$ to a short root for G_2, a long root for G_2 and regular unipotent element for G_2. The main aspect that is relevant here is that ι_s (resp. ι_l, resp. ι_r) associates the weight 4 (resp. weight 6, resp. weight 2) discrete series of $\mathrm{PGL}_2(\mathbb{R})$ to the trivial representation of $G(\mathbb{R})$.

We remark that corresponding lifts to split G_2 associated to ι_s and ι_l have been studied, e.g., see [GG06] and [GG09].

Now, by Proposition 3.10, if f is an elliptic or Hilbert modular form of (parallel) weight k and is congruent to the Eisenstein series E_k mod p for $k \in \{2,4,6\}$, then the appropriate functorial lift to $G(\mathbb{A})$, if automorphic, will be Hecke congruent to $\mathbb{1}$ mod p. As in the case of SO(5) and unitary groups over fields, we expect that the Eisenstein congruences from Theorem 7.1 arise from PGL(2) Eisenstein congruences coming from a suitable generalization of Proposition 6.3 in weight 6. Moreover, we expect such congruences to arise as lifts from PGL(2) cusp forms which have Atkin–Lehner sign $+1$ at each $v \in S_2$.

8. GL(2)

In this section, we discuss weight 2 Eisenstein congruences in the case of GL(2) (or rather PGL(2)). This was treated in [Mar17] over totally real number fields F originally under the assumption that $h_F = h_F^+$. However, as pointed out to us by Jack Shotton, the published argument only gives cuspidal congruences mod p when $p \nmid h_F$ and h_F is odd.[3]

Here we explain how to remove this class number condition by working with PGL(2) rather than GL(2) and using congruence modules as in Section 5.3.1. Moreover, even in the case that $p \nmid h_F$ and h_F is odd, we slightly refine our earlier result by making use of [Mar] together with congruence modules.

Let F be a totally real number field of degree d, and B/F be a definite quaternion algebra. Let $\mathcal{O}$ be a special order of B (in the sense of Hijikata–Pizer–Shemanske) of the following type. For a prime v split in B, assume $\mathcal{O}_v$ is an Eichler order of level $\mathfrak{p}_v^{r_v}$ (with $r_v = 0$ for almost all v). For v a finite prime at which B ramifies, assume $\mathcal{O}_v$ is of the form $\mathfrak{o}_{E,v} + \mathfrak{P}_v^{2m}$ where m is a non-negative integer, $\mathfrak{o}_{E,v}$ is the ring of integers of the unramified quadratic extension E_v/F_v and $\mathfrak{P}_v$ is the unique prime ideal for B_v. In the latter case we say $\mathcal{O}_v$ is a special order of level $\mathfrak{p}_v^{2m+1}$ (of unramified quadratic type). Let $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$) be $\prod_v \mathfrak{p}_v^{r_v}$ where v ranges over the finite primes such that B/F splits (resp. ramifies) and $\mathfrak{p}_v^{r_v}$ is the level of $\mathcal{O}_v$. Let $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2$. Let $E_{2,\mathfrak{N}}$ be a parallel weight 2 Eisenstein eigenform over F of level $\mathfrak{N}$ which has Hecke eigenvalue q_v (resp. 1) for $v|\mathfrak{N}_1$ (resp. $v|\mathfrak{N}_2$), and Hecke eigenvalue $q_v + 1$ for finite $v \nmid \mathfrak{N}$.

Theorem 8.1. *Suppose p is a rational prime which divides*

$$2^{1-d-e-|\{v|\mathfrak{N}_1\}|}|\zeta_F(-1)| \prod_{v|\mathfrak{N}_1} q_v^{r_v-1}(q_v-1) \prod_{v|\mathfrak{N}_2} q_v^{r_v-1}(q_v+1), \tag{8.1}$$

where e is the 2-exponent of the narrow class group $\mathrm{Cl}^+(F)$. Then there exists a parallel weight 2 cuspidal Hilbert eigenform f of level $\mathfrak{N}$ and trivial nebentypus such that f is Hecke congruent to $E_{2,\mathfrak{N}}$ mod p at all finite v such that $r_v \leq 1$. Moreover, for $v|\mathfrak{N}_1$ we may take f such that the v-part of the exact level of f is $\mathfrak{p}_v^{s_v}$, where (i) s_v is odd; (ii) $s_v = 1$ if $p \nmid q_v$; and (iii) $s_v = r_v$ for any single chosen $v|\mathfrak{N}_1$ lying above p (if such a v exists).

