TRANSFER FROM $\mathrm{GL}(2, D)$ TO $\mathrm{GSp}(4)$

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Abstract. An integral criterion is known for when a cuspidal representation of GL(4) transfers to a generic representation of GSp(4). We review this transfer and discuss an integral criterion for when a representation of $\mathrm{GL}(2, D)$, D a quaternion algebra, transfers to GSp(4) (or an inner form). Then in Section 3 we provide an introduction to the relative trace formula, with a bent to the study of GSp(4). In the last section, we explain how one can use the relative trace formula to obtain a transfer criterion.

1. Transfer from GL(4)

Fix a base field F, being a number field. The algebraic groups below should be regarded as groups over F.

Recall that the dual groups $\widehat{\mathrm{GSp}}(4) = \mathrm{GSp}_4(\mathbb{C})$ and $\widehat{\mathrm{GL}}(4) = \mathrm{GL}_4(\mathbb{C})$. So one has a map of dual groups

$$\widehat{\mathrm{GSp}}(4) \to \widehat{\mathrm{GL}}(4)$$

given by inclusion. This induces a map of L-groups:

$${}^L\mathrm{GSp}(4) \to {}^L\mathrm{GL}(4).$$

(In this case the L-groups are simply direct products of the dual groups with the absolute Galois group of the base field.) Thus functoriality conjectures predict that there should be a transfer of automorphic representations from GSp(4) to GL(4). This setup is discussed elsewhere in this volume in more detail (see [Ito] for a statement of functoriality conjectures and [Hir] and [Whi] for this particular transfer).

Many years ago, Jacquet, Piatetski-Shapiro and Shalika announced a proof, using the theta correspondence, of a transfer

$$\tau : \mathcal{A}_{0,\mathrm{gen}}(\mathrm{GSp}(4)) \to \mathcal{A}(\mathrm{GL}(4))$$

where the space on the left is the cuspidal generic representations of GSp(4) and on the right is the set of automorphic representations of GL(4). In terms of L-functions, $\pi \in \mathcal{A}_{0,\mathrm{gen}}(\mathrm{GSp}(4))$ transfers to $\pi' = \tau(\pi) \in \mathcal{A}(\mathrm{GL}(4))$ means

$$L(s, \pi) = L(s, \pi')$$

where $L(s, \pi)$ denotes the degree 4 spinor L-function for GSp(4). They never published this (or even wrote it down as far as I know), but Asgari and Shahidi ([AS06a], [AS06b]) have proven this transfer very recently by quite different methods.

In fact, one would like to know that under this transfer a (generic) cuspidal representation π of GSp(4) transfers to a representation π' of GL(4) which is also cuspidal. Asgari and Shahidi [AS06b] in fact prove that this is so unless π is a Weil lift from GSO(4), in which case π' is an isobaric sum of two cuspidal representations.

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Conversely, suppose one starts with a cuspidal representation π' of GL(4). One can think of the reverse transfer from GL(4) back to GSp(4). Of course this is only a partial transfer as not all representations of GL(4) come from GSp(4). Then it is natural to ask the following.

Question 1. *When does π' transfer to (i.e., come from) a cuspidal representation π of* GSp(4)*? In other words, when is π' in the image of τ?*

Let me remark that π' being in the image of τ only means that π comes from a *generic* cuspidal representation of GSp(4). However, what should be true is that packets of representations of GSp(4) should transfer to packets of representations of GL(4). Now each packet for GSp(4) should contain a generic representation and each packet for GL(4) contains only a single representation. Thus if π' comes from a representation of GSp(4), in principle it should also come from a generic representation of GSp(4). We will only address transfer among generic cuspidal representations in these notes. The transfer of non-generic representations cannot be tackled with the methods of [JPSS] or [AS06a], [AS06b] but requires an approach with the trace formula. See [Whi] in this volume for a discussion of the trace formula approach.

