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AN ON-AVERAGE MAEDA-TYPE CONJECTURE IN THE LEVEL ASPECT

KIMBALL MARTIN

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ABSTRACT. We present a conjecture on the average number of Galois orbits of newforms when fixing the weight and varying the level. This conjecture implies, for instance, that the central L-values (resp. L-derivatives) are nonzero for 100% of even weight prime level newforms with root number $+1$ (resp. -1).

1. Introduction

Let $S_k(N)$ (resp. $S_k^{\mathrm{new}}(N)$) be the space of weight k elliptic cusp (resp. new) forms of level $\Gamma_0(N)$. For S a Hecke-stable subspace of $S_k^{\mathrm{new}}(N)$ such that the subset of newforms is closed under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, denote by $\mathrm{Orb}(S)$ the set of Galois orbits of newforms in S.

Maeda's conjecture asserts that for any even k such that $S_k(1) = S_k^{\mathrm{new}}(1) \neq 0$, there is a single Galois orbit, i.e., $\#\mathrm{Orb}(S_k(1)) = 1$, and in fact T_2 (or any T_p) acts irreducibly (over $\mathbb{Q}$) on $S_k(1)$ with Galois group of type S_n. More generally, let Sq_r denote the set of squarefree positive integers with exactly r prime factors. Then for $N \in \mathrm{Sq}_r$, $S_k(N)$ has 2^r Atkin–Lehner eigenspaces. Tsaknias' [Tsa14] generalization of Maeda's conjecture in the case of squarefree levels states that, for fixed $N \in \mathrm{Sq}_r$, one has $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N)) = 2^r$ for all k larger than some $k_0(N)$. It follows from trace formula methods that $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N)) \geq 2^r$ for all sufficiently large k, and this can be made effective in terms of N [Mar18]. Thus we may think of this generalized Maeda conjecture as saying that the number of Galois orbits is almost always the minimum possible.

On the other hand, if we fix $k = 2$ and vary $N \in \mathrm{Sq}_r$, one expects a strict inequality $\#\mathrm{Orb}(S_2(N)) > 2^r$ infinitely often due to the existence of sufficiently many elliptic curves of squarefree level. However, we predict the following on-average analogue of Maeda's conjecture in the level aspect.

Conjecture A. *Let $k \geq 2$ be even. Then the average number of Galois orbits of $S_k^{\mathrm{new}}(N)$ over all $N \in \mathrm{Sq}_r$ is 2^r, i.e.,*

$$\lim_{X\to\infty} \frac{\sum_{N\in \mathrm{Sq}_r(X)} \#\mathrm{Orb}(S_k^{\mathrm{new}}(N))}{\#\mathrm{Sq}_r(X)} = 2^r, \tag{1.1}$$

where $\mathrm{Sq}_r(X) = \{N \in \mathrm{Sq}_r : N \leq X\}$. In fact, for a fixed prime p, T_p acts irreducibly on each of the 2^r Atkin–Lehner eigenspaces of $S_k^{\mathrm{new}}(N)$ for 100% of $N \in \mathrm{Sq}_r$ which are coprime to p.

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In particular, this asserts that each Atkin–Lehner eigenspace is spanned by a single Galois orbit 100% of the time, analogous to the usual Maeda conjecture.

One application of Maeda's conjecture is to non-vanishing of central L-values for newforms of full level [CF99] (see also [KZ81, Corollary 2]). The above conjecture similarly has applications to non-vanishing L-values and derivatives. We recall that if a newform $f \in S_k(N)$ has root number -1, then the central L-value $L(k, f)$ is forced to vanish by the functional equation.

Theorem 1. *Let $k \geq 2$ be even, and assume Conjecture A. Let $\mathcal{F}'_k$ denote the collection of newforms in $\bigcup_{N \text{ prime}} S_k(N)$, partially ordered by level. Then*

(1) *for 100% of $f \in \mathcal{F}'_k$ with root number $+1$, we have $L(k, f) \neq 0$; and*
(2) *for 100% of $f \in \mathcal{F}'_k$ with root number -1, we have $L'(k, f) \neq 0$.*

Now we expand on these statements as well as the content of the paper.

1.1. **Statistics of Galois orbits.** First, Maeda's conjecture for full level was formulated on the basis of computational data. To date (for T_2 as well as for most other T_p's), Maeda's conjecture has been verified for $k \leq 14000$ [GM12]. See [Tsa14] and [DPT] for evidence towards the analogue for squarefree level as well as a possible generalization for non-squarefree level. See also [MS16] for additional indirect analytic evidence.

Our conjecture is based on both heuristics in Section 2 and data in Section 3. Our primary heuristics come from applying general heuristics about factorizations of random integer polynomials to Hecke polynomials. This leads to three heuristics in Section 2.1 which, roughly stated, are the following: (wt) suggests the probability of small Galois orbits decreases rapidly as the weight increases; (lev) suggests the probability of small Galois orbits decreases rapidly as the level increases; and (lin) suggests most small Galois orbits arise from rational newforms, i.e., most small Galois orbits are size 1. These heuristics are in accord with data from the LMFDB [LMFDB], and our computational progress here is to compute the number and sizes of Galois orbits for $S_2(N)$ where $N < 60000$ is prime. (LMFDB calculations cover $N \leq 10000$ when $k = 2$.) Note that according to the heuristic (wt), $k = 2$ should be the key case in which to test Conjecture A.

