EXACT DOUBLE AVERAGES OF TWISTED L-VALUES

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Abstract. Consider central L-values of even weight elliptic or Hilbert modular forms f twisted by ideal class characters χ of an imaginary quadratic extension K. Fixing χ, and assuming K is inert at each prime dividing the level, one knows simple exact formulas for averages over newforms f of squarefree levels satisfying a parity condition on the number of prime factors. These averages stabilize when the level is large with respect to K (the "stable range").

In weight 2, we obtain exact formulas for a simultaneous average over both f and χ. We allow for non-squarefree levels with any number of prime factors, and ramification or splitting of K above the level. Under elementary conditions on the level, these double averages are "stable" in all ranges. Two consequences are generalizations of the aforementioned stable (single) averages and effective results on nonvanishing of central L-values.

1. Introduction

1.1. **Prime power level.** We first explain our results in the case of prime power levels over $\mathbb{Q}$. Denote by $S_2(N)$ the space of weight 2 elliptic modular forms of level $\Gamma_0(N)$. Let $\mathcal{F}^{\mathrm{new}}(N)$ denote the set of level N newforms in $S_2(N)$. Denote by $L(s,-)$ a completed L-function with center $s=\frac{1}{2}$, and by $L_{\mathrm{fin}}(s,-)$ the nonarchimedean part of $L(s,-)$.

Suppose N is a prime, $K/\mathbb{Q}$ an imaginary quadratic field of discriminant D_K which is inert at N, and χ a character of $\mathrm{Cl}(K)$. Assuming D_K is odd, [MR12] proved an exact formula for weighted averages of the central values $L(\frac{1}{2},f,\chi)$, where f ranges over $\mathcal{F}^{\mathrm{new}}(N)$. In the *stable range* $N>|D_K|$, these averages simplify to yield:

$$\frac{u_K^2\sqrt{-D_K}}{8\pi^2}\sum_{f\in\mathcal{F}^{\mathrm{new}}(N)}\frac{L_{\mathrm{fin}}(\frac{1}{2},f,\chi)}{(f,f)}=h_K\left(u_K-\delta_{\chi,1}\frac{12h_K}{N-1}\right). \tag{1.1}$$

Here $\delta_{\chi,1}$ is the Kronecker delta and $u_K=\frac{\#\mathfrak{o}_K^\times}{2}$. The work [MR12] also allows for weighting by Fourier coefficients of f and arbitrary even weights. From the generalization in [FW09], (1.1) still holds when D_K is even.

Our main theme is that the arithmetic of quaternion algebras allows us to provide exact formulas for any range of $(N,|D_K|)$ when we average over both f and χ. These double average value formulas are strikingly simple for "balanced" levels, and yield stable single average value formulas generalizing (1.1).

Theorem 1.1. *Suppose $N=p^{2r+1}\geq 11$ for a prime p with $N\neq 27$, and $K/\mathbb{Q}$ is an imaginary quadratic field which is not split at p. If $r>0$, assume $K/\mathbb{Q}$ is inert at p. Put*

$$C_K=\frac{1}{2}e_p(K/\mathbb{Q})u_K^2\sqrt{-D_K},$$

where $e_p(K/\mathbb{Q})=1$ if $K/\mathbb{Q}$ is inert at p and $e_p(K/\mathbb{Q})=2$ if $K/\mathbb{Q}$ is ramified at p.

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(i) If either $p \equiv 1 \bmod 12$ or $u_K > 1$, we have the exact double average formula

$$\frac{C_K}{4\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{f \in \mathcal{F}^{\mathrm{new}}(N)} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} = \begin{cases} h_K^2(u_K - \frac{12}{p-1}) & r = 0 \\ h_K^2 u_K (1 - \frac{1}{p^2}) & r \geq 1. \end{cases} \tag{1.2}$$

(ii) If $p > |D_K|$, we have the stable (single) average formula

$$\frac{C_K}{4\pi^2} \sum_{f \in \mathcal{F}^{\mathrm{new}}(N)} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} = \begin{cases} h_K(u_K - \delta_{\chi,1}\frac{12h_K}{p-1}) & r = 0 \\ h_K u_K (1 - \frac{1}{p^2}) & r \geq 1. \end{cases} \tag{1.3}$$

Since the formula in (i) only depends on h_K and u_K, by analogy with [MR12], we think of this double average as being "stable" in all ranges when $p \equiv 1 \bmod 12$ (in which case we call $N = p^{2r+1}$ balanced) or if $u_K > 1$. If we do not assume that $p \equiv 1 \bmod 12$, then one still has a double average value formula but it involves height pairings and is less elementary. One can also weight the L-values by Fourier coefficients of f as in [MR12], but then the formulas also involve certain representation numbers of quadratic forms. We also treat levels $N = p^2$ when K is ramified at p, and allow splitting of K above the level when N has multiple prime factors. The issue with levels $N = p^{2r}$ with $r > 1$ is explained below.

The most interesting, or at least classical, case of (ii) is when $\chi = 1_K$ is trivial, so that our twisted L-function factors as $L(s, f, 1_K) = L(s, f)L(s, f \otimes \eta_K)$. In this case if $N = p^{2r+1}$ and K is split at p, then we are forced to have $L(\frac{1}{2}, f, 1_K) = 0$ because the root number is -1. So for prime power level N and imaginary quadratic K, the cases of most interest are when K is inert or ramified at p, as in the theorem.

Using the extension of (i) to allow $p \not\equiv 1 \bmod 12$, we still have elementary upper and lower bounds on the double average. We leverage this to get lower (and upper) bounds for the single average (1.3) when $\chi = 1_K$ is trivial. These bounds are valid outside of the stable range $p > |D_K|$ in (ii), and yield the following sharper *effective* non-vanishing result.

Corollary 1.2. *Fix an imaginary quadratic field K. Then for any odd power of a prime $N = p^{2r+1}$ such that p is inert in K, there exists a newform $f \in S_2(N)$ such that $L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \eta_K) \neq 0$, provided that (i) $p > \frac{12h_K}{u_K} + 1$ if $N = p$ is prime; or (ii) $p^2 > \frac{3h_K}{u_K}$ if $N > 27$ is not prime.*

The prime level case is already in [MR12, Theorem 3]. See Proposition 5.9 for an analogue in level p^2. We remark that effective non-vanishing results of this type in levels p^2 and $2p^2$ have been applied to generalized Fermat equations in [Ell05] and [BEN10].

The prime level case of Theorem 1.1(i) (and its extension to general p) follows easily from Gross's L-value formula [Gro87]. (Gross's formula was also the basis of the proof of the weight 2 case of [MR12].) However, to our knowledge, this had not been observed before. The main results of [MR12] have been extended to squarefree levels N which have an odd number of prime factors in [FW09] (which also treats totally real base fields). Note [FW09] also requires K to be inert at each prime dividing the level. The approach there is via a relative trace formula, and is based on the central L-value formula in terms of periods on quaternion algebras from [MW09].

Our approach also uses the relative trace formula from [MW09], but instead of computing geometric orbital integrals as in [FW09], we directly compute averages of periods using the arithmetic of quaternion algebras. The main technical difficulties in this work arise from treating levels which are both not squarefree and which may have an even number of prime factors. A novel feature of our approach is that it allows K to be split or ramified at some primes dividing the level when $\chi = 1_K$. Moreover, our approach to compute period averages is much simpler than the geometric trace formula calculations in [FW09]. On the other hand, we do not get an independent proof of the stable range formulas, nor is it clear how well our method could treat higher weights. (However, the case of weight 2 where our method readily applies is the precisely the case in which the geometric trace formula calculations are the most complicated.) That said, we expect that blending our methods with those of [FW09] should allow one to extend [FW09] to more general levels and ramification behavior of K in arbitrary even weight. (E.g., see Remark 1.6.)

1.2. **A more general class of levels.** In this paper, we treat a broad class of level structures for parallel weight 2 Hilbert modular forms over a totally real base field of class number 1. However, we will restrict to elliptic modular forms in the introduction, as well as for some of our more explicit results at the end.

By a special level type (N_1, N_2, M) we mean pairwise coprime positive integers N_1, N_2 and M such that (i) each prime dividing N_1 occurs to an odd power; (ii) N_2 is the square of a squarefree number; (iii) $\#\{p|N_1N_2\}$ is odd; and (iv) M is squarefree. We say an imaginary quadratic field K is (N_1, N_2, M)-admissible if $K/\mathbb{Q}$ is non-split at each $p|N_1N_2$, split at each $p|M$, ramified at each $p|N_2$, and unramified at each p such that $p^3|N_1$.

From now on, assume that $N = N_1N_2M$ with (N_1, N_2, M) a special level type, and that K is an (N_1, N_2, M)-admissible imaginary quadratic field. Our main results for elliptic modular forms extend and refine Theorem 1.1 such N and K. We address the reason for these restrictions in the outline of the proof below.