Proof. Let $G = PB^\times$ and $K = \prod K_v$, where K_v the image of $\mathcal{O}_v^\times$ in $PB^\times$ for $v < \infty$ and $K_v = G_v$ for $v|\infty$. From the SO(3) case of the mass formula in

[3] See `arXiv:1601.03284v4` for a corrected version of [Mar17].

[GHY01], one deduces that (8.1) is $2^{-e}m(K)$ (compare with the mass formula in [Mar17]). As explained in [Mar17], the constant function $\mathbb{1}$ on $\mathrm{Cl}(K)$ is a Hecke eigenfunction of all Hecke operators T_v (v finite), with the same Hecke eigenvalues as the modular form $E_{2,\mathfrak{N}}$ for any v with $r_v \le 1$. Then by Proposition 2.1, there exists an eigenform $\phi \in \mathcal{A}_0(G,K)$ such that ϕ is Hecke congruent to $\mathbb{1}$ mod p. This congruence is also valid for ramified Hecke eigenvalues when $r_v = 1$ (again, see *op. cit.*).

Now we want to show we can take ϕ to be non-abelian. The abelian forms in $\mathcal{A}_0(G,K)$, viewed as functions on $\mathbb{A}^\times \backslash B^\times(\mathbb{A})/B^\times(F_\infty)$, are generated by the forms $\psi \circ N$, where $N : B^\times \to F^\times$ is the reduced norm and ψ is a quadratic character of $\mathrm{Cl}^+(F)$. Necessarily, such a form can only be congruent to $\mathbb{1}$ mod p if $p = 2$. Using the same argument as in Proposition 5.6 (the relevant congruence module for the space of abelian forms orthogonal to $\mathbb{1}$ has 2-exponent e, whereas the congruence module for $\mathcal{A}_0(G,K)$ has 2-exponent $v_2(m(K))$), gives such a non-abelian ϕ.

Let $\mathcal{S}(G,K)$ be the orthogonal complement of the abelian subspace of $\mathcal{A}(G,K)$. By the Jacquet–Langlands correspondence for modular forms from [Mar], we have an isomorphism of Hecke modules, for the Hecke algebras away from the set of $v|\mathfrak{N}_1$ with $r_v > 1$,

$$\mathcal{S}(G,K) \simeq \bigoplus S_2^{\mathfrak{M}\text{-new}}(\mathfrak{M}\mathfrak{N}_2), \quad \mathfrak{M} = \prod_{v|\mathfrak{N}_1} \mathfrak{p}_v^{2m_v+1},\ 1 \le 2m_v + 1 \le r_v.$$

The spaces on the right are the spaces of parallel weight 2 Hilbert cusp forms of level $\mathfrak{M}\mathfrak{N}_2$ which are locally new at each $v|\mathfrak{M}$ (the associated local representation of $\mathrm{PGL}_2(F_v)$ has conductor $\mathfrak{p}_v^{2m_v+1}$), and $\mathfrak{M}$ runs over divisors of $\mathfrak{N}_1$ which have odd exponent at every $v|\mathfrak{N}_1$. This shows (i).

Let f be a Hilbert modular form corresponding to ϕ. If $v|\mathfrak{N}_1$ such that $p \nmid q_v$, then if necessary we may enlarge K by taking $K_v = \mathcal{O}_{B,v}^\times$ at v so that $r_v = 1$. This forces f to have exact level $\mathfrak{p}_v$ at v, i.e., we may assume (ii).