We also explore statements stronger than Conjecture A in higher weight. A conjecture of Roberts [Rob18], in the case of squarefree level, asserts that there are only finitely many rational newforms of weight $k \geq 6$, and none with $k \geq 52$. The heuristic (lin) suggests Conjecture B: *if k is such that there are only finitely many weight k rational newforms of squarefree level, then in fact $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N)) = 2^r$ for all but finitely many $N \in \mathrm{Sq}_r$.* Combining Roberts' conjecture with Conjecture B and (lin) suggests Conjecture B+: *for $k \geq 6$ (resp. k sufficiently large), $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N))$ is exactly the number of nonzero Atkin–Lehner eigenspaces in $S_k^{\mathrm{new}}(N)$ for all but finitely many (resp. all) squarefree N.*

See Table 1 for a rough comparison of the kinds of statements these conjectures make about the number of certain Galois orbits in $S_k^{\mathrm{new}}(N)$ for squarefree N. The "weight aspect" column refers to fixed N and varying k, and vice versa for the "level aspect" columns. Here "rational orbits" refers to counting only Galois orbits of size 1, i.e., Galois orbits consisting of rational newforms. The "all" row consists of assertions of the form that the number of Galois orbits is always the minimum possible (meaning the number of nonzero Atkin–Lehner eigenspaces in the case of all orbits, and 0 in the case of rational orbits), whereas the "a.a." (almost all) row

TABLE 1. Conjectures on numbers of Galois orbits in squarefree level

	weight aspect	level aspect	
	all orbits	rational orbits	all orbits
all	Maeda ($N = 1$)	Roberts ($k \geq 52$)	Conjecture B+ (k large)
a.a.	Tsaknias	Roberts ($k \geq 6$)	Conjecture B+ ($k \geq 6$)
100%			Conjecture A

allows for a finite number of possible exceptions, and the "100%" row allows for an infinite but density 0 set of exceptions.

While we have presented Roberts' conjecture and Conjecture B+ in this table as statements in the level aspect, since there should be no exceptions for almost all k, one can view these statements as allowing both the weight and level to vary provided k is not too small. Consequently, Conjecture B+ is stronger than Tsaknias' generalized Maeda conjecture for squarefree levels as it asserts that the $k_0(N)$ in that conjecture can be taken to be independent of N, i.e., there is a uniform bound k_0 after which there are no "extra" Galois orbits in any $S_k^{\mathrm{new}}(N)$ with $k \geq k_0$ and N squarefree.

We also briefly discuss how often Hecke polynomials are irreducible or Galois groups are of type S_n in Sections 2.3 and 3.4. In addition, we raise some other questions related to [Rob18], [Mur99] and [KSW08] based on our data (Questions 3 and 4).

1.2. **Rationality fields of large degree.** Lipnowski and Schaeffer [LS] formulated a conjecture in a similar vein as Conjecture A, that the rational Hecke modules of each Atkin–Lehner eigenspace are asymptotically simple. Restricting to N prime, this means that for fixed $\varepsilon = \pm 1$ the maximal size of an orbit in the Atkin–Lehner ε-eigenspace $S_k^{\mathrm{new},\varepsilon}(N)$ should be asymptotic to the dimension of $S_k^{\mathrm{new},\varepsilon}(N)$, i.e.,

$$\lim_{N\to\infty} \frac{\max\{\#\mathcal{O} : \mathcal{O} \in \mathrm{Orb}(S_k^{\mathrm{new},\varepsilon}(N))\}}{\dim S_k^{\mathrm{new},\varepsilon}(N)} = 1. \tag{1.2}$$

While neither Conjecture A nor the Lipnowski–Schaeffer conjecture imply the other, Conjecture A is morally stronger in that it is suggestive of the Lipnowski–Schaeffer conjecture but not conversely. Namely, Conjecture A would imply that (1.2) holds for a density 1 subsequence of prime levels N, but (1.2) alone does not imply any bounds on the number of Galois orbits, even restricting to some density 1 subset of prime levels.

Both Conjecture A and the Lipnowski–Schaeffer conjecture assert very strong statements about the growth of degrees of rationality fields of newforms, namely that the growth of rationality fields is must be generically linear in the level. At present, only much weaker results are known. For instance, as the level grows, one knows that the proportion of newforms with rationality fields of bounded degree tends to 0 [Bin17]. Moreover, there exist sequences of newforms with rationality fields with degrees growing at least on the order of $\log N$ [BM16], and $N^{\frac{1}{2}-\epsilon}$ if one admits class number heuristics [LS]. See also [LS] and [BPGR] for $\log\log N$ type bounds for more general sequences.