Let $\mathcal{F}(N)$ be the set of normalized eigenforms $f \in S_2(N)$ which are newforms of some level $N'|N$ such that (i) $\mathrm{ord}_p(N')$ is odd for each $p|N_1$, and (ii) the local representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to f is discrete series for each $p|N_2$. For a character χ of $\mathrm{Cl}(K)$, denote by $\mathcal{F}(N;\chi)$ the subset of $f \in \mathcal{F}(N)$ such that $a_p(f) = \chi_p(\sqrt{-D_K})$ for each $p|N$ which ramifies in K.

Put

$$C(K;N) = 2^{\#\{p|\gcd(D_K,N)\}-1} u_K^2 \sqrt{-D_K} \prod_{p|N_2} \frac{1}{1+p^{-1}}. \tag{1.4}$$

Note that since $\{p|N_2\} = \{p|N : \mathrm{ord}_p(N) = 2\}$, we see $C(K;N)$ only depends on N and not the precise choice of triple (N_1, N_2, M), which in general is not uniquely determined by N. Set

$$c(N_1, N_2, M) = \frac{12}{N} \prod_{p|N_1N_2} \frac{1}{1-p^{-1}} \prod_{p|N_2M} \frac{1}{1+p^{-1}}.$$

We also define a weight $\Lambda_N(f,\chi) = \prod_{p|N} \Lambda_p(f,\chi)$. Here the local factor $\Lambda_p(f,\chi) = 1$ if $p|N_1$, is defined by (5.8) when $p|N_2$, and by (5.4) or equivalently (5.11) when $p|M$. When $p|N_2$, the factor $\Lambda_p(f,\chi)$ does not actually depend on χ, only whether f is p-new and whether f is p-minimal.

Let $N_1' = \prod p^{\mathrm{ord}_p(N_1)}$ where p runs over primes such that $\mathrm{ord}_p(N_1) > 1$. Let $\omega'(n)$ be the number of odd prime divisors of n. Denote by $\deg T_m$ the degree of the Hecke operator on T_m, which is 1 if $m = 1$ and $p+1$ if $m = p$ is a prime not dividing N. (See Section 4.1 for our precise definition.)

For a newform f, let N_f denote the (exact) level of f, and $(f,f) = (f,f)_{N_f}$ be the usual Petersson norm of f with respect to the standard measure on $X_0(N_f)$.

Our main result, Theorem 5.5, specialized to the case $F = \mathbb{Q}$ says the following, with remaining notation defined below.

Theorem 1.3. *Let $m \geq 1$ be coprime to $N_1' N_2$. Then*

$$\frac{C(K;N)}{4\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{f \in \mathcal{F}(N;\chi)} a_m(f) \Lambda_N(f,\chi) \frac{N_f}{N} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} = h_K \left(\sum w_i h_i a_{ii}(m) - \delta^+(N,m) \deg T_m 2^{\omega'(N_2)} h_K c(N_1, N_2, M) \right).$$

In particular, if $m = 1$, we have

$$\frac{C(K;N)}{4\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{f \in \mathcal{F}(N;\chi)} \Lambda_N(f,\chi) \frac{N_f}{N} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} = h_K \left(\sum w_i h_i - 2^{\omega'(N_2)} h_K c(N_1, N_2, M) \right).$$

The weighting of our average L-values by $\Lambda_N(f,\chi)\frac{N_f}{N}$ is simply for two reasons: to account for oldforms at $p|M$, and because we wrote the above average terms of the Petersson norm (f,f) rather than $L(1,f,\mathrm{Ad})$. The factors $\Lambda_p(f,\chi)$ are just 1 for $p|M$ when f is p-new, and the factors $\frac{N_f}{N}$ and $\Lambda_p(f,\chi)$ for $p|N_2$ are not needed if one replaces (f,f) with $L(1,f,\mathrm{Ad})$ (see Theorem 5.5.)

We will now explain the remaining quantities in this theorem, but the main point is that the quantity $\sum w_i h_i$ is approximately h_K (or exactly $u_K h_K$ if $u_K > 1$), and is exactly h_K under either elementary conditions on (N_1, N_2, M) or when we are in the stable range.

Let $B/\mathbb{Q}$ be the definite quaternion algebra with discriminant $D_B = \prod_{p|N_1N_2} p$. Then the condition that K is (N_1, N_2, M)-admissible implies that there is a special order (in the sense of Hijikata–Pizer–Shemanske [HPS89]) $\mathcal{O} \subset B$ of level type (N_1, N_2, M) in which $\mathfrak{o}_K$ embeds. In particular, if N is squarefree, we simply mean that $\mathcal{O}$ is an Eichler order of level N in B. See Section 3.1 for the general definition. We remark that $c(N_1, N_2, M)$ is simply the reciprocal of the mass of $\mathcal{O}$.

Let $\mathrm{Cl}(\mathcal{O}) = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ denote the set of (invertible) right $\mathcal{O}$-ideal classes of B. Let w_i be one half of the number of units in the left order of $\mathcal{I}_i$. We usually have $w_i = 1$, and if $N \geq 5$ then each $w_i \leq 3$ (see Lemma 5.7). Fix an embedding $\mathfrak{o}_K \subset \mathcal{O}$, which induces an ideal class map $\mathrm{Cl}(K) \to \mathrm{Cl}(\mathcal{O})$. Let h_i be the size of the preimage of $\mathcal{I}_i$ under this map. Hence $\sum h_i = h_K$, and we see that $h_K \leq \sum w_i h_i \leq 3h_K$. We note that $\sum w_i h_i$ can be interpreted in terms of heights of special cycles as in [Gro87]. This explains all of the notation in the $m = 1$ case.

For $m \geq 1$, $\delta^+(N,m)$ is either 0 or 1. It is always 1 if $m = 1$ or if $N_2 = 1$. See Proposition 4.1 for the full definition, where it is denoted $\delta^+(\mathcal{O}, m)$. Finally, $a_{ii}(m)$ denotes the i-th diagonal element of the m-th Brandt matrix associated to the given ordering of $\mathrm{Cl}(\mathcal{O})$. This can be expressed in terms of the number of ways a quadratic form associated to (the left order of) $\mathcal{I}_i$ represents m—see Section 3.3.

We say a triple (N_1, N_2, M) is balanced if it satisfies the conditions of Corollary 4.10. These are simple elementary conditions on the primes dividing N_1, N_2 and M, that force $\sum w_i h_i = h_K$ for any (N_1, N_2, M)-admissible K. For instance, the triple (N_1, N_2, M) is balanced if $N_2 > 9$, or if there exists a $p \equiv 1 \bmod 12$ dividing N_1, or if there exists a

$p \equiv 11 \bmod 12$ dividing M. Hence for balanced level types, the above double average formula, at least for $m = 1$, is completely elementary.

To get from Theorem 1.3 to Theorem 1.1(i), we want to isolate the newform contribution at level N. This is possible, at least for $m = 1$, when $\sum w_i h_i = h_K$ in both level N as well as any relevant level $N'|N$ for which we need to remove the contribution of oldforms. (This is the reason for the at-first-glance curious exclusion of $N = 27$—cf. Example 6.3.) This idea leads to a generalization of Theorem 1.1(i) to where one restricts to forms which are new away from M. (The factors $\Lambda_N(f,\chi)$ prevent us from exactly isolating forms which are new also at M, but we can at least approximate the contribution from forms new at M—e.g., Proposition 5.10.)

Let $\mathcal{F}_0(N)$ be the set of normalized eigenforms $f \in \mathcal{F}(N)$ which are newforms for some level $N'|N$ such that $\mathrm{ord}_p(N') = \mathrm{ord}_p(N)$ for $p|N_1N_2$, i.e., f is N_1N_2-new. In particular, if $M = 1$ each $f \in \mathcal{F}_0(N)$ is new of level N (though one does not get all newforms if $N_2 > 1$). For a character χ of $\mathrm{Cl}(K)$, put $\mathcal{F}_0(N;\chi) = \mathcal{F}_0(N) \cap \mathcal{F}(N;\chi)$.

Theorem 1.4. *Assume that one of the following holds:*

(1) the triple $(D_B, 1, M)$ is balanced;
(2) we are in the stable range $D_B > |D_K|$ with $\gcd(D_B, D_K) = 1$; or
(3) $u_K > 1$ and $N \geq 11$ with $N \neq 27$.

Then

$$\frac{C(K;N)}{4\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{f \in \mathcal{F}_0(N;\chi)} \Lambda_N(f,\chi) \frac{N_f}{N} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} =$$

$$h_K^2 \left(u_K \prod_{p|N_1'} \left(1 - \frac{1}{p^2}\right) \cdot \prod_{p|N_2} \frac{p}{p+1} - \delta \frac{12}{N} \prod_{p|N_1N_2} \frac{1}{1-p^{-1}} \prod_{p|N_2M} \frac{1}{1+p^{-1}} \right),$$

where $\delta = 1$ if N_1 is squarefree and N_2 is odd; otherwise $\delta = 0$.

Now one can ask what these results tell us about single averages over f as in (1.1). The above theorems clearly give upper bounds on averages for fixed χ which are exact, though certainly suboptimal. More interesting are exact formulas and lower bounds. First we state a stable average value formula generalizing Theorem 1.1(ii).