1.1. **The exterior-square L-function.** Suppose π and π' are cuspidal representations of GSp(4) and GL(4) which correspond in the sense that $\tau(\pi) = \pi'$. Then the central characters of π and π' are related by $\omega_{\pi'} = \omega_\pi^2$. One also has a map of dual groups

$$\widehat{\mathrm{GSp}}(4) = \mathrm{GSp}_4(\mathbb{C}) \to \mathrm{PGSp}_4(\mathbb{C}) \simeq \mathrm{SO}_5(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$$

which predicts a transfer of π to a representation π'' of GL(5). Indeed, suppose that $\omega_\pi^2 = 1$ and π restricts to a generic cuspidal automorphic representation of Sp(4). The dual group of Sp(4) is $\mathrm{SO}_5(\mathbb{C})$ which embeds in $\mathrm{GL}_5(\mathbb{C})$ as above. Then lifting results of Cogdell, Piatetski-Shapiro and Shahidi [CKPSS01] say π transfers to a representation π'' of $\mathrm{GL}_5(\mathbb{C})$. Now let us study how the transfers π' and π'' on GL(4) and GL(5) are related.

Then the following diagram of dual groups

$$\begin{tikzcd} \mathrm{GSp}_4(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{GL}_4(\mathbb{C}) \arrow[d, "\Lambda^2"'] \\ \mathrm{GL}_5(\mathbb{C}) \arrow[r] & \mathrm{GL}_6(\mathbb{C}) \end{tikzcd}$$

predicts the following transfer diagram

$$\begin{tikzcd} \pi \in \mathcal{A}(\mathrm{GSp}(4)) \arrow[r] \arrow[d] & \pi' \in \mathcal{A}(\mathrm{GL}_4(\mathbb{C})) \arrow[d, "\Lambda^2"'] \\ \pi'' \in \mathcal{A}(\mathrm{GL}_5(\mathbb{C})) \arrow[r] & \pi'' \boxplus \omega_{\pi'} \in \mathcal{A}(\mathrm{GL}_6(\mathbb{C})). \end{tikzcd}$$

The exterior-square transfer $\Lambda^2 : \mathrm{GL}(4) \to \mathrm{GL}(6)$ was established by Kim [Kim03]. Thus if $\omega_{\pi'} = 1$, $\Lambda^2(\pi') = \pi'' \boxplus 1$ and we should have an equality of degree 6 L-functions:

$$\zeta_F(s)L(s, \pi; \mathrm{std}) = L(s, 1)L(s, \pi'') = L(s, \pi'; \Lambda^2),$$

where $L(s, \pi; \mathrm{std}) = L(s, \pi'')$ is the standard L-function for GSp(4) (cf. [Ich]). Indeed, if $\pi' = \tau(\pi)$ in the context of [AS06b], then $L(s, \pi'; \Lambda^2)$ has a pole at $s = 1$. Furthermore, [JPSS] asserted that the converse is also true:

Theorem 1. ([JPSS], *unpublished*) *Suppose* $\omega_{\pi'} = 1$. *Then* π' *transfers to* GSp(4) *(generic) if and only if* $L(s, \pi'; \Lambda^2)$ *has a pole at* $s = 1$.

If $\omega_{\pi'} \neq 1$, then the L-function criterion should be that the twisted L-function $L(s, \pi'; \Lambda^2 \otimes \omega_{\pi'}^{-1})$ has a pole at $s = 1$.

This provides a complete answer to our question above in terms of L-functions. Now we will move on to answering the question in terms of period integrals.

1.2. **The Shalika period.** Define the *Shalika subgroup*

$$S = \left\{ \begin{pmatrix} A & X \\ & A \end{pmatrix} \right\} \subseteq \mathrm{GL}(4),$$

and fix a nontrivial additive character ψ of $\mathbb{A} := \mathbb{A}_F$. Define a character θ of S by

$$\theta\left(\begin{pmatrix} A & \\ & A \end{pmatrix} \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \right) = \psi(\mathrm{tr} X).$$

Let π' be a cuspidal repesentation of $\mathrm{GL}_4(\mathbb{A})$ with trivial central character. Define a linear form on the space of π' by

$$P_\theta(\phi) = \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} \phi(s)\theta(s) ds$$

for $\phi \in \pi'$. (Here Z denotes the center of GL(4).) The period integral $P_\theta(\phi)$ is called the *Shalika period*.

Using an integral representation for $L(s, \pi'; \Lambda^2)$, Jacquet and Shalika proved the following result.

Theorem 2. ([JS90]) $L(s, \pi'; \Lambda^2)$ *has a pole at* $s = 1$ *if and only if* P_θ *is nonzero on the space of* π'.