1.3. ***L*-values.** The conclusion of Theorem 1 is expected from minimalist-type conjectures (see [Bru95] for $k = 2$) and the Katz–Sarnak philosophy. The non-vanishing of central L-values $L(k, f)$ and L-derivatives $L'(k, f)$ is already known for a positive proportion of such f in prime levels by [Van99], [KM00] and [IS00]. In [IS00], a connection is made between large proportions of non-vanishing of central values (via a more refined density conjecture) and Landau–Siegel zeroes. In particular, suitable lower bounds on central L-values $L(k, f)$ for strictly greater than 50% of newforms $f \in S_k(N)$ with root number $+1$ implies the nonexistence of Landau–Siegel zeros of Dirichlet L-functions. This suggests a potential connection between the average number of Galois orbits and Landau–Siegel zeroes.

The proof of Theorem 1 in Section 4 is an immediate application of the above non-vanishing results together with the behavior of L-values and L-derivatives under the Galois action. The obstruction to extending this to squarefree level is that one needs the existence of non-vanishing L-values and L-derivatives in 100% of Atkin–Lehner eigenspaces. While this may very well be accessible by current analytic methods, to our knowledge it has not been considered. One could also apply Conjecture A to questions such as non-vanishing of twisted L-values (e.g., see [MW20]).

1.4. **Final remarks.** In [MW20], we give another application of Conjecture A to zeroes of automorphic forms on definite quaternion algebras.

Finally, we remark that one could attempt to generalize Conjecture A to non-squarefree levels. In the case of non-squarefree level, the naive expectation for the number of non-CM Galois orbits should be given by the number of possible local representation types at ramified places, which varies with both the primes $p|N$ as well as $v_p(N)$. We do not consider this here, in part because it would be difficult to generate a convincing amount of data for non-squarefree level and partly because even the correct generalization of Maeda's conjecture to non-squarefree level is not clear (see [DPT]). Similarly, we do not consider nontrivial nebentypus or odd weights, but it is reasonable to expect an analogue of Conjecture A in these settings as well.

2. Heuristics

2.1. **Random polynomials.** Let $k \geq 2$ be even, $N \in \mathrm{Sq}_r$ and $n = \dim S_k^{\mathrm{new}}(N) \sim \frac{(k-1)\varphi(N)}{12}$. Let ε denote a sign pattern for N, i.e., a collection $(\varepsilon_p)_{p|N}$ such that $\varepsilon_p = \pm 1$ for each $p|N$. For $p|N$, let W_p be the Atkin–Lehner involution at p on $S_k^{\mathrm{new}}(N)$. Denote by $S_k^{\mathrm{new},\varepsilon}(N) = \{f \in S_k^{\mathrm{new}}(N) : W_p f = \varepsilon_p f \text{ for } p|N\}$ the Atkin–Lehner eigenspace of newforms associated to ε. By [Mar18], we know that $\dim S_k^{\mathrm{new},\varepsilon}(N) \sim \frac{n}{2^r}$ (asymptotically as $kN \to \infty$ for $(k, N) \in 2\mathbb{Z}_{>0} \times \mathrm{Sq}_r$), and in fact we know good error estimates.

Consider a Hecke operator T_p ($p \nmid N$) acting on $S_k^{\mathrm{new}}(N)$ or some $S_k^{\mathrm{new},\varepsilon}(N)$. Then each Galois orbit $\mathcal{O}$ of newforms in these spaces corresponds to a factor $g_{\mathcal{O},p}(x)$ of the characteristic polynomial $c_{T_p}(x) \in \mathbb{Z}[x]$ of T_p. More precisely, we must have $g_{\mathcal{O},p}(x) = m_{\mathcal{O},p}(x)^j$ for some irreducible polynomial $m_{\mathcal{O},p}(x) \in \mathbb{Z}[x]$, and the roots of $g_{\mathcal{O},p}(x)$ are in one-to-one correspondence with the T_p-eigenvalues of the newforms in $\mathcal{O}$.

Thus a random model for the number and degrees of factors of $c_{T_p}(x)$ will provide a simple heuristic for an upper bound on the number of and a lower bound on the sizes of the Galois orbits of $S_k^{\mathrm{new}}(N)$ or $S_k^{\mathrm{new},\varepsilon}(N)$. In fact, the main result of [KSW08] implies that for a given Galois orbit $\mathcal{O}$, $g_{\mathcal{O},p}(x) = m_{\mathcal{O},p}(x)$ for 100% of primes p. Hence we may use a model for the factorization of $c_{T_p}(x)$ for an arbitrary $p \nmid N$ to model the Galois orbits of $S_k^{\mathrm{new}}(N)$.

We recall two heuristic principles on random polynomials, which are quite robust to the model being considered.

(RP1) In the absence of simple reasons for nontrivial factors, the probability that a well-behaved random polynomial in $\mathbb{Z}[x]$ is irreducible tends to 1 (typically quickly) as the size of the polynomial grows.

(RP2) Asymptotically, the probability that a well-behaved random polynomial in $\mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ is proportional to the probability that it has a linear factor over $\mathbb{Q}$.