Corollary 1.5. *Suppose $N_2 = 1$. Fix a character χ of $\mathrm{Cl}(K)$. Then, in the stable range $D_B > |D_K|$ with $\gcd(D_B, D_K) = 1$, we have*

$$\frac{C(K;N)}{4\pi^2} \sum_{f \in \mathcal{F}_0(N;\chi)} \Lambda_M(f,\chi) \frac{N_f}{N} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)} =$$

$$h_K \left(u_K \prod_{p|N_1'} \left(1 - \frac{1}{p^2}\right) - \delta \frac{12 h_K}{N} \prod_{p|N_1} \frac{1}{1-p^{-1}} \prod_{p|M} \frac{1}{1+p^{-1}} \right),$$

where $\delta = 1$ if both N_1 is squarefree and $\chi = 1$, and $\delta = 0$ otherwise.

The novel aspects of this formula are that N is not required to be squarefree, N may be divisible by an even number of primes, and K may be split at primes dividing N. The restriction to $N_2 = 1$ is not due to a real obstruction in the method, but simply to make the second term on the right (which comes from Eisenstein series) easy to describe.

Our final results are effective lower bounds for average L-values $L_{\mathrm{fin}}(\frac{1}{2}, f, 1_K)$, where f runs over level N newforms in $\mathcal{F}(N; 1_K)$. For simplicity, we just treat 2 cases:

(1) $N = p^r N_0$, where N_0 is a squarefree product of an even number of primes, r is odd or 2, and K is inert or ramified at each prime dividing N (necessarily ramified at p if $r = 2$, and inert at p if $r > 2$; here $M = 1$)

(2) $N = N_1 p$ is a squarefree product of an even number of primes, and K is inert or ramified at each prime dividing N_1 and K is split at p (here $M = p$)

In each of these situations, the lower bound implies a generalization of Corollary 1.2: when p or p^2 is larger than an explicit multiple of h_K, there exists a newform $f \in S_2^{\mathrm{new}}(N)$ such that $L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \eta_K) \neq 0$. See Section 5.7 for precise statements.

As a check on our formulas, we note that our single stable average formulas match with those in [MR12] and [FW09] when $N = N_1$ is squarefree and K is inert at each $p|N$. We also present a few examples in Section 6 that provide a numerical check on our formulas in other situations.

1.3. **Methods.** The proof of the exact double average formula has two main steps. First, one computes averages of squares of periods for trivial weight quaternionic modular (i.e., automorphic) forms associated to an order $\mathcal{O}$ in a definite quaternion algebra B. These quaternionic modular forms are $\mathbb{C}$-valued functions the a finite set $\mathrm{Cl}(\mathcal{O})$ which have a Hecke action, and we can think of a basis of quaternionic Hecke eigenforms as being analogous to the complex irreducible characters of a finite group G. The analogue of column orthogonality of characters of G (or a generalization when one weights by Hecke eigenvalues) leads easily to an exact double average formula for periods. This much is carried out for general orders $\mathcal{O}$ in definite quaternion algebras over arbitrary totally real fields in Section 4.

Second, one relates periods of quaternionic modular forms to central L-values with the relative trace formula. For this, one first needs to precisely relate the quaternionic modular forms for $\mathcal{O}$ to classical modular forms, which is a refinement of the Jacquet–Langlands correspondence. We established this correspondence for special orders $\mathcal{O}$ of level $N = N_1 N_2 M$ in [Mar], under the stated conditions (i)–(iii) for (N_1, N_2, M) to be a special level type. Here M need not be squarefree. The condition that N_2 must be the square of a squarefree number arises due to complications in the precise description of the space of quaternionic oldforms when higher even powers of primes divide N_2.

To finish the second step, we need to precisely relate the quaternionic periods to L-values of modular forms. This is essentially carried out in [MW09], but the technical complication is that in general one also needs to relate periods of quaternionic *oldforms* to L-values. This requires an understanding of the local quaternionic oldforms at primes $p|N_1N_2$ from [Mar] and a calculation of local spectral distributions for oldforms when $p|M$. The latter is possibly quite complicated in general, and we only carry this out when M is squarefree and K is split at each $p|M$. However, in principle, one could extend this to allow M to be non-squarefree and K to be split or ramified at $p|M$.

The remaining conditions on the admissibility of K and the restriction of $f \in \mathcal{F}(N; \chi)$ are what we need to see that the relevant periods appear when working with special orders of level N in B. In general, provided that an L-function $L(s, f, \chi)$ has root number $+1$, the central value corresponds to a period on a unique quaternion algebra, which may or may not be B.

This leads to Theorem 1.3. Then one derives Theorem 1.4 by isolating the newform contribution using the inclusion-exclustion principle.

The restriction to base fields of class number 1 in Theorem 5.5 for Hilbert modular forms is simply due to the fact that the relative trace formula of Jacquet used in the L-value formula of [MW09] was only established in for representations whose base change to K has trivial central character.

Remark 1.6. Our calculation of local spectral distributions for $p|M$ is based off of a similar calculation from [FMP17] at a prime p sharply dividing M such that K is inert at p and χ_p is *ramified*. If K is inert at $p|M$ and χ_p is unramified (as in the present paper), one does not see the relevant L-values with periods on B, but on a quaternion algebra which is ramified at p. It should be possible to use our calculations in Section 5.3 to extend the average value formulas in [FW09] and [FMP17] to levels which are products of even numbers of primes, and allow for K to be split at primes dividing the level, with the caveat that now one needs to weight the L-values of oldforms by the factors $\Lambda_N(f, \chi)$.

Now we briefly explain how to derive the lower bounds and stable formulas for single averages. Fixing χ, the average of L-values $L(\frac{1}{2}, f, \chi)$ for $f \in \mathcal{F}(N; \chi)$ plus an Eisenstein contribution is essentially the height $\langle c_{K,\chi}, c_{K,\chi}\rangle$ of a special divisor $c_{K,\chi}$. These heights are maximized when $\chi = 1_K$ is the trivial character, hence the single average of $L(\frac{1}{2}, f, 1_K)$ must be at least $\frac{1}{h_K}$ times the double average over f and χ, minus the Eisenstein contribution. This leads to Corollary 1.2 and the generalizations in Section 5.7.

We *define* (N, K) to be in stable range if $\langle c_{K,\chi}, c_{K,\chi}\rangle$ is dependent only on K (and our triple (N_1, N_2, M)) but not χ. Then in the stable range, the average over $f \in \mathcal{F}(N; \chi)$ for any fixed χ is given by $\frac{1}{h_K}$ times the double average over f and χ, minus the Eisenstein contribution. The converse is also true, and so by comparing our double average formulas with the stable averages in [MR12] and [FW09], we can conclude that their stable range is contained in our definition of stable range. In other words, we use [MR12] and [FW09] to verify that $D_B > |D_K|$ implies (D_B, K) is in the stable range by our definition. Then we use this to deduce (N, K) is also in the stable range. This leads to Theorem 1.1(ii) and Corollary 1.5.

1.4. **Related work.** Here we briefly discuss some additional related work.

First, we know of 2 works on stable averages which allow non-squarefree level. In [Nel13], Nelson established a quite general formula for Rankin–Selberg averages $L(\frac{1}{2}, f \times g)$ over $\mathbb{Q}$. Here g is a fixed form and one averages over an orthogonal basis of cusp forms f of some weight $k \geq 4$. The full formula is quite complicated, but it stabilizes in various situations, including large prime-power levels. However, Nelson does not get averages over newforms (or in weight 2).

Recently, Pi [Pi18] proved a certain stable average formula similar to that of [FW09], which allows for squares and cubes dividing the level. More precisely, Pi averages over representations which are *fixed* depth 0 or simple supercuspidals π_v at a given set of places. This restriction on the type of local representations forces the bound for the stable range to be quite large. Moreover, the character χ is prescribed according to the local representation type of π_v, so one cannot always take $\chi = 1_K$. Consequently, even for level p^3 it does not give a stable average result as in Theorem 1.1(ii).

We remark that there are also a number of asymptotic results on averages for a fixed χ. See for instance [ST16] for weights ≥ 6 when $\chi = 1_K$.

The main consequences of our double average value formula that we considered here are about single averages over f for fixed K and χ. In particular, we thought of K as being fixed and N varying. However one nice feature of our double average formula, especially the stable double average in Theorem 1.4, is that we may simultaneously vary N and K.

For a fixed form f with $|D_K| \to \infty$, an asymptotic for averages over $\chi \in \widehat{\mathrm{Cl}}(K)$ was established in [MV07], which led to a quantitative lower bound on the number of non-vanishing twists $L(\frac{1}{2}, f, \chi)$. The authors assumed f is weight 2 of prime level over $\mathbb{Q}$ for simplicity, though their method (using equidistribution of special cycles) clearly generalizes. This asymptotic average was extended in [LMY15] where now one can vary both f and K. Our double average value formula at least provides exact upper bounds on such averages in more general situations, though it is too crude to obtain averages for a fixed f (unless $\#\mathcal{F}_0(N) = 1$).