Combining this with Theorem 1 gives an alternative answer to our question above: a cuspidal representation of of GL(4) with trivial central character transfers to GSp(4) (generic) if and only if it has a nonvanishing Shalika period. In the case π' has nontrival central character, one needs to multiply by the integrand of P_θ by $\omega_{\pi'}^{-1}$. (Otherwise the integral does not even make sense.)

2. Transfer from GL$(2, D)$

If D is a quaternion algebra over F, then $\mathrm{GL}(2, D)$ is an inner form of GL(4) (meaning they are isomorphic over the algebraic closure of F). All inner forms of GL(4) are of the form $\mathrm{GL}(r, D)$ where $r = 1$, 2 or 4, and D is a division algebra of rank 16, 4, or 1. On the other hand, the inner forms of GSp(4) are of the form GSp(4) or

$$\left\{ g \in \mathrm{GL}(2, D) : {}^t g J g = J \right\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and D is a quaternion algebra. Since two inner forms have the same dual group, functoriality predicts transfers between any inner form G of GSp(4) and any inner form G' of GL(4). One way to obtain such a transfer is by using (generalized) Jacquet-Langlands correspondences. The relevant Jacquet-Langlands correspondences would be the conjectural transfers of representations from G to GSp(4) and G' to GL(4). Then one could transfer between G and G' via

$$G \overset{\dashleftarrow}{\rightarrow} \mathrm{GSp}(4) \overset{\dashleftarrow}{\rightarrow} \mathrm{GL}(4) \underset{\dashrightarrow}{\leftarrow} G',$$

where the dashed arrows indicate partial transfers. In fact, the Jacquet-Langlands transfers here are already known in some cases due to Sorensen [Sor] for G and Badulescu [Bad] for G'. Regardless, we will restrict ourselves to considering transfer between $\mathrm{GSp}(4)$ and $\mathrm{GL}(2,D)$ and simply assume that representations of $\mathrm{GL}(2,D)$ are in one-to-one correspondence with representations of $\mathrm{GL}(4)$ which satisfy certain local conditions at the places where D ramifies.

More precisely, fix a quaternion algebra D. We assume that for each cuspidal automorphic representation π of $\mathrm{GL}_2(D(\mathbb{A}))$, there is a unique automorphic representation $\pi' = \mathrm{JL}(\pi)$ of $\mathrm{GL}_4(\mathbb{A})$ such that $\pi_v = \pi'_v$ wherever $D_v^\times = \mathrm{GL}_2(F_v)$. We remark that this transfer is the subject of ongoing work of Badulescu, and hope that it will be established in the near future.

Now one can ask:

Question 2. *Given a cuspidal representation π of $\mathrm{GL}(2,D)$, when does it transfer to $\mathrm{GSp}(4)$?*

We may try to answer the question as we did for $\mathrm{GL}(4)$. Again, let us assume that the central character of π is trivial. Let $\pi' = \mathrm{JL}(\pi)$ be the corresponding representation of $\mathrm{GL}(4)$. Note that, because their local factors agree almost everywhere, $L(s,\pi;\Lambda^2)$ has a pole at $s=1$ if and only if $L(s,\pi';\Lambda^2)$ does. Thus we get for free an answer to our question in terms of L-functions from Theorem 1.

Let us now consider an answer to Question 2 in terms of period integrals. As before, define the *Shalika subgroup*

$$S = \left\{ \begin{pmatrix} A & X \\ & A \end{pmatrix} \right\} \subseteq \mathrm{GL}(4),$$

and fix a nontrivial additive character ψ of $\mathbb{A} := \mathbb{A}_F$. Define a character $\theta : S \to \mathbb{C}^\times$ by

$$\theta\left(\begin{pmatrix} A & \\ & A \end{pmatrix} \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \right) = \psi(\mathrm{tr} X),$$

and set

$$P_\theta(\phi) = \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} \phi(s)\theta(s)ds$$

for $\phi \in \pi$.

It is not clear how one might directly prove an analogue of Theorem 2 in the context of $\mathrm{GL}(2,D)$ because one no longer has an integral representation for the exterior-square L-function. Instead, Jacquet has suggested the following conjecture to relate Shalika periods for $\mathrm{GL}(2,D)$ and $\mathrm{GL}(4)$.