We do not attempt to define the notions of "well-behaved" or the "size" of the polynomial—indeed we use them in a somewhat vague sense here—but just remark that by size we have in mind some combination of the sizes of the degree, the coefficients and the roots of the polynomial. The principle (RP1) has been long studied, and there are many results in this direction. See, e.g., [BBB+18] for a recent study of (RP2), which the authors term universality.

Apart from the decomposition of $S_k^{\mathrm{new}}(N)$ into Atkin–Lehner eigenspaces, there are no obvious reasons why the characteristic polynomials of T_p's acting on $S_k^{\mathrm{new}}(N)$ should factor for $p \nmid N$. Thus (RP1) suggests that, for $p \nmid N$, T_p acts irreducibly on each Atkin–Lehner eigenspace 100% of the time. This is our first heuristic why Conjecture A should be true.

To be more precise, we recall a simple model for Hecke polynomials on Atkin–Lehner eigenspaces recently proposed by Roberts [Rob18].

Consider the collection $\mathcal{P}_d(t)$ of degree d monic polynomials in $\mathbb{Z}[x]$ all roots real size at most t. It follows from [AP14a, Theorem 4.1] that $|\mathcal{P}_d(t)|$ is approximately

$$\mathcal{R}_d(t) := (2t)^{\frac{d(d+1)}{2}} \prod_{j=1}^{d} \frac{(j-1)!^2}{(2j-1)!} = 2^d t^{\frac{d(d+1)}{2}} \prod_{j=1}^{d-1} \left(\frac{j}{2j+1}\right)^{d-j} \tag{2.1}$$

when $d+t$ is large (cf. [AP14b, Theorem 3.1]). Thus the probability that a random polynomial in $\mathcal{P}_d(t)$ has a factor of degree $e \le \frac{d}{2}$ is approximately

$$\begin{aligned} P_{d,e}(t) &:= \frac{\mathcal{R}_e(t)\mathcal{R}_{d-e}(t)}{2^\delta \mathcal{R}_d(t)} \\ &= \frac{1}{2^\delta t^{e(d-e)}} \prod_{j=1}^{e-1} \left(\frac{2j+1}{j}\right)^{j} \prod_{j=e}^{d-e-1} \left(\frac{2j+1}{j}\right)^{e} \prod_{j=d-e}^{d-1} \left(\frac{2j+1}{j}\right)^{d-j}, \end{aligned} \tag{2.2}$$

where $\delta = 1$ if $d = 2e$ and 0 otherwise.

Note (for $d > 2$)

$$P_{d,1}(t) = \frac{1}{t^{d-1}} \prod_{j=1}^{d-1} \frac{2j+1}{j}$$

and

$$P_{d,e}(t) < \left(\frac{3}{t}\right)^{e(d-e)}.$$

Consequently, we see that both $P_{d,e}(t) \to 0$ and $P_{d,1}(t) \gg \sum_{e=2}^{\lfloor d/2 \rfloor} P_{d,e}(t)$ as $d+t \to \infty$, provided $t > 3$. In fact, by refining the above bound, this analysis works for fixed $t > 2$ and $d \to \infty$. Thus (RP1) and (RP2) hold for this model with fixed $t > 2$ and $d \to \infty$.

Let $n_\varepsilon = \dim S_k^{\mathrm{new},\varepsilon}(N) \approx \frac{(k-1)\varphi(N)}{12 \cdot 2^r}$. Then a crude model for the characteristic polynomial of T_p acting on $S_k^{\mathrm{new},\varepsilon}(N)$ is a random polynomial in $\mathcal{P}_{n_\varepsilon}(2p^{(k-1)/2})$. One obvious defect is that, even for fixed N and ε, the probability of reducibility decreases rapidly as $p \to \infty$.

For odd N, we will just use a model for the factorization of c_{T_2} as a model for the Galois orbits. In light of both [KSW08] and observed data (see Table 7), the factorization of c_{T_2} does seems to be a very good model for the sizes of Galois orbits. Further, as in [Rob18], we may view $P_{n_\varepsilon,e}(2^{\frac{k+1}{2}})$ as a rough model for the probability that $S_k^{\mathrm{new},\varepsilon}(N)$ has a Galois orbit of size e (or a collection of smaller Galois orbits whose sizes sum to e), even when N is even.

This model is not very accurate—as pointed out in [Rob18], it severely underpredicts the actual number of factorizations of c_{T_p}'s. In fact it suggests all but finitely many Atkin–Lehner eigenspaces consist of a single Galois orbit as $N \to \infty$ along Sq_r, which should not be true at least in weight 2 (cf. Section 2.2 and Question 3). We will partially address this by also considering arithmetic statistics in Section 2.2 below. However, the model at least suggests the following principles which we believe in accordance with the data and general expectations about randomness. In the following statements, we consider $r \geq 1$ fixed, $N \in \mathrm{Sq}_r$, $k \geq 2$ even, ε a sign pattern for N and $p \nmid N$.