For (iii), suppose there exists $v|\mathfrak{N}_1$ such that $p|q_v$. If $r_v = 1$, there is nothing to show, so assume $r_v \ge 3$. Then we may use the above decomposition of $\mathcal{S}(G,K)$ together with the argument from Proposition 5.6. Namely, for a sufficiently large rationality field L, we may decompose $\mathcal{A}_0^L(G,K) = X_1(L) \oplus X_2(L)$, where $X_1(L)$ is generated by abelian forms together with cuspidal eigenforms which have level at most $\mathfrak{p}_v^{r_v-2}$ at v, and $X_2(L)$ is generated by cuspidal eigenforms which have exact level $\mathfrak{p}_v^{r_v}$ at v. Now $X_1(L) = A_0^L(G,K')$ where K' is defined in the same way as K except replacing r_v with $r_v - 2$. Then the p-exponent of the congruence module for X_1 is simply $v_p(m(K'))$. But this is strictly less than $v_p(m(K))$, so the argument of Proposition 5.6 gives an eigenform in $X_2(L)$ which is Hecke congruent to $\mathbb{1}$ mod p. $\square$

Remark 8.2. If $v|\mathfrak{N}_2$ such that $p|(q_v + 1)$ if $r_v = 1$ (resp. $p|q_v$ if $r_v > 1$), we expect that we can also assume the f in the theorem is locally new at v. Similarly, we expect we can impose (iii) for all v such that $p|q_v$. This is because then the local factor at v contributes to the $v_p(m(K))$, i.e., contributes to the p-exponent of the relevant congruence module. Alternatively, this factor contributes to the depth of the congruence mentioned in Remark 2.3. In order to prove this along the lines of our argument for (iii), we would need to know the p-exponent of the congruence module for the v-old forms. We do not attempt to study this here.

Remark 8.3. Ribet and Yoo (see [Yoo19]) have studied weight 2 Eisenstein congruences with fixed Atkin–Lehner signs for elliptic modular forms of squarefree level under some conditions. If $p > 2$ and $\mathfrak{N}$ is squarefree, then f as in the theorem necessarily has Atkin–Lehner sign -1 at each $v|\mathfrak{N}_1$, and Atkin–Lehner sign $+1$ at each $v|\mathfrak{N}_2$ such that the v-part of the exact level of f is $\mathfrak{p}_v$.

Corollary 8.4. *Let $F = \mathbb{Q}$ and p be prime. Then for any $m \geq 1$ (resp. $m \geq 3$) if p is odd (resp. $p = 2$), there exists a newform $f \in S_2(p^{2m+1})$ which is Hecke congruent to $E_{2,p}$ $\mathrm{mod}\ p$ away from p.*

9. Special mod p congruences for $\mathrm{U}(p)$

Given a weight 2 cuspidal newform f on $\mathrm{PGL}(2)$ whose p-th Fourier coefficient is -1 for a p dividing the level (i.e., locally is the unramified quadratic twist of Steinberg at p), one can use quaternionic modular forms to construct a newform g of the same weight and level which is congruent to f mod 2 and has Fourier coefficient $+1$ at p (i.e., locally is the untwisted Steinberg at p), at least in the case that the level is a squarefree product of an odd number of primes [Mar18b]. In general g may be Eisenstein, but under some simple explicit conditions it can be chosen to be cuspidal. Here we extend this to higher rank in the setting of unitary groups.

Let E/F be a CM extension of number fields. Let S be a non-empty finite set of finite places of F which split in E. Consider a definite unitary group $G = \mathrm{U}_A(n)$, where A/E is a degree n central division algebra such that, for each finite $v \in S$, $G(F_v) \simeq D_v^\times$ for some division algebra D_v/F_v. Let $K \subset G(\mathbb{A})$ be as in the beginning of Section 5.3 such that $K_v \simeq \mathcal{O}_{D_v}^\times$ for $v \in S$.

If π occurs in $\mathcal{A}(G, K; 1)$, then, for $v \in S$, π_v is 1-dimensional, and thus of the form $\mu_v \circ \det$ for some unramified character $\mu_v : F_v^\times \to \mathbb{C}^\times$ such that $\mu_v^n = 1$. (Here det denotes the reduced norm from D_v to F_v.) Consider a collection $\mu_S = (\mu_v)_{v \in S}$ of such μ_v. We denote by $\mathcal{A}(G, K; 1)^{\mu_S}$ the subspace of $\mathcal{A}(G, K; 1)$ generated by π^K where π runs over all π contributing to $\mathcal{A}(G, K; 1)$ such that $\pi_v \simeq \mu_v \circ \det$ for all $v \in S$. When $\mu_v = 1$ for all $v \in S$, we write this as $\mathcal{A}(G, K; 1)^{1_S}$. Let $\zeta_m = e^{2\pi i/m}$.