Conjecture 1. ([JM]) *Let π be a cuspdial representation of $\mathrm{GL}(2,D)$ and $\pi' = \mathrm{JL}(\pi)$. Then π has a nonvanishing Shalika period if and only if π' does.*

This would then provide an answer Question 2 in terms of period integral. Using the relative trace formula, Jacquet and I have proven the following partial result.

Theorem 3. ([JM]) *Suppose $D_{v_1}^\times \neq \mathrm{GL}_2(F_{v_1})$ and π_{v_2} is supercuspidal for some places $v_1 | \infty$ and $v_2 < \infty$. Then if π has a nonvanishing Shalika period, π' does also. In particular, $L(s,\pi;\Lambda^2)$ has a pole at $s=1$ and π transfers to $\mathrm{GSp}(4)$.*

We remark that our proof in fact establishes a weak transfer of π from $\mathrm{GL}(2,D)$ to $\mathrm{GL}(4)$, i.e. we get the existence of π', *provided* π satisfies a strong multiplicity one condition. It may even be possible to remove this requirement, but that is not our intent.

The condition at v_1 in the theorem may be removable with a relatively small amount of work—namely, by studying truncation on GL(2). However, removing the cuspidality condition at v_2 seems much more difficult. It may be possible to lift the condition to the considerably weaker statement that π_{v_2} be a discrete series representation, but to get a complete proof of the conjecture following our approach, one needs much more sophisticated relative trace formulas than what we develop. Specifically, one will need to analyze the contribution of continuous spectrum for trace formulas on GL$(2, D)$ and GL(4) relative to the Shalika subgroup. As of yet, there is no general machinery for doing this or a standard way of truncating. In this sense, the current state of the relative trace formula could be likened to the pre-Arthurian age of the Selberg trace formula.

3. The relative trace forumla

Let G be a connected reductive algebraic group over F. Denote the center of G by Z. For a function $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$, we associate to f a kernel function

$$K(x, y) := \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma y).$$

This is a smooth function on $(G(F)\backslash G(\mathbb{A}))^2$. To keep our trace formulas as simple as possible, let us make the following assumptions:

Assumption 1. *The integrals below converge.*

Assumption 2. *All representations π that occur in $R(f)$ below are cuspidal.*

These assumptions will be satisfied for instance if either i) $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is compact, or ii) f satisfies some restrictive hypotheses. For our application in the next section we will choose f so that these assumptions are satisfied.

The function $K(x, y)$ is called a kernel because it is one. Specifically, consider the right regular representation R of the group $Z(\mathbb{A})\backslash G(\mathbb{A})$ on the Hilbert space $L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ given by

$$(R(y)\phi)(x) = \phi(xy).$$

Then for $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$, we define an operator

$$R(f) : L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) \to L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$$

by

$$(R(f)\phi)(x) := \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)R(y)\phi(x)dy = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\phi(xy).$$

Simple manipulations show that $R(f)$ is an integral operator with kernel $K(x, y)$:

$$(R(f)\phi)(x) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K(x, y)\phi(y)dy.$$

Just as R decomposes into irreducible representations $\{\pi\}$ (not necessarily all cuspidal), $R(f)$ decomposes into a direct sum of (cuspidal by assumption) operators with finite multiplicities,

$$R(f) = \oplus m_\pi \pi(f).$$

Here the operator $\pi(f)$ is defined in the same way as $R(f)$:

$$(\pi(f)\phi)(x) := \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\pi(y)\phi(x)dy = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\phi(xy)$$

for ϕ in the subrepresentation space π of $L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$. Associated to this decomposition of $R(f)$ into $\pi(f)$'s, one gets an alternative expression for the kernel

$$K(x,y) = \sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma y) = \sum_{\pi} K_\pi(x,y), \tag{1}$$

where

$$K_\pi(x,y) = \sum_{\{\phi_i\}} (\pi(f)\phi_i)(x)\overline{\phi_i(y)}$$

and ϕ_i runs over an orthornormal basis for the space of π. The expression occurring on the left hand side (by which I mean the middle) of (1) is called the *geometric expansion* for the kernel and the expression on the right is called *spectral expansion* of $K(x,y)$.