- (wt) Given N, ε and p, the probability that T_p acts reducibly on $S_k^{\mathrm{new},\varepsilon}(N)$ decreases rapidly as k becomes large.
- (lev) Given ε and p, the probability that T_p acts reducibly on $S_k^{\mathrm{new},\varepsilon}(N)$ decreases rapidly as N grows coprime to p.
- (lin) If $k + N$ is large, the probability that T_p acts reducibly on $S_k^{\mathrm{new},\varepsilon}(N)$ is roughly equal to the probability that T_p acting on $S_k^{\mathrm{new},\varepsilon}(N)$ has a rational eigenvalue.

Note (wt) and (lev) follow from (RP1) since increasing $k + N$ increases the dimensions of the Atkin–Lehner subspaces. Similarly, (lin) follows from (RP2). Specifically, (lev) suggests Conjecture A.

Before we discuss other heuristics and data, we discuss the relation of the above heuristics with the following recent conjecture of [Rob18] restricted to our setting of squarefree level. (In particular, we do not need to account for quadratic twist classes or CM forms.) By a rational newform, we mean a newform in $S_k(N)$ with rational Fourier coefficients, i.e., a newform whose Galois orbit has size 1. Let $\mathcal{F}_k$ be the set of newforms which lie in some $S_k(N)$ with N squarefree.

Conjecture 2 (Roberts)**.** *Fix $k \geq 6$. There are only finitely many rational newforms in $\mathcal{F}_k$. Further, if $k \geq 52$, there are no rational newforms in $\mathcal{F}_k$.*

Roberts' support for his conjecture comes from his rough heuristic model together with an apparent lack of motivic sources for rational newforms in higher weight and computations in weights $k \leq 50$ and levels $N \leq C_k$. Here the bound C_k depends on k—e.g., $C_6 = 1000$, $C_{10} = 450$, $C_{20} = 150$, $C_{30} = 100$ and $C_{40} = 30$.

Of course when $k = 2$, rational newforms correspond to isogeny classes of elliptic curves, so we expect infinitely many rational newforms (cf. Section 2.2). For

$k = 4$, Roberts remarks that it is unclear if there should be infinitely many rational newforms or not, and that this is related to the existence of suitable Calabi–Yau threefolds.

Now (lin) suggests the following more speculative conjecture, which implies something much stronger than Conjecture A if Roberts' conjecture is true for k.

Conjecture B. *Fix k. Suppose there are only finitely many rational newforms in $\mathcal{F}_k$. Then $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N)) = 2^r$ for all but finitely many $N \in \mathrm{Sq}_r$.*

As some numerical evidence for this conjecture, Roberts observed that in his calculations for $k \geq 6$ that there were only four squarefree levels where an Atkin–Lehner eigenspace has multiple Galois orbits with no orbits of size 1, which is in line with (lin) and the above conjecture.

Moreover, in light of the second part of Roberts' conjecture, the above heuristics suggest that for k sufficiently large, each Atkin–Lehner eigenspace may only be a single Galois orbit for arbitrary squarefree N, which is what we labeled Conjecture B+ in the "all" row of Table 1. Put another way, it is possible that the generalized Maeda conjecture for squarefree level is true with a uniform bound on the weight: there exists some absolute k_0 such that for any $r \geq 0$, $\#\mathrm{Orb}(S_k^{\mathrm{new}}(N)) = 2^r$ for all $k \geq k_0$ and all $N \in \mathrm{Sq}_r$. (Recall that the conjecture in [Tsa14] only asserts the existence of a lower bound $k_0(N)$ depending on N.)

2.2. **Elliptic curves.** As mentioned before, random polynomial models as above seem too crude to accurately predict the frequency of small Galois orbits. One perspective is that the existence of small Galois orbits is due to the existence of suitable motives, which seem to be hard to model without a deep understanding of arithmetic. However, we can supplement the random polynomial model above with heuristics from arithmetic geometry.

In particular, based on the principle (wt), we expect that Conjecture A should be true if it is true in weight 2. Our data below corroborate this idea, by indicating the average number of Galois orbits converges to 2^r faster the higher the weight is (see Section 3.1). Moreover, by the principle (lin), we expect that Conjecture A should be true if, as $N \to \infty$ in Sq_r, 0% of weight 2 newforms are rational.

By dimension formulas, we know that the number of weight 2 newforms of level $N \leq X$, $N \in \mathrm{Sq}_r$, grows at least on the order of $\frac{X^2}{\log X}$ (up to a constant, this is the asymptotic for $r = 1$). However the number of isogeny classes of elliptic curves of arbitrary conductor less than X is $O(X^{1+\varepsilon})$ [DK00]. (Heuristics of Watkins [Wat08] suggest it is actually $O(X^{5/6})$.) Consequently, 0% of weight 2 newforms along levels in Sq_r are rational. This gives arithmetic support for our belief in Conjecture A.

In fact, generalizing work of Serre, Binder [Bin17] showed that, for any weight k and fixed degree A, 0% of weight k newforms of levels N have a rationality field of degree $\leq A$ for any sequence of $N \to \infty$ with a bounded number of prime factors.