Finally, we remark that many earlier average value results have yielded subconvexity results. One might wonder if it is possible to obtain some form of subconvexity from our double average value formula. The problem is that our double average involves too many L-values: the upper bound one naively from gets Theorem 1.3 is that $L(\frac{1}{2}, f, \chi) \ll (f, f)|D_K|^{\frac{1}{2}+\varepsilon}$, which is no better than convexity.

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2. Notation

Let F be a number field with integer ring $\mathfrak{o}_F$ and discriminant D_F. We denote a place of F by v. If A is an F- or $\mathfrak{o}_F$-algebra, denote by A_v the localization at v, and by $\hat{A} = \prod'_{v<\infty} A_v$ the restricted direct product with respect to $\{R_v\}_v$ where $R \subset A$ is a maximal $\mathfrak{o}_F$-order. In particular, $\hat{\mathfrak{o}}_F = \prod_{v<\infty} \mathfrak{o}_{F,v}$ and the adele ring is $\mathbb{A}_F = \prod'_v F_v = \hat{F} \times F_\infty$. The ideal class group of F is denoted by $\mathrm{Cl}(F)$, which we identify with both $F^\times \backslash \mathbb{A}_F^\times / \hat{\mathfrak{o}}_F^\times F_\infty^\times$ and $F^\times \backslash \hat{F}^\times / \hat{\mathfrak{o}}_F^\times$. Put $h_F = \#\mathrm{Cl}(F)$ and let $\widehat{\mathrm{Cl}}(F)$ denote the character group of $\mathrm{Cl}(F)$.

For a finite place v of F, we use $\mathfrak{p}_v$ to denote both the associated prime ideal of $\mathfrak{o}_F$ and the maximal ideal of $\mathfrak{o}_{F,v}$—when the distinction matters, the context will make this notation clear. For an ideal $\mathfrak{N}$ in $\mathfrak{o}_F$, let $\mathrm{ord}_v(\mathfrak{N})$ be the $\mathfrak{p}_v$-adic valuation of $\mathfrak{N}$. Let $\varpi_{F,v}$ denote a uniformizer in $\mathfrak{o}_{F,v}$ and q_v be the size of the residue field of $\mathfrak{o}_{F,v}$. When we specialize to $F = \mathbb{Q}$, we typically use the roman form of the corresponding fraktur letters to denote the positive generator of an ideal, e.g., N is the positive generator of a non-zero ideal $\mathfrak{N} \subset \mathbb{Z}$.

In what follows, F is a totally real number field, and we often write $\mathfrak{o} = \mathfrak{o}_F$, $\mathfrak{o}_v = \mathfrak{o}_{F,v}$, etc. In Section 5, we further assume $h_F = 1$. Denote by B a definite quaternion algebra over F, and $\mathcal{O}$ an $\mathfrak{o}$-order in B. Let $\mathrm{Cl}(\mathcal{O})$ be the set of (invertible) right $\mathcal{O}$-ideal classes in B. As with number fields, we identify $\mathrm{Cl}(\mathcal{O}) = B^\times \backslash B^\times(\mathbb{A}_F)/\hat{\mathcal{O}}^\times B^\times(F_\infty) = B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times$. Write $n = h(\mathcal{O}) = \#\mathrm{Cl}(\mathcal{O})$ for the class number of $\mathcal{O}$.

For a simple algebra A/F (we have in mind a quaternion algebra B/F or a quadratic field extension K/F), let $N_{A/F}$ denote the reduced norm map from A to F. When understood or unimportant, we often omit the subscript, and simply write N for the norm map. Further, denote by $\mathrm{Ram}(A)$ the set of *finite* primes v of F at which A ramifies, and $\mathfrak{D}_A = \prod_{v \in \mathrm{Ram}(A)} \mathfrak{p}_v$ the (reduced) relative discriminant of A (over F).

For quadratic extension K/F of number fields, we denote by $\eta_K = \eta_{K/F}$ the associated quadratic idele class character of F, and analogously for quadratic extensions of local fields.

Excluding the right regular representation, all representations are assumed to be irreducible. We often use 1 to denote a trivial character. However, in the case of a trivial local or global Hecke character over a field k, we sometimes write 1_k for clarity.

3. Quaternionic modular forms

In this section, we discuss quaternionic modular forms for arbitrary orders $\mathcal{O} \subset B$, but for some results we need to restrict to special orders.

3.1. **Special orders.** Fix a definite quaternion algebra B/F. A class of quaternionic orders $\mathcal{O} \subset B$ generalizing Eichler orders was introduced was introduced in [HPS89], which the authors termed special. First we recall the local notion of special orders.

Let v be a finite place of F. If B_v is split, a local special order $\mathcal{O}_v \subset B_v \simeq M_2(F_v)$ of level r (or level $\mathfrak{p}_v^r$) is a local Eichler order of level r, i.e., an order conjugate to $\begin{pmatrix} \mathfrak{o}_v & \mathfrak{o}_v \\ \mathfrak{p}_v^r & \mathfrak{o}_v \end{pmatrix}$. Now suppose B_v is division and E_v/F_v is a quadratic extension of local fields. Let $\mathcal{O}_{B,v}$ denote the unique maximal order of B_v and $\mathfrak{P}_v$ its unique maximal (2-sided) ideal. Set

$$\mathcal{O}_r(E_v) = \mathfrak{o}_{E,v} + \mathfrak{P}_v^{r-1}, \quad r \geq 1.$$

A local order $\mathcal{O}_v \subset B_v$ is special if $\mathcal{O}_v \simeq \mathcal{O}_r(E_v)$ for some r, E_v/F_v, in which case we say the level of $\mathcal{O}_v$ is r (or $\mathfrak{p}_v^r$) if r is minimal such that $\mathcal{O}_v \simeq \mathcal{O}_r(E_v)$. We recall some facts about local special orders.

Lemma 3.1 ([HPS89])**.** *Suppose B_v is division. Then:*

(i) Every special order in B_v is of the form $\mathcal{O}_r(E_v)$ where either $r \geq 1$ is odd and E_v/F_v is unramified, or $r \geq 1$ is arbitrary and E_v/F_v is ramified.

(ii) $\mathcal{O}_1(E_v) = \mathcal{O}_{B,v}$ is the maximal order for every E_v/F_v.

(iii) Suppose E_v/F_v and E'_v/F_v are quadratic extensions and $r, r' \geq 1$. If $r \neq r'$, then $\mathcal{O}_r(E_v) \simeq \mathcal{O}_{r'}(E'_v)$ implies $E_v \simeq E'_v$ is unramified and $|r - r'| = 1$ with $\min\{r, r'\}$ odd.

(iv) If E_v/F_v is unramified and E'_v/F_v is ramified, then $\mathcal{O}_r(E_v) = \mathcal{O}_{r'}(E'_v)$ if and only if $r = r' = 1$.

(v) Suppose E_v, E'_v are two non-isomorphic ramified quadratic extensions of F_v. If $r \leq 2$, then $\mathcal{O}_r(E_v) \simeq \mathcal{O}_r(E'_v)$; the converse also holds if v is odd.

See [HPS89, Theorem 3.10] for the converse of (v) when v is dyadic.

Following [HPS89], we say a global order $\mathcal{O} \subset B$ is special of level $\mathfrak{N}$ if $\mathcal{O}_v$ is special of level $\mathrm{ord}_v(\mathfrak{N})$ for each $v < \infty$. However, $\mathfrak{N}$ alone does not the local isomorphism type of $\mathcal{O}$. We will always place the following additional assumption on a global special order $\mathcal{O}$ of level $\mathfrak{N}$: for $v < \infty$ such that B_v is division and $r_v := \mathrm{ord}_v(\mathfrak{N})$ is odd, $\mathcal{O}_v \simeq \mathcal{O}_r(E_v)$ where E_v/F_v is the unramified quadratic extension. Further, when we write $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$, it will always mean that $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{M}$ are pairwise coprime such that (i) for finite $v|\mathfrak{N}_1\mathfrak{N}_2$ if and only if B_v is division, and (ii) for $v|\mathfrak{N}_1\mathfrak{N}_2$, $v|\mathfrak{N}_1$ if and only if $\mathrm{ord}_v(\mathfrak{N}_1)$ is odd. Thus $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$) is divisible exactly by the finite primes v ramifying in B such that $\mathcal{O}_v$ is defined by an unramified (resp. ramified) quadratic extension E_v/F_v, and $\mathfrak{M}$ is divisible exactly by the finite primes v splitting B where $\mathcal{O}_v$ is a non-maximal local Eichler order.

Sometimes to emphasize the above conventions, we will say $\mathcal{O}$ is a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. Note that by Lemma 3.1, if $\mathrm{ord}_v(\mathfrak{N}_2) = 2$ for all $v|\mathfrak{N}_2$, then any two special orders of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ are locally isomorphic, i.e., in the same genus.