3.1. **The Selberg trace formula.** We first recall the Selberg trace formula for purposes of comparison. Computing the integral of the kernel over the diagonal

$$\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} K(x,x)dx$$

in two ways using the two expansions for $K(x,y)$ yields the following simple form of the Selberg trace formula:

$$\sum_{\gamma\in G(F)} \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx = \mathrm{tr}\pi(f).$$

The left and right hand sides are respectively called the *geometric* and *spectral sides* of the trace formula. The geometric side is often written in the form

$$\sum_{\gamma\in\{G(F)\}} \mathrm{vol}(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx,$$

where $\{G(F)\}$ denotes the set of conjugacy classes of $G(F)$ and G_γ denotes the centralizer of γ in G. These integrals in the sum are known as *orbital integrals*, making it more clear why the left hand side is called the geometric side: it is a sum of integrals over orbits of $G(\mathbb{A})$. So the Selberg trace formula expresses the sum of the traces of the operators $\{\pi(f)\}$ as a sum of orbital integrals. Some of its many applications (transfer, dimensions of spaces of cusp forms, etc.) are mentioned elsewhere in this volume.

3.2. **The relative trace formula.** Let H be a closed subgroup of G (typically the fixed points of an involution) which contains the center Z, and let θ be a character of H. Suppose you want to study period integrals

$$P_\theta(\phi) := \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\theta(h)dh.$$

Jacquet's idea was the following. Instead of integrating $K(x,y)$ over the diagonal, integrate $K(x,y)$ against θ over the subgroup $H\times H$. (More generally, one can integrate over a product of two different subgroups $H_1 \times H_2$ with different characters

θ_1 and θ_2, but we will stick to the situation $H_1 = H_2$ and $\theta_1 = \theta_2$.) Precisely, use both expressions for the kernel to integrate

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} K(h_1, h_2)\theta(h_1 h_2) dh_1 dh_2.$$

Then one obtains the relative trace formula

$$RTF(f) := \sum_{\gamma \in Z(F)\backslash G(F)} \int \int f(h_1^{-1}\gamma h_2)\theta(h_1 h_2) dh_1 dh_2 = \sum_{\pi} B_\pi(f)$$

As before, we call the left (or rather middle here) hand side the geometric side and the right hand side the spectral side. We can rewrite the geometric side as a sum over double cosets of integrals over the double cosets $H\gamma H$. These integrals are called by analogy *(relative) orbital integrals*. The distributions $B_\pi(f)$ appearing on the spectral side, called *Bessel distributions*, are given by

$$B_\pi(f) = \sum_{\{\phi_i\}} P_\theta(\pi(f)\phi_i)\overline{P_\theta(\phi_i)}.$$

Thus the relative trace formula provides an expression for sums of (products of) period integrals in terms of integrals over double cosets. The name relative trace formula comes from the fact that $B_\pi(f)$ is formally the trace of $\pi(f)$ relative to the Hermitian form on H given by $(\phi, \phi') = P_\theta(\phi)\overline{P_\theta(\phi')}$. We can summarize the analogy with the Selberg trace formula in the following table.

	Geometric Side	Spectral Side
Selberg Trace Formula	conjugacy classes	traces
Relative Trace Formula	double cosets	period integrals

Thus just as the Selberg trace formula provides a way to study traces of representations, the relative trace formula provides a way to study period integrals. In particular, for Conjecture 1 we wanted to compare period integrals on two different groups, GL(2, D) and GL(4). I will now outline a general relative trace formula approach to comparing period integrals on different groups, which mimics the way the (Arthur-)Selberg trace formula is used to compare traces of representations on different groups.

3.3. **Strategy.** Suppose H and H' are subgroups of G and G' with characters θ and θ'. Suppose we want to compare period integrals P_θ and $P_{\theta'}$ on the two groups G and G'. The general strategy is as follows.

i. Find matching functions f and f' on G and G' such that

$$\int_H \int_H f(h_1^{-1}\gamma h_2)\theta(h_1 h_2) dh_1 dh_2 = \int_{H'} \int_{H'} f((h_1')^{-1}\gamma' h_2')\theta'(h_1' h_2') dh_1' dh_2'$$

for some correspondence of double cosets $H\gamma H \leftrightarrow H'\gamma' H'$. This correspondence need not be injective or surjective in either direction.