Lemma 9.1. *Fix $p|n$. Suppose μ_v has prime power order $p^{r_v}|n$ for all $v \in S$. Let $\mathcal{O}$ be the ring of integers of some number field containing ζ_{p^r}, and $\mathfrak{p}$ a prime of $\mathcal{O}$ above p. Then for any nonzero $\phi \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{\mu_S}$, there exists a nonzero $\phi' \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{1_S}$ such that $\phi' \equiv \phi \bmod \mathfrak{p}$.*

Proof. Let $\bar{G} = G/Z$ and $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Then we may view ϕ as a function on $\mathrm{Cl}(\bar{K})$. For $v \in S$, fix a uniformizer $\varpi_{D,v}$ of D_v such that $\det \varpi_{D,v} = \varpi_v$. Then $\varpi_{D,v}$ acts on $\mathrm{Cl}(\bar{K})$ via right multiplication with order dividing n. Denote this action by σ_v. Let $Y_1, \ldots, Y_t$ be the orbits of the ensuing action of $\Gamma = \prod_{v\in S} \langle \varpi_{D,v} \rangle$ on $\mathrm{Cl}(\bar{K})$.

Note that for $\phi \in \mathcal{A}(G, K; 1)$, we have $\phi \in \mathcal{A}(G, K; 1)^{\mu_S}$ if and only if $\phi(\sigma_v(y)) = \mu_v(\varpi_v)\phi(y)$ for all $y \in \mathrm{Cl}(\bar{K})$, $v \in S$. Fix some orbit Y_i and write $Y_i = \{y_1, \ldots, y_s\}$. Then for any $y_j \in Y_i$, there is some sequence of σ_v's (with $v \in S$) whose composition sends y_1 to y_j. Hence $\phi(y_j) = \zeta\phi(y_1)$ for some p-power root of unity ζ. Since $\zeta \equiv 1 \bmod \mathfrak{p}$, defining $\phi'(y_j) = \phi(y_1)$ for $1 \leq j \leq s$ gives a function on Y_i which is congruent to $\phi \bmod \mathfrak{p}$. Defining ϕ' this way on each orbit completes the proof. □

The following is a partial analogue of [Mar18b, Theorem 1.3] in higher rank, and the proof is similar in spirit.

Theorem 9.2. *Let $n = p$ be an odd prime, and assume (EC-U) for n. Let S be a finite set of finite places of F which are split in E/F. Suppose p does not divide $|\mathrm{Cl}(\mathrm{U}(1))|$ nor*

$$\prod_{r=1}^{p} L(1-r, \chi_{E/F}^r) \times \prod_{v\in S} \left(\prod_{r=1}^{p-1} (q_v^r - 1) \right).$$

For each $v \in S$, let μ_v be an unramified character of $F_v^\times$ of order 1 or p. For finite $v \notin S$, assume K_v is a hyperspecial maximal compact open subgroup of $\mathrm{U}_p(F_v)$.

Let π be an automorphic representation of $G'(\mathbb{A}) = \mathrm{U}_p(\mathbb{A})$ holomorphic of parallel weight p with trivial central character such that π_E is cuspidal, K_v-spherical for all finite $v \notin S$, and $\pi_v \simeq \mathrm{St}_v \otimes \mu_v$ for all $v \in S$. Then there exists an automorphic representation of π' of $G'(\mathbb{A})$, also holomorphic of parallel weight p with trival central character and π'_E cuspidal, such that π'_v is K_v-spherical for all finite $v \notin S$, $\pi'_v \simeq \mathrm{St}_v$ for all $v \in S$ and π is Hecke congruent to π' $\bmod p$.