If $\mathcal{O}$ is special of level type $(\mathfrak{N}_1, \mathfrak{o}, \mathfrak{M})$, we say $\mathcal{O}$ is of unramified quadratic type. This means that $\mathcal{O}_v$ contains the maximal order of an unramified or split quadratic extension of

F_v for each $v < \infty$. Such orders are nice to work with because one has that $N(\hat{\mathcal{O}}^\times) = \hat{\mathfrak{o}}^\times$. Note that an order $\mathcal{O}$ is Eichler if and only if $\mathcal{O}$ is of unramified quadratic type with $\mathfrak{N}_1$ squarefree, i.e., $\mathfrak{N}_1 = \mathfrak{D}_B$ and $\mathfrak{N}_2 = \mathfrak{o}$.

3.2. **Automorphic forms.** Fix an arbitrary $\mathfrak{o}_F$-order $\mathcal{O} \subset B$. We define the space of quaternionic modular forms (or automorphic forms) of level $\mathcal{O}$ on B with trivial weight to be $M(\mathcal{O}) = \{\varphi : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}\}$. Set $n = h(\mathcal{O}) = \#\mathrm{Cl}(\mathcal{O})$, and write $\mathrm{Cl}(\mathcal{O}) = \{x_1, \dots, x_n\}$. By abuse of notation, we use also use x_i to denote an element of $\hat{B}^\times$ representing the corresponding element of (idelic quotient) $\mathrm{Cl}(\mathcal{O})$. Let $\mathcal{I}_i$ denote a classical right $\mathcal{O}$-ideal representing x_i. Let $\mathcal{O}_\ell(\mathcal{I}_i) = \{\alpha \in B : \alpha\mathcal{I}_i \subset \mathcal{I}_i\} = x_i\hat{\mathcal{O}}x_i^{-1} \cap B$ be the left order of $\mathcal{I}_i$. Put $w_i = [\mathcal{O}_\ell(\mathcal{I}_i)^\times : \mathfrak{o}^\times]$, which is always finite. We define an inner product on $M(\mathcal{O})$ by

$$(\varphi, \varphi') = \sum_{i=1}^{n} \frac{1}{w_i} \varphi(x_i)\overline{\varphi(x_i)}. \tag{3.1}$$

This inner product can also be defined by integrating φ against $\bar{\varphi}$ over $\hat{F}^\times B^\times \backslash \hat{B}^\times$ with an appropriate normalization of measure (e.g., see [Mar17]).

Let $\omega : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be an idele class character of F which is trivial at each infinite place. Let $L^2(B^\times \backslash \hat{B}^\times, \omega)$ be the space L^2-functions $\varphi : B^\times \backslash \hat{B}^\times \to \mathbb{C}$ such that $\varphi(zx) = \omega(z)\varphi(x)$ for all $z \in \hat{F}^\times$, $x \in \hat{B}^\times$. Setting $M(\mathcal{O}, \omega) = M(\mathcal{O}) \cap L^2(B^\times \backslash \hat{B}^\times, \omega)$, we see that

$$M(\mathcal{O}) = \bigoplus M(\mathcal{O}, \omega), \tag{3.2}$$

where ω ranges over idele class characters which are trivial on $\mathfrak{o}_F^\times \times F_\infty^\times = \mathbb{A}_F^\times \cap (\hat{\mathcal{O}}^\times \times B_\infty^\times)$, i.e., ω runs over the group $\widehat{\mathrm{Cl}}(F)$ of ideal class characters of F.

By an automorphic representation π of $B^\times$ with trivial weight and (central) character ω, we mean an irreducible unitary subrepresentation of the right regular representation of $\hat{B}^\times$ on $L^2(B^\times \backslash \hat{B}^\times, \omega)$. Call π cuspidal if π is not 1-dimensional. We say that π occurs in $M(\mathcal{O})$ if $\pi \cap M(\mathcal{O}) \neq 0$, which is equivalent to $\pi^{\hat{\mathcal{O}}^\times} \neq 0$, and speak similarly for $M(\mathcal{O}, \omega)$. Accordingly we get decompositions

$$M(\mathcal{O}) = \bigoplus \pi^{\hat{\mathcal{O}}^\times}, \quad M(\mathcal{O}, \omega) = \bigoplus \pi^{\hat{\mathcal{O}}^\times}, \tag{3.3}$$

where π respectively runs over automorphic representations occurring in $M(\mathcal{O})$ and $M(\mathcal{O}, \omega)$.

We define the Eisenstein subspace of $M(\mathcal{O})$ to be $\mathrm{Eis}(\mathcal{O}) = \bigoplus \pi^{\hat{\mathcal{O}}^\times}$, where π runs over 1-dimensional automorphic representations occurring in $M(\mathcal{O})$. All such π must be of the form $\mu \circ N_{B/F}$ for some character $\mu : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. We may view $N = N_{B/F}$ as a map from $B^\times \backslash B^\times(\mathbb{A}_F)/\hat{\mathcal{O}}^\times B_\infty^\times$ to

$$\mathrm{Cl}^+(N(\hat{\mathcal{O}})) := F^\times \backslash \mathbb{A}_F^\times / N(\hat{\mathcal{O}}^\times) F_\infty^+,$$

where F_∞^+ denotes the totally positive elements of F_∞. Thus we may take μ to be a character of $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$. If $\mathcal{O}$ is special of unramified quadratic type, then $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$ is just the narrow class group $\mathrm{Cl}^+(F)$.

Put $\mathrm{Eis}(\mathcal{O}, \omega) = \mathrm{Eis}(\mathcal{O}) \cap M(\mathcal{O}, \omega)$. Note $\mathrm{Eis}(\mathcal{O}, 1)$ always contains the constant function $\mathbb{1}$. In general, a basis of $\mathrm{Eis}(\mathcal{O}, \omega)$ is given by the characters $\mu \circ N$, where μ runs over the characters of $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$ such that $\mu^2 = \omega$. Note that for any character $\mu \circ N \in \mathrm{Eis}(\mathcal{O})$, its norm $(\mu \circ N, \mu \circ N)$ equals the mass of $\mathcal{O}$,

$$m(\mathcal{O}) := \sum_{i=1}^{n} \frac{1}{w_i}.$$

Lemma 3.2. *Let $\mathcal{O}$ be a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. Then*

$$m(\mathcal{O}) = m(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) := 2^{1-[F:\mathbb{Q}]} h_F |\zeta_F(-1)| N(\mathfrak{N}) \prod_{v|\mathfrak{N}_1\mathfrak{N}_2} (1 - q_v^{-1}) \prod_{v|\mathfrak{N}_2\mathfrak{M}} (1 + q_v^{-1}).$$

Proof. Let $\mathcal{O}'$ be a special order of level type $(\mathfrak{N}_1', \mathfrak{o}, \mathfrak{M})$ containing $\mathcal{O}$ where $\mathfrak{N}_1' = \mathfrak{N}_1 \prod_{v|\mathfrak{N}_2} \mathfrak{p}_v$. A formula for $m(\mathcal{O}')$ is given in [Mar17, (1.6)]. Since one can interpret masses of orders as volumes in $\hat{B}^\times$ with respect to suitable Haar measures, one has that $m(\mathcal{O}) = m(\mathcal{O}') \prod_{v|\mathfrak{N}_2} [\mathcal{O}_v^\times : (\mathcal{O}_v')^\times]$. From [HPS89, Proposition 2.6], one knows $[\mathcal{O}_v^\times : (\mathcal{O}_v')^\times] = (q_v + 1) q_v^{e_v - 2}$ where $e_v = \mathrm{ord}_v(\mathfrak{N}_2)$. These two calculations combine to give the lemma. (The $F = \mathbb{Q}$ case is already stated in [HPS89, Theorem 6.8].) □

Lemma 3.3. *Let $\mathcal{O}$ be a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ such that* $\mathrm{ord}_v(\mathfrak{N}_2) = 2$ *for each $v|\mathfrak{N}_2$. Then there is a basis of* $\mathrm{Eis}(\mathcal{O}, 1)$ *consisting of $\mathbb{1}$ and the characters $\eta_{E/F} \circ N$, where E/F runs over quadratic extensions such that each finite v ramified in E/F is odd and divides $\mathfrak{N}_2$.*

Proof. Consider a character $\mu \circ N$ occurring in $M(\mathcal{O}, 1)$ and let $F_v^{(2)}$ denote the subgroup of squares of $F_v^\times$. Then $\mu^2 = 1$ and μ_v factors through $F_v^\times / N(\mathcal{O}_v^\times) F_v^{(2)}$ for $v < \infty$. (Note for $v|\infty$, the only requirement on μ_v is that it factors through $\mathbb{R}^\times/\mathbb{R}_{>0} \simeq \{\pm 1\}$). For a finite $v \nmid \mathfrak{N}_2$, we have $N_{B/F}(\mathcal{O}_v^\times) = \mathfrak{o}_v^\times$, so $F_v^\times / N_{B/F}(\mathcal{O}_v^\times) F_v^{(2)} \simeq \langle \varpi_v \rangle / \langle \varpi_v^2 \rangle$ and we see μ_v can be trivial or the unramified quadratic character of $F_v^\times$.