2.3. **Galois groups.** Conjecture A is an on-average analogue of two aspects of Maeda's conjecture in the level aspect: the number of Galois orbits and the irreducibility of the action of T_p. The remaining aspect of Maeda's conjecture is the assertion that the rationality fields of newforms are of type S_n. Since random polynomials tend to have Galois groups of type S_n, it is reasonable to expect that Conjecture A also holds with the added statement that c_{T_p} has Galois group of type S_n for 100% of Atkin–Lehner eigenspaces.

TABLE 2. Average numbers $A_{k,1}(X)$ of Galois orbits for $r = 1$

	$X = 250$	500	1000	2500	5000	10000
$k = 2$	2.038	2.484	2.679	2.684	2.577	2.483
4	2.057	2.042	2.030	2.016		
6	1.981	2.000	2.000			
8	2.000	2.000				
10	1.981					
12	1.943					

TABLE 3. Average numbers $A_{k,2}(X)$ of Galois orbits for $r = 2$

	$X = 250$	500	1000	2500	5000	10000
$k = 2$	3.243	4.386	5.292	5.615	5.608	5.442
4	4.405	4.352	4.250	4.135		
6	4.108	4.069	4.042			
8	3.973	3.986				
10	4.013					
12	4.000					

TABLE 4. Average numbers $A_{k,3}(X)$ of Galois orbits for $r = 3$

	$X = 250$	500	1000	2500	5000	10000
$k = 2$	3.708	5.885	8.652	11.34	12.30	12.29
4	7.500	8.902	9.237	8.701		
6	8.167	8.557	8.348			
8	8.292	8.197				
10	8.083					
12	7.958					

There are examples of newforms with rationality fields whose Galois group is not a full symmetric group. For instance, in LMFDB [LMFDB] one finds 351 Galois orbits of weight 2 newforms in squarefree levels $N < 10000$ whose rationality field is the degree 3 cyclic extension $\mathbb{Q}(\zeta_{14})^+/\mathbb{Q}$. Analogous to questions about the finitude of rational newforms, one might ask if all or almost all Galois orbits have rationality fields of type S_n when one restricts to sufficiently large weights. Unfortunately, we do not have precise enough heuristics or abundant enough data to speculate about this.

3. DATA

3.1. **LMFDB data.** First we present some numerical evidence for Conjecture A using data from LMFDB [LMFDB]. LMFDB contains data for the newforms in $S_k^{\text{new}}(N)$ whenever $Nk^2 \leq 40000$. Using these data, we computed the average number $A_{k,r}(X)$ of Galois orbits over the spaces $S_k^{\text{new}}(N)$ where $N \in \mathrm{Sq}_r$ with $N < X$ for numerous values of $X \leq 10000$, $2 \leq k \leq 12$ and $1 \leq r \leq 3$. The data are summarized in Tables 2 to 4, corresponding respectively to $r = 1$, 2 and 3. Blank spaces in the tables denote the situations where LMFDB lacks sufficient data to compute these averages.

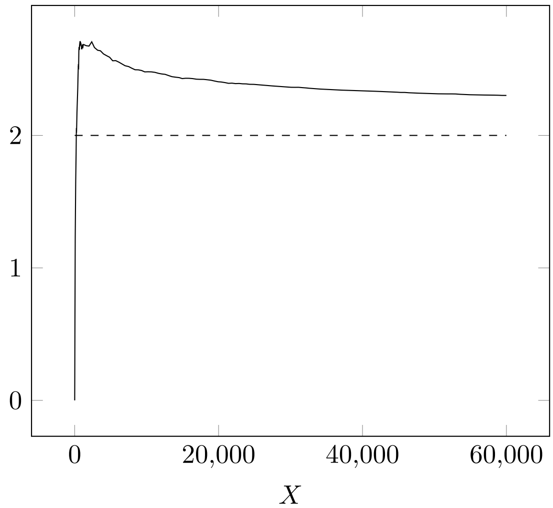

FIGURE 1. The average number $A_{2,1}(X)$ of Galois orbits for $S_2(N)$, $N < X$ prime

Our first remark about the data is that, for fixed k, r the rough shape of the graph $A_{k,r}(X)$ as a function of X initially increases with N (corresponding to the range where some Atkin–Lehner spaces are 0) and then is essentially decreasing. In accordance with our heuristics, it appears that $A_{k,r}(X)$ tends to 2^r faster the larger k is and the smaller r is, as the dimensions of the Atkin–Lehner eigenspaces are larger in these situations.

In particular, the data for the case $k = 2$ and $r = 3$ are not sufficient to make it numerically apparent whether the average tends to 2^3. However, the data on the whole seems to be in support of Conjecture A, and also the notion that Conjecture A should be true if it is true for $k = 2$. In addition, it seems reasonable to expect that, for given k, the distribution of sizes of Galois orbits along a sequence of Atkin–Lehner eigenspaces depends primarily on dimension of the Atkin–Lehner eigenspaces and not to any significant amount on the number r of prime factors of N. Thus, at least to our mind, we can be confident about Conjecture A if we are in the special case of $k = 2$ and $r = 1$, which is what we focus on below.