Proof. Let $G = \mathrm{U}_A(p)$ be a totally definite inner form of G' which is locally isomorphic to G' at all finite places outside of S and compact at each $v \in S$. Now π corresponds to a simple generic formal parameter ψ, which we may think of as the cuspidal representation π_E of $\mathrm{GL}_p(\mathbb{A}_E)$. Then there exists an automorphic representation $\sigma \in \Pi_\psi(G)$ such that $\sigma_v \simeq \pi_v$ for all finite $v \notin S$, $\sigma_v \simeq \mu_v \circ \det$ for $v \in S$, and σ_v is trivial for $v|\infty$.

For $v \in S$, let D_v/F_v be a division algebra isomorphic to A_w/E_w for some $w|v$ and put $K_v = \mathcal{O}_{D_v}^\times$. For $v|\infty$, put $K_v = G_v$. Set $K = \prod K_v$. Then σ occurs in $\mathcal{A}_0(G, K; 1)$ and we may take a nonzero $\phi \in \sigma^K$ to have values in the ring of integers $\mathcal{O}$ of some number field L. Let $\mathfrak{p}$ be a prime of $\mathcal{O}$ above p.

If $\phi \equiv 0 \bmod \mathfrak{p}$, we may consider the Hilbert class field H_L of L so that $\mathfrak{p}$ is unramified and principal in H_L. Thus we may scale ϕ by an element of H_L to assume that $\phi \not\equiv 0 \bmod \mathfrak{p}$, and moreover $\phi \not\equiv 0 \bmod \mathfrak{P}$ for some prime $\mathfrak{P}$ of H_L

above p. Hence by replacing L with H_L and $\mathfrak{p}$ with $\mathfrak{P}$ if necessary, we may and will assume $\phi \not\equiv 0 \bmod \mathfrak{p}$.

By Lemma 9.1, there exists a nonzero $\phi' \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{1_S}$ such that $\phi' \equiv \phi \bmod \mathfrak{p}$. We claim ϕ' is non-abelian. First note that, since $p \nmid |\mathrm{Cl}(\mathrm{U}(1))|$, the only non-abelian forms in $\mathcal{A}(G, K; 1)$ are constant functions. However, if $\phi' = c\mathbb{1}$ for some $c \in \mathcal{O}$, then $\phi \in \mathcal{A}_0(G, K; 1)$ implies $0 = (\phi, \mathbb{1}) \equiv c(\mathbb{1}, \mathbb{1}) \equiv cm(K) \bmod \mathfrak{p}$. This would mean $\mathfrak{p}|m(K)$, since $\phi' \equiv \phi \not\equiv 0 \bmod \mathfrak{p}$ implies $c \not\equiv 0 \bmod \mathfrak{p}$. But this is impossible by our indivisibility assumption together with Proposition 5.8.

Then, as in the proofs of Proposition 2.1 and Theorem 5.7, we can transfer this to a mod $\mathfrak{p}$ Hecke congruence with a non-abelian eigenform ϕ'' on G, and obtain a congruent π' on G' as asserted. $\square$

Remark 9.3. It is clear from the proof that one can allow K_v to be a finite index subgroup of a hyperspecial maximal compact K_v^0 at a finite number of $v \notin S$ by also imposing the conditions $p \nmid [K_v^0 : K_v]$. At such v, then the appropriate statement is that both π_v and π'_v have nonzero K_v-fixed vectors.

Remark 9.4. In the case of weight 2 elliptic modular forms of squarefree level, we showed in [Mar18a] that there is a strict (though small) bias towards local ramified factors being Steinberg as opposed to the unramifed quadratic twist of Steinberg. In [Mar18b], this bias was shown to be related to the existence of mod 2 congruences of forms which are twisted Steinberg at certain places to untwisted Steinberg at these places. Similarly, the above congruence result suggests a bias towards local untwisted Steinberg representations on $\mathrm{U}_p(\mathbb{A})$. Specifically, in the notation of the proof, we expect that the number of representations occurring in $\mathcal{A}(G, K; 1)^{1_S}$ is always at least the number of representations occurring in $\mathcal{A}(G, K; 1)^{\mu_S}$. The above result implies the analogous statement is true for mod p Hecke congruence classes of representations.

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