Now suppose $v|\mathfrak{N}_2$. To elucidate the assumption on $\mathfrak{N}_2$, first just suppose $\mathrm{ord}_v(\mathfrak{N}_2) = 2r$ with $r \geq 1$, and write $\mathcal{O}_v = \mathcal{O}_{2r}(L_v)$ where L_v/F_v is a ramified quadratic extension. Then $\mathcal{O}_v^\times = \mathfrak{o}_{L,v}^\times (1 + \mathfrak{P}_v^{2r-1})$. One has $N_{B/F}(1 + \mathfrak{P}_v^{2r-1}) = 1 + \mathfrak{p}_v^r$, and thus $F_v^\times / N_{B/F}(\mathcal{O}_v^\times) F_v^{(2)} \simeq (\mathfrak{o}_v^\times / N(\mathfrak{o}_{L,v}^\times)(1 + \mathfrak{p}_v^r)) \times \langle \varpi_v \rangle / \langle \varpi_v^2 \rangle$. Thus μ_v can be unramified or any ramified quadratic character η_{E_v/F_v} such that $N(\mathfrak{o}_{E,v}^\times) \supset N(\mathfrak{o}_{L,v}^\times)(1 + \mathfrak{p}_v^r)$. Now assume $r = 1$. It is well known that $N(\mathfrak{o}_{E,v}^\times) \supset 1 + \mathfrak{p}_v$ if and only if v is odd, and thus if v is even μ_v must be unramified. If v is odd, then $\mathcal{O}_2(L_v) \simeq \mathcal{O}_2(E_v)$, so we may take $L_v \simeq E_v$ to see that μ_v can be any quadratic character. □

Remark 3.4. Let $\mathcal{O}$ be as in Lemma 3.3. Since the 1-dimensional automorphic representations occurring in $M(\mathcal{O})$ form a group, the above lemma yields a complete description of $\mathrm{Eis}(\mathcal{O})$ in terms of $N(\hat{\mathcal{O}})$ and $\mathrm{Cl}(F)$. Namely, if $\omega = \mu^2$ for some $\mu \in \widehat{\mathrm{Cl}}^+(N(\hat{\mathcal{O}}))$, then $\mathrm{Eis}(\mathcal{O}, \omega) = \{(\mu \circ N)\varphi : \varphi \in \mathrm{Eis}(\mathcal{O}, \mathbb{1})\}$. Otherwise $\mathrm{Eis}(\mathcal{O}, \omega) = 0$.

We define the cuspidal subspaces $S(\mathcal{O})$ and $S(\mathcal{O}, \omega)$ of $M(\mathcal{O})$ and $M(\mathcal{O}, \omega)$ to be the orthogonal complements of the Eisenstein subspaces. Hence we have decompositions

$$S(\mathcal{O}) = \bigoplus \pi^{\hat{\mathcal{O}}^\times}, \quad S(\mathcal{O}, \omega) = \bigoplus \pi^{\hat{\mathcal{O}}^\times},$$

where π respectively runs over cuspidal representations occurring in $M(\mathcal{O})$ and $M(\mathcal{O}, \omega)$.

3.3. **Brandt matrices and representation numbers.** We may realize $M(\mathcal{O})$ as $\mathbb{C}^n$ via $\varphi \mapsto (\varphi(x_1), \dots, \varphi(x_n))$. Let $[\varphi]$ denote the column vector ${}^t(\varphi(x_1) \dots \varphi(x_n))$. Then we can define a Hecke action in terms of Brandt matrices as in [DV13] and [Mar].

For a nonzero integral ideal $\mathfrak{m}$, we define the $n \times n$ Brandt matrix $A_{\mathfrak{m}} = (a_{ij}(\mathfrak{m}))$ via

$$a_{ij}(\mathfrak{m}) = \# \left(\{\gamma \in \mathcal{I}_i\mathcal{I}_j^{-1} : N(\gamma)\mathfrak{o} = \mathfrak{m}N(\mathcal{I}_i\mathcal{I}_j^{-1})\}/\mathcal{O}_\ell(\mathcal{I}_j)^\times \right). \tag{3.4}$$

Define the Hecke operator $T_{\mathfrak{m}} : M(\mathcal{O}) \to M(\mathcal{O})$ via matrix multiplication: $[T_{\mathfrak{m}}\varphi] = A_{\mathfrak{m}}[\varphi]$. We can also express

$$(T_{\mathfrak{m}}\varphi)(x) = \sum \varphi(x\beta),$$

where β runs over the integral right $\mathcal{O}$-ideals of norm $\mathfrak{m}$. Note $T_{\mathfrak{o}}$ is the identity operator. The collection of $T_{\mathfrak{m}}$'s is a commuting family of real self-adjoint operators on $M(\mathcal{O})$.

Under a class number 1 assumption, the Brandt matrix entries can be expressed as classical representation numbers of quadratic forms as follows. For a (full) $\mathfrak{o}_F$-lattice $\Lambda \subset B$, let $\mathcal{O}_r(\Lambda) = \{\alpha \in B : \Lambda\alpha \subset \Lambda\}$ and $\mathcal{O}_r(\Lambda)^1$ be the subset of norm 1 elements in $\mathcal{O}_r(\Lambda)$. Let $\mathfrak{o}_+$ be the subset of totally positive elements of $\mathfrak{o}$, and $\mathfrak{o}_+^\times$ be the totally positive units. For $y \in \mathfrak{o}_+$, define the representation number

$$r_\Lambda(y) = \#\{\lambda \in \Lambda : N(\lambda) = y\}. \tag{3.5}$$

Now suppose $h_F^+ = 1$. Since $h_F = h_F^+$, every element of $\mathfrak{o}_+^\times$ is a square, and thus $N_{B/F}(\mathfrak{o}^\times) = \mathfrak{o}_+^\times$. Fix $y \in \mathfrak{o}_+$. Then $N(\lambda)\mathfrak{o} = y\mathfrak{o}$ is equivalent to $N(\lambda) \in y\mathfrak{o}_+^\times$. Since $\mathcal{O}_r(\Lambda)^\times = \mathfrak{o}^\times\mathcal{O}_r(\Lambda)^1$, we see there is a bijection of sets,

$$\{\lambda \in \Lambda : N(\lambda) {\in} y\mathfrak{o}_+^\times\}/\mathcal{O}_r(\Lambda)^\times \simeq \{\lambda \in \Lambda : N(\lambda) {=} y\}/\mathcal{O}_r(\Lambda)^1.$$

Put $\Lambda_{ij} = \mathcal{I}_i\mathcal{I}_j^{-1}$ and let α_{ij} be a totally positive generator of $N(\mathcal{I}_i\mathcal{I}_j^{-1})$. Note $\mathcal{O}_r(\Lambda_{ij}) = \mathcal{O}_\ell(\mathcal{I}_j)$. Also $\mathfrak{o}^\times \cap \mathcal{O}_r(\Lambda_{ij})^1 = \{\pm 1\}$, which implies $\#\mathcal{O}_r(\Lambda_{ij})^1 = 2w_j$. Hence, assuming $h_F^+ = 1$ and $y \in \mathfrak{o}_+$, we can rewrite our Brandt matrix entries as

$$a_{ij}(y\mathfrak{o}) = \frac{1}{2w_j} r_{\Lambda_{ij}}(y\alpha_{ij}). \tag{3.6}$$

Note that when $i = j$, we get $\Lambda_{ii} = \mathcal{O}_\ell(\mathcal{I}_i)$, so the diagonal Brandt matrix entries are simply

$$a_{ii}(y\mathfrak{o}) = \frac{1}{2w_i} r_{\mathcal{O}_\ell(\mathcal{I}_i)}(y). \tag{3.7}$$

3.4. **Jacquet–Langlands correspondence.** The Jacquet–Langlands correspondence is a dictionary between automorphic representations of $B^\times$ and $\mathrm{GL}(2)$. Here we present a refinement of it at the level of modular forms from [Mar].

Let $\mathcal{O} \subset B$ be a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ and write $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$. Let $S_{\mathbf{2}}(\mathfrak{N}, \omega)$ denote the space of adelic holomorphic Hilbert cusp forms of level $\mathfrak{N}$, parallel weight 2, and central character ω. (By level $\mathfrak{N}$, we mean "level $W(\mathfrak{N})$," which is Shimura's analogue of $\Gamma_0(N)$—see, e.g., [Mar, Section 5.1].) For relatively prime integral ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of $\mathfrak{o}$ such that $\mathfrak{a}|\mathfrak{N}_1$ and $\mathfrak{b}\mathfrak{c}|\mathfrak{N}_2$, let $S_{\mathbf{2}}^{[\mathfrak{a},\mathfrak{b},\mathfrak{c}]}(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{M}, \omega)$ be the subspace of $S_{\mathbf{2}}(\mathfrak{N}, \omega)$ generated by eigenforms which are $\mathfrak{p}$-new for each $\mathfrak{p}|\mathfrak{a}\mathfrak{b}\mathfrak{c}$ and whose corresponding local representations of $\mathrm{GL}_2(F_v)$ are: (i) local discrete series for $v|\mathfrak{a}\mathfrak{b}\mathfrak{c}$, (ii) supercuspidal for $v|\mathfrak{b}$, and (iii) special (i.e., twisted Steinberg) for $v|\mathfrak{c}$.