3.2. **Data for $S_2(N)$, N prime.** To gather more numerical evidence for Conjecture A, we computed $A_{2,1}(X)$ for $X \leq 60000$. Raw data on the number and size of Galois orbits for $S_2(N)$ for prime $N \leq 60000$ are available on the author's website.[1] These calculations were carried out in parallel using Sage [Sage] on the University of Oklahoma's supercomputing facilities (OSCER) over the course of several weeks. See Figure 1 for a graph of $A_{2,1}(X)$, which appears to be eventually decreasing to 2, as conjectured. We remark that $A_{2,1}(60000) \approx 2.3016$ (compare with Table 2). The apparent slow rate of convergence is expected in light of the well-known numerical phenomenon that the proportion of weight 2 newforms accounted for by elliptic curves tends to 0 quite slowly (e.g., see [BHK+16]).

[1] `https://math.ou.edu/~kmartin/data/`

Table 5. Counts of number of Galois orbits for $S_2(N)$ with N prime

	number of orbits						% with
	2	3	4	5	6	7+	2 orbits
$0 < N < 10000$	777	331	67	25	9	6	63.2%
$10000 < N < 20000$	786	193	39	9	5	1	76.1%
$20000 < N < 30000$	769	176	31	5	1	1	78.2%
$30000 < N < 40000$	768	158	23	7	2	0	80.2%
$40000 < N < 50000$	750	162	14	3	1	0	80.6%
$50000 < N < 60000$	765	121	28	6	4	0	82.8%

Table 6. Counts of small Galois orbits for $S_2(N)$ with N prime

	size of orbits						
	1	2	3	4	5	6	7
$0 < N < 10000$	329	212	76	28	20	11	18
$10000 < N < 20000$	200	104	16	3	0	0	0
$20000 < N < 30000$	176	80	5	2	1	0	0
$30000 < N < 40000$	171	56	5	1	0	0	0
$40000 < N < 50000$	140	56	7	0	0	0	0
$50000 < N < 60000$	152	57	2	0	0	0	0

One consequence of Conjecture A would be that, in 100% of prime levels, $S_2(N)$ has exactly 2 Galois orbits. (For $N > 59$, $S_2(N)$ has at least 2 orbits.) In fact, provided the number of Galois orbits does not grow too fast along any subsequence, this is equivalent to the $k = 2$, $r = 1$ case of Conjecture A. Table 5 summarizes how often we get exactly 2 (or 3, or 4, etc.) Galois orbits in weight 2 in certain ranges. These numerics suggest that indeed there are exactly 2 Galois orbits 100% of the time. We remark that for prime $N < 60000$, the maximum number of Galois orbits is 10.

One of our heuristics for Conjecture A uses the idea (lin), that most of the time an Atkin–Lehner eigenspace has multiple Galois orbits, the multiple orbits are accounted for by the existence of a rational newform. In Table 6, we summarize the number of small Galois orbits in various ranges, and observe that indeed most of the time there is a small Galois orbit, it is of size 1. In fact, there are no orbits of "moderate" size: for $10000 < N < 60000$, there are no Galois orbits of size $6 \leq d \leq 300$. This suggests the following question:

Question 3. Fix k, r, d. Are there infinitely many Galois orbits of size d in the union of spaces $S_k^{\mathrm{new}}(N)$ with $N \in \mathrm{Sq}_r$?

Note that for $d = 1$, this is just asking about the infinitude of rational newforms, which is the topic of Roberts' conjecture discussed above. In fact, Conjecture B+ would imply that for $k \geq 6$, the answer is negative for all r, d. On the other hand, when $k = 2$ and d is small, we expect this question has a positive answer. So the most novel case of this question is when k is small but d is not. At least for $k = 2$, $r = 1$ and d sufficiently large, our data suggest the answer may be no.

3.3. **The method.** Now we describe our method to compute Galois orbits. For an odd prime N, let $B = B_N$ be the definite quaternion algebra of discriminant

TABLE 7. Frequency that p is the smallest prime such that T_p acts on $S_2(N)$ with no repeated eigenvalues ($N < 60000$ prime)

p	2	3	5	7	11	13	17	19	23	47
frequency	5815	158	42	14	15	2	3	4	3	1

N. Then we computed the Brandt matrix T_2^B for a maximal order $\mathcal{O}_B$ of B, which acts on the space of M quaternionic modular forms associated to $\mathcal{O}_B$. This space of quaternionic modular forms is Hecke isomorphic to $M_2(N) \simeq \mathbb{C}E_{2,N} \oplus S_2(N)$, where $E_{2,N}$ is the normalized weight 2 level N holomorphic Eisenstein series. The Eisenstein eigenvalue of T_2^B is 3, and thus the eigenvalues of T_2^B acting on B_N are 3 together with the eigenvalues of T_2 acting on $S_2(N)$. We compute the characteristic polynomial $c_{T_2^B}(x) = (x-3)c_{T_2}(x)$.