ii. Establish relative trace formulas $RTF(f)$ and $RTF(f')$ on G and G'. By (1), the geometric sides should be equal—at least if the correspondence of double cosets above is one-to-one. Looking at the spectral sides, we see

$$\sum_{\pi} B_\pi(f) = \sum_{\pi'} B_{\pi'}(f'). \tag{2}$$

iii. Use Equation (2) to obtain the desired relation of period integrals. On the level of Bessel distributions, this relation typically amounts to

$$B_\pi(f) = B_{\pi'}(f') \tag{3}$$

where π' is the functorial image of π. When G and G' are inner forms, equality of local distributions $B_{\pi_v}(f_v) = B_{\pi'_v}(f'_v)$ for almost all places v. Then a linear independence argument of Langlands shows that (2) implies (3) for $\pi \leftrightarrow \pi'$. However in a more general setting, one does not expect (3), but rather an equality of the form (2) where the sum will be over certain "packets" of representations π and π'.

3.4. **Special values.** I would like to mention one other application of the relative trace formula for GSp(4). Let $G = \mathrm{GSp}(4)$, and let π be a cuspidal representation of G. Then Börcherer's conjecture essentially says

$$L(1/2, \pi \otimes \chi) = c \left| \int_H \phi(h)\theta(h)dh \right|^2, \tag{4}$$

where H is the Bessel subgroup of a certain inner form G' of G, θ is a certain character of H, χ is a quadratic character related to G' and ϕ is a certain cusp form on G'. This is a higher-dimensional analogue of a classical result of Waldspurger for GL(2). Furusawa and Shalika have an approach for Börcherer's conjecture using the relative trace formula [FS03]. They have proven the relevant fundamental lemmas, which is an important step for constructing matching functions f and f' on G and G'.

While much work still needs to be done, we remark that the relative trace formula has recently been used to obtain explicit formulas of a form similar to (4) in the cases where $G = \mathrm{GL}(n)$ and H is unitary ([LO]), and where $G = \mathrm{GL}(2)$ and H is a torus ([MW]). The relative trace formulas that we discuss in the next section are related to the residue at $s = 1$ of the exterior-square L-function, i.e., they are related to the value of the standard L-function for GSp(4) at $s = 1$. More specifically, if π is a representation of GSp(4) which transfers to a representation π' of $\mathrm{GL}(2, D)$, then it seems reasonable to expect an identity of the form

$$L(1, \pi; \mathrm{std}) = c \int_S \phi(s)\theta(s)ds$$

for a certain $\phi \in \pi'$.

4. On the proof

Let us return to the setting of Theorem 3. Thus $G = \mathrm{GL}(2, D)$ and $G' = \mathrm{GL}(4)$ with the Shalika subgroups $H = S$ and $H' = S'$. We say a function f_v (on G_v or G'_v) is *restricted* (resp. *elliptically restricted*) if

$$\mathrm{supp}(f_v) \subseteq \left\{ s_1 \begin{pmatrix} & \gamma \\ 1 & \end{pmatrix} s_2 \middle| s_1, s_2 \in S \text{ (resp. and } \gamma \text{ is elliptic)} \right\}.$$

i. (Local matching.) Let us work over a local field F_v. Recall that two semisimple elements γ and γ' of $D^\times$ and GL(2) correspond if they have equal eigenvalues. We define a correspondence of double cosets by

$$S \begin{pmatrix} & \gamma \\ 1 & \end{pmatrix} S \leftrightarrow S' \begin{pmatrix} & \gamma' \\ 1 & \end{pmatrix} S'$$

if and only if $\gamma \leftrightarrow \gamma'$. Double cosets of this form will be called *good*. Some double cosets are left out of this correspondence, but the relative orbital integrals over those double cosets will be zero and we need not worry about them. Observe that with a change of variables I can write

$$\begin{aligned}
&\int_{Z\backslash S}\int_{Z\backslash S} f\left(s_1^{-1}\begin{pmatrix} & \gamma \\ 1 & \end{pmatrix} s_2\right)\theta(s_1s_2)ds_1ds_2 \\
=\ &\int_{Z\backslash S}\int_{Z\backslash S} f\left(\begin{pmatrix} A & X \\ & A\end{pmatrix}\begin{pmatrix} & \gamma \\ 1 & \end{pmatrix}\begin{pmatrix} B & Y \\ & B\end{pmatrix}\right)\theta(s_1s_2)dAdXdBdY \\
=\ &\int_{Z\backslash S}\int_{Z\backslash S} f\left(\begin{pmatrix} A & \\ & A\end{pmatrix}\begin{pmatrix} X & XY+B^{-1}\gamma B \\ 1 & Y\end{pmatrix}\right)\theta(s_1s_2)dAdXdBdY \\
=\ &\int_{Z\backslash D^\times}\phi(B^{-1}\gamma B)dB,
\end{aligned}$$