For an integral ideal $\mathfrak{A} \subset \mathfrak{o}$, let $\mathcal{H}^{\mathfrak{A}}$ be the Hecke algebra generated by the $T_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathfrak{A}$. (For our purposes, it does not matter if we work with Hecke algebras over $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{C}$.)

Theorem 3.5 ([Mar]). *Assume* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}_2) = 2$ *for each* $\mathfrak{p}|\mathfrak{N}_2$. *Let* $\mathfrak{N}' = \mathfrak{N}\mathfrak{D}_B^{-1}$. *We have the following isomorphism of* $\mathcal{H}^{\mathfrak{N}'}$*-modules:*

$$S(\mathcal{O}, \omega) \simeq \bigoplus 2^{\#\{\mathfrak{p}|\mathfrak{b}\}} S_{\mathbf{2}}^{[\mathfrak{a},\mathfrak{b},\mathfrak{c}]}(\mathfrak{abc}\mathfrak{M}, \omega),$$

where (i) $\mathfrak{a}$ *runs over divisors of* $\mathfrak{N}_1$ *such that* $v_{\mathfrak{p}}(\mathfrak{a})$ *is odd for all* $\mathfrak{p}|\mathfrak{N}_1$, *and (ii)* $\mathfrak{b}, \mathfrak{c}$ *run over relatively prime divisors of* $\mathfrak{N}_2$ *such that* $\mathfrak{p}|\mathfrak{bc}$ *for each* $\mathfrak{p}|\mathfrak{N}_2$.

Proof. The result [Mar, Corollary 5.5] shows the stated isomorphism holds for $\mathcal{H}^{\mathfrak{N}}$-modules. Note $\mathcal{H}^{\mathfrak{N}'}$ is the algebra generated by $\mathcal{H}^{\mathfrak{N}}$ and the $T_{\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{N}_1$ such that $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}_1) = 1$. Since any eigenform occurring in either side of the isomorphism is $\mathfrak{p}$-new for $\mathfrak{p}$ sharply dividing $\mathfrak{N}_1$, the above isomorphism also preserves the action of each such $T_{\mathfrak{p}}$ (e.g., see [Mar18] or [Mar, Remark 5.2]). □

We remark that [Mar] establishes the corresponding Hecke isomorphism for *newspaces* without assuming $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}_2) = 2$ when $\mathfrak{p}|\mathfrak{N}_2$. The difficulty in removing this assumption in the above theorem boils down to an issue in the local determination of oldspaces.

3.5. **Average values of quaternionic modular forms.** The following simple linear-algebraic relation forms the basis of our calculation of period averages in the next section.

Proposition 3.6. *Let* $\mathcal{O}$ *be an arbitrary order in* B, *and* $T = (a_{ij})$ *a real self-adjoint operator on* $M(\mathcal{O})$. *Let* Φ *be an orthogonal basis for* $M(\mathcal{O})$. *Then, for* $1 \leq i, j \leq n$, *we have*

$$\sum_{\varphi \in \Phi} \frac{(T\varphi)(x_i)\overline{\varphi(x_j)}}{(\varphi, \varphi)} = w_j a_{ij}.$$

Proof. Let $e_i \in M(\mathcal{O})$ be the indicator function of x_i. Then the left hand side of the equation above equals

$$\sum_{\varphi \in \Phi} \frac{(T\varphi, w_i e_i)\overline{(\varphi, w_j e_j)}}{(\varphi, \varphi)} = \sum_{\varphi \in \Phi} \frac{(\varphi, Tw_i e_i)\overline{(\varphi, w_j e_j)}}{(\varphi, \varphi)} = \overline{(Tw_i e_i, w_j e_j)} = w_i a_{ji} = w_j a_{ij}.$$

□

Remark 3.7. In the special case $T = T_{\mathfrak{o}}$ is the identity, we get

$$\sum_{\varphi \in \Phi} \frac{\varphi(x_i)\overline{\varphi(x_j)}}{(\varphi, \varphi)} = \delta_{ij} w_i. \tag{3.8}$$

One can think of quaternionic eigenforms, which are functions on finite class sets associated to irreducible representations, as a kind of analogue of characters of finite groups. With this point of view, we can think of (3.8) as an analogue of column orthogonality for characters of finite groups. Here is a proof of (3.8) which is perhaps more parallel to a typical proof of column orthogonality:

Write $\Phi = \{\varphi_1, \ldots, \varphi_n\}$. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = \frac{\varphi_i(x_j)}{\sqrt{w_j(\varphi_i, \varphi_i)}}$. Then $A^t \bar{A} = I$, and (3.8) follows from orthogonality of the columns of A.

4. Periods and embeddings

In this section, we study periods of quaternionic modular forms and their averages. As in the previous section, we will often work with arbitrary orders $\mathcal{O} \subset B$, but in certain cases restrict to special orders.

4.1. **Periods and averages.** Let $\mathcal{O}$ be a $\mathfrak{o}$-order of B, and K/F a quadratic subfield of B such that $\mathfrak{o}_K \subset \mathcal{O}$. Define the ideal class map $t \mapsto x(t)$ from $\mathrm{Cl}(K)$ to $\mathrm{Cl}(\mathcal{O})$ by $K^\times t\hat{\mathfrak{o}}_K^\times \mapsto B^\times t\hat{\mathcal{O}}^\times$. This map depends on the specific embedding of K into B. Let h_K be the class number of K.

Let $\mathrm{Pic}_{\mathbb{C}}(\mathrm{Cl}(\mathcal{O}))$ be the set of formal $\mathbb{C}$-linear combinations of elements of $\mathrm{Cl}(\mathcal{O})$. For $c = \sum c_i x_i, d = \sum d_i x_i \in \mathrm{Pic}_{\mathbb{C}}(\mathcal{O})$, define the height pairing

$$\langle c, d \rangle = \sum_{i=1}^n w_i c_i \overline{d_i},$$

and let $\check{c} \in M(\mathcal{O})$ be the dual element given by $\check{c}(x_i) = w_i c_i$ for $1 \le i \le n$. Then $c \mapsto \check{c}$ defines an isometry of $\mathrm{Pic}_{\mathbb{C}}(\mathrm{Cl}(\mathcal{O}))$ with $M(\mathcal{O})$.

For $\chi \in \widehat{\mathrm{Cl}}(K)$, consider the complex divisor $c_{K,\chi} = \sum \chi(t) x(t) \in \mathrm{Pic}_{\mathbb{C}}(\mathrm{Cl}(\mathcal{O}))$, which we also denote by c_K when $\chi = 1$. For $\varphi \in M(\mathcal{O})$, we define the period along K against χ to be

$$P_{K,\chi}(\varphi) = (\varphi, \check{c}_{K,\chi}) = \sum_{t \in \mathrm{Cl}(K)} \varphi(x(t))\chi^{-1}(t). \tag{4.1}$$

We also denote this by $P_K(\varphi)$ when $\chi = 1$. Define the ideal class embedding numbers $h_i = h_{K,i} = \#\{t \in \mathrm{Cl}(K) : x(t) = x_i\}$ for $1 \le i \le n$. Thus $\sum h_i = h_K$, and $P_K(\varphi) = \sum h_i \varphi(x_i)$.

We consider averages of absolute squares of periods in two directions. First, note that orthogonality of characters of $\mathrm{Cl}(K)$ implies that

$$\sum_{\chi \in \widehat{\mathrm{Cl}}(K)} |P_{K,\chi}(\varphi)|^2 = h_K \sum_{t \in \mathrm{Cl}(K)} |\varphi(x(t))|^2 = h_K \sum_{i=1}^n h_i |\varphi(x_i)|^2. \tag{4.2}$$

This expression was used in [MV07] to study such averages asymptotically (for simplicity, restricted to prime level over $\mathbb{Q}$).

Second, for $T = (a_{ij})$ a real self-adjoint operator on $M(\mathcal{O})$ and Φ an orthogonal basis for $M(\mathcal{O})$, we have

$$\sum_{\varphi \in \Phi} \frac{P_{K,\chi}(T\varphi)\overline{P_{K,\chi}(\varphi)}}{(\varphi,\varphi)} = \sum_{\varphi \in \Phi} \frac{(\varphi, T\check{c}_{K,\chi})(\check{c}_{K,\chi}, \varphi)}{(\varphi,\varphi)} = (T\check{c}_{K,\chi}, \check{c}_{K,\chi}). \tag{4.3}$$

This expression (in terms of the height pairing) was used for the exact average formula in [MR12].

Let $\mathfrak{N}' = \mathfrak{N}\mathfrak{D}_B^{-1}$, where $\mathfrak{N}$ is the (reduced) discriminant of $\mathcal{O}$. Thus $\mathfrak{p} | \mathfrak{N}'$ if and only if $\mathcal{O}_\mathfrak{p}$ is not a maximal order. Then for a nonzero integral ideal $\mathfrak{m} \subset \mathfrak{o}$ which is coprime to $\mathfrak{N}'$, and for any π occurring in $M(\mathcal{O})$, $T_\mathfrak{m}$ acts on $\pi^{\hat{\mathcal{O}}^\times}$ by a scalar $\lambda_\mathfrak{m}(\pi)$ since the local space $\pi_v^{\mathcal{O}_v^\times}$ of invariants is 1-dimensional for $v | \mathfrak{m}$. For $\varphi \in \pi^{\hat{\mathcal{O}}^\times}$, set $\lambda_\mathfrak{m}(\varphi) = \lambda_\mathfrak{m}(\pi)$.