If $c_{T_2}(x)$ has no repeated factors, then the number of Galois orbits in $S_2(N)$ is simply the number of irreducible factors of $c_{T_2}(x)$. If $c_{T_2}(x)$ has repeated factors, then we repeat the above calculation with successive T_p's until this method succeeds. We performed these calculations in parallel by treating different N on different cores. Most of the calculation time is spent computing the characteristic polynomials, and the computational complexity increases both with N and with p. For N close to 60000, this calculation for a single T_p can take over 24 hours of wall time.

3.4. **Irreducibility of Hecke polynomials.** In most cases, T_2 acts on $S_2(N)$ with no repeated eigenvalues. Even when T_2 does not, we typically do not have to try many T_p's to find one that does. Table 7 shows for how many primes $N < 60000$ a given p is minimal such that T_p has no repeated eigenvalues.

More generally, given p we can ask how many prime $N \neq p$ are there such that c_{T_p} has no repeated roots for $S_2(N)$? Note that if we have multiple rational newforms occurring in $S_2(N)$, the naive probability that two given such newforms f_1 and f_2 have the same a_p is approximately $\frac{1}{4\sqrt{p}}$ by Deligne's bound. Since we expect that there are infinitely many prime levels N where $S_2(N)$ has 2 rational newforms (e.g., coming from Neumann–Setzer elliptic curves [Set75]) we expect that c_{T_p} has repeated roots for infinitely many N.

To avoid this situation, let us examine the case where $S_2(N)$ has exactly 2 Galois orbits. Out of the 4615 prime levels $N < 60000$ such that $S_2(N)$ has exactly 2 Galois orbits, there is only one level N such that T_2 has repeated eigenvalues on $S_2(N)$, namely $N = 251$. Here T_2 acts irreducibly on the 17-dimensional root number $+1$ subspace of $S_2(251)$ but acts reducibly on the 4-dimensional root number -1 subspace. (Incidentally, a newform in the latter space has rationality field with Galois group D_8.) Based on the rarity of repeated eigenvalues of T_2, we guess that there may be no other such N.

This is related to a question studied in [Mur99] and [KSW08]: given a non-CM newform $f \in S_k(N)$ with rationality field K, how often is $\mathbb{Q}(a_p(f))$ a proper subfield of K? Restricted to our setting of squarefree level and trivial nebentypus, [Mur99, Conjecture 3.4] asserts that this happens for infinitely many p exactly in the following cases: $k = 2$ and K is quadratic, cubic, or quartic with a quadratic subfield. See [VH17] for more precise heuristics in the case $k = [K : \mathbb{Q}] = 2$. Also, the results and heuristics in [MS16] suggest an affirmative answer does not happen too often. In the cases where we expect $\mathbb{Q}(a_p(f)) = K$ for all but finitely many p, we can ask if something stronger is true.

Question 4. Given k and N, let $f \in S_k(N)$ be a non-CM newform with rationality field K. Is $\mathbb{Q}(a_p(f)) = K$ for all $p \nmid N$ assuming k or K is "sufficiently large"?

By K being sufficiently large, we mean either that $[K : \mathbb{Q}]$ is sufficiently large or the Galois group of K is not too degenerate. When $k = 2$, just requiring that $[K : \mathbb{Q}] \geq 5$ to avoid the cases in Murty's conjecture is not sufficient to guarantee a positive answer. Here is an example of this, pointed out to us by Alex Cowan. There is a degree 9 newform $f \in S_2(1223)$, such that both the $a_2(f)$ and $a_{13}(f)$ Hecke eigenvalues generate isomorphic degree 3 extensions with Galois group S_3. (This does not happen for any other $a_p(f)$ with $p < 10000$, $p \neq N$.) We note that the rationality field K of f has Galois group of order 1296, which is much smaller than S_9.

Note that an affirmative answer to Question 4 under appropriate conditions would imply that T_p acts irreducibly on each Galois orbit of $S_k(N)$ for all but finitely many squarefree N coprime to p.

4. Proof of Theorem 1

Fix $k \geq 2$ even. For a prime level N, the Atkin–Lehner eigenspaces of $S_k^{\mathrm{new}}(N)$ are simply the spaces with fixed root number ± 1.

Let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. It follows from an algebraicity result of Shimura [Shi76, Theorem 1] that $L(k, f^\sigma) \neq 0$ if and only if $L(k, f) \neq 0$. Thanks to the extension of the Gross–Zagier formula by Zhang, we also know that $L'(k, f^\sigma) \neq 0$ if and only if $L'(k, f) \neq 0$ [Zha97, Corollary 0.3.5].

Consequently to show Theorem 1, it suffices to know that for N sufficiently large, there exist $f \in S_k(N)$ with $L(k, f) \neq 0$ (so f necessarily has root number $+1$) and $g \in S_k(N)$ with root number -1 such that $L'(k, g) \neq 0$. This follows, e.g., from the works [Van99], [KM00] and [IS00] mentioned in the introduction.

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Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019
Email address: kimball.martin@ou.edu