where $\phi \in C_c^\infty(Z\backslash D^\times)$. Thus we have written relative orbital integrals of f (for good double cosets) as ordinary orbital integrals of some function ϕ on $D^\times$. We may do the same for f' to reduce to ordinary orbital integrals of some ϕ' on GL(2). The maps $f \mapsto \phi$ and $f' \mapsto \phi'$ are surjective. Hence the matching of ordinary orbital integrals on $D^\times$ and GL(2) provides us with restricted functions f and f' whose relative orbital integrals match (at least for good double cosets).

With this local matching, global matching for functions of a restricted type is now easy. Consider a function $f = \prod f_v \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$ with f_v restricted for each v where $G_v \neq G'_v$. Then we choose a matching function $f' = \prod f'_v \in C_c^\infty(Z(\mathbb{A})\backslash G'(\mathbb{A}))$ by $f_v \leftrightarrow f'_v$ when $G_v \neq G'_v$ and $f_v = f'_v$ when $G_v = G'_v$.

ii. Here we are back in the global setting. We establish the following relative trace formulas in [JM].

Proposition 1. *Suppose $f = \prod f_v$ on $G = \mathrm{GL}(2, D)$ such that f_{v_1} is restricted and f_{v_2} is a supercusp form for some places v_1 and v_2. Then $RTF(f)$ holds.*

Proposition 2. *Suppose $f' = \prod f'_v$ on $G' = \mathrm{GL}(4)$ such that f'_{v_1} is elliptically restricted and f'_{v_2} is a supercusp form for some places v_1 and v_2. Then $RTF(f')$ holds.*

I will not say anything about the proofs except that they are not so difficult. One can do all the necessary formal manipulations because the assumptions guarantee convergence. Thus if f and f' are matching functions which satisfy the hypotheses of Propositions 1 and 2, we have

$$\sum_{\pi \text{ cusp}} B_\pi(f) = \sum_{\pi' \text{ cusp}} B_{\pi'}(f').$$

iii. Applying Langlands' linear independence argument to the above equality yields in fact

$$B_\pi(f) = B_{\pi'}(f'),$$

where $\pi' = \mathrm{JL}(\pi)$. This is what we want to prove our theorem.

Let D and π be as in Theorem 3, i.e., D is ramified at an infinite place v_1 and π_{v_2} is supercuspidal at a place where D is split. Essential to the proof is the following fact: π (resp. π') has a nonvanishing Shalika period if and only if B_π (resp. $B_{\pi'}$) is nonzero. The "if" direction is obvious, and the other direction is not hard. Suppose π has a nonvanishing Shalika period, i.e., $B_\pi \neq 0$.

The difficulty now is showing that this is not always zero for f of our restricted type. Specifically, one must show that there exists an $f = \prod f_v \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}))$ such that

(a) f_v is restricted where $G_v \neq G'_v$;
(b) f_{v_1} is elliptically restricted;
(c) f_{v_2} is a supercusp form;
(d) $B_\pi(f) \neq 0$.

This, except for (c), is quite involved. One first shows that B_π factors into local Bessel distributions: $B_\pi(f) = \prod B_{\pi_v}(f_v)$. Then the above amounts to proving that the local Bessel distributions B_{π_v} are not supported on certain bad sets.

Condition (a) guarantees we have a locally matching f'_v as above. While condition (b) is stronger than what we require for the trace formula of Proposition 1, it allows to choose a matching f'_{v_1} which is also elliptically restricted, which is needed for Proposition 2. Condition (c) similarly guarantees $f'_{v_2} = f_{v_1}$ is a supercusp form. Thus with these conditions we have matching functions f and f' which satisfy the hypotheses we need for our simple relative trace formulas. Finally, condition (d) tells us what we really want: $B_{\pi'}(f') = B_\pi(f) \neq 0$, i.e., π' has a nonvanishing Shalika period. ☐

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