Proposition 4.1. *Let Φ be an orthogonal basis of eigenforms for $M(\mathcal{O})$. For a nonzero integral ideal $\mathfrak{m} \subset \mathfrak{o}$ coprime to $\mathfrak{N}'$, consider the divisor $a(\mathfrak{m}) = \sum_{i=1}^n a_{ii}(\mathfrak{m}) x_i$ associated to the diagonal of the Brandt matrix $A_\mathfrak{m}$. Then we have*

$$\sum_{\varphi \in \Phi} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \frac{\lambda_\mathfrak{m}(\varphi)|P_{K,\chi}(\varphi)|^2}{(\varphi,\varphi)} = h_K \langle c_K, a(\mathfrak{m}) \rangle = h_K \sum_{i=1}^n w_i h_i a_{ii}(\mathfrak{m}). \tag{4.4}$$

Proof. This follows directly from (4.2) and Proposition 3.6. □

For $\mathfrak{m}$ as above, define the degree of $\deg T_{\mathfrak{m}}$ on $M(\mathcal{O})$ to be the number of integral right $\mathcal{O}$-ideals of norm $\deg T_{\mathfrak{m}}$. Note each row of the Brandt matrix $A_{\mathfrak{m}}$ sums to $\mathfrak{m}$. For a prime $\mathfrak{p} \nmid \mathfrak{N}$, note $\deg T_{\mathfrak{p}} = q_{\mathfrak{p}} + 1$, where $q_{\mathfrak{p}} = \#(\mathfrak{o}/\mathfrak{p})$.

Corollary 4.2. *Let Φ_0 be an orthogonal basis of eigenforms for $S(\mathcal{O})$. If $\mathfrak{m}$ is coprime to $\mathfrak{N}'$, then*

$$\sum_{\varphi \in \Phi_0} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \frac{\lambda_{\mathfrak{m}}(\varphi)|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K \langle c_K, a(\mathfrak{m}) \rangle - \delta^+(\mathcal{O}, \mathfrak{m}) \deg T_{\mathfrak{m}} \frac{h_K^2 |\mathrm{Cl}^+(N(\hat{\mathcal{O}}))|}{m(\mathcal{O})},$$

where $\delta^+(\mathcal{O}, \mathfrak{m})$ is 1 if the class of $\mathfrak{m}$ is trivial in $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$ and 0 otherwise.

Proof. Recall that a basis of $\mathrm{Eis}(\mathcal{O})$ is given by $\{\mu \circ N\}$, where μ runs over the characters of $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$. For $\varphi = \mu \circ N$, we see that $P_{K,\chi}(\varphi) = \sum_{t \in \mathrm{Cl}(K)} \mu(N_{K/F}(t))\chi^{-1}(t)$ is h_K if $(\mu \circ N)|_{\mathrm{Cl}(K)} = \chi$ and 0 otherwise. Since each such φ has norm $(\varphi, \varphi) = m(\mathcal{O})$ and $\lambda_{\mathfrak{m}}(\mu \circ N) = \mu(\mathfrak{m})\lambda_{\mathfrak{m}}(\mathbb{1}) = \mu(\mathfrak{m}) \deg T_{\mathfrak{m}}$, we see that the Eisenstein contribution to (4.4) is

$$\sum_{\mu} \lambda_{\mathfrak{m}}(\mu \circ N) \frac{h_K^2}{m(\mathcal{O})} = \frac{h_K^2}{m(\mathcal{O})} \sum_{\mu} \deg T_{\mathfrak{m}} \, \mu(\mathfrak{m}),$$

where μ runs over the characters of $\mathrm{Cl}^+(N(\hat{\mathcal{O}}))$. □

This double average formula simplifies in various situations. We explicate two now. We call $\mathcal{O}$ *balanced* if $w_i = 1$ for all $1 \leq i \leq n$, i.e., if $n = m(\mathcal{O})$. (Balanced implies that each ideal class in $\mathrm{Cl}(\mathcal{O})$ has the same volume with respect to a Haar measure, though it is not exactly equivalent.) In Section 4.4, we give elementary criteria for orders to be balanced.

Corollary 4.3. *Let Φ_0 be an orthogonal basis of eigenforms for $S(\mathcal{O})$. Then:*

(i) If $\mathcal{O}$ is balanced, then

$$\sum_{\varphi \in \Phi_0} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K^2 \left(1 - \frac{|\mathrm{Cl}^+(N(\hat{\mathcal{O}}))|}{m(\mathcal{O})} \right).$$

(ii) If $h_F^+ = 1$, $\mathcal{O}$ is special of unramified quadratic type, and $\mathfrak{m}$ is coprime to $\mathfrak{N}'$, then

$$\sum_{\varphi \in \Phi_0} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \frac{\lambda_{\mathfrak{m}}(\varphi)|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K \sum_{i=1}^{n} w_i h_i a_{ii}(\mathfrak{m}) - \deg T_{\mathfrak{m}} \frac{h_K^2}{m(\mathcal{O})}.$$

4.2. **Exact averages and stability.** Here we will use the double average formula in the previous section to obtain some precise results about the single averages (4.3) for fixed χ. We continue the notation of the previous section. In particular, $\mathfrak{o}_K \subset \mathcal{O}$, where $\mathcal{O}$ is an arbitrary order in B.

Lemma 4.4. *We have $\langle c_{K,\chi}, c_{K,\chi} \rangle \leq \langle c_K, c_K \rangle$ for all $\chi \in \widehat{\mathrm{Cl}}(K)$. Further, $\langle c_{K,\chi}, c_{K,\chi} \rangle = \langle c_K, c_K \rangle$ for all χ if and only if $h_i \leq 1$ for all $1 \leq i \leq n$.*

Proof. Note that

$$\langle c_{K,\chi}, c_{K,\chi} \rangle = \sum_{i=1}^{n} w_i \Big| \sum_{x(t) = x_i} \chi(t) \Big|^2,$$

where in the inner sum t runs over the elements of $\mathrm{Cl}(K)$ which map to x_i via the ideal class map. Clearly this is maximized for χ being the trivial character, and we have equality among all χ when $h_i = \#\{t \in \mathrm{Cl}(K) : x(t) = x_i\} \le 1$ for all i. Conversely, if $h_i > 1$ for some i, then there exist distinct $t, t' \in \mathrm{Cl}(K)$ such that $x(t) = x(t') = x_i$. Now $\chi(t) \ne \chi(t')$ for some $\chi \in \widehat{\mathrm{Cl}}(K)$, and we obtain a strict inequality for such χ. ☐

We say that the pair $(\mathcal{O}, K)$ lies in the *semistable range* if $h_i \le 1$ for all $1 \le i \le n$, and in the *stable range* if in addition $h_i = 1$ implies $w_i = u_K := [\mathfrak{o}_K^\times : \mathfrak{o}^\times]$. Note that if $h_i > 0$, then $\mathfrak{o}_K \subset \mathcal{O}_\ell(\mathcal{I}_i)$, and thus $u_K | w_i$. If $\mathcal{O}$ is balanced, then the stable range is the same as the semistable range.

Corollary 4.5. *Let Φ_0 be an orthogonal basis of eigenforms for $S(\mathcal{O})$ and $\chi \in \widehat{\mathrm{Cl}}(K)$. Then we have the bounds*

$$\sum_{i=1}^n w_i h_i - m^+(\mathcal{O}, 1)\frac{h_K^2}{m(\mathcal{O})} \le \sum_{\varphi \in \Phi_0} \frac{|P_K(\varphi)|^2}{(\varphi, \varphi)} \le h_K \sum_{i=1}^n w_i h_i - m^+(\mathcal{O}, 1)\frac{h_K^2}{m(\mathcal{O})},$$

where $m^+(\mathcal{O}, \chi)$ is the number of characters μ of $\widehat{\mathrm{Cl}}^+(N(\hat{\mathcal{O}}))$ such that $\mu \circ N_{K/F} = \chi$. The lower bound is an equality if and only if $(\mathcal{O}, K)$ in the semistable range, in which case we have

$$\sum_{\varphi \in \Phi_0} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = \sum_{i=1}^n w_i h_i - m^+(\mathcal{O}, \chi)\frac{h_K^2}{m(\mathcal{O})}$$

for all $\chi \in \widehat{\mathrm{Cl}}(K)$.

Proof. We extend Φ_0 to an orthogonal basis Φ of $M(\mathcal{O})$ by adjoining $\mu \circ N$ for each $\mu \in \widehat{\mathrm{Cl}}^+(N(\hat{\mathcal{O}}))$. Then comparing Proposition 4.1 (with $\mathfrak{m} = \mathfrak{o}$), (4.3) (for $T = T_\mathfrak{o}$ the identity) and Lemma 4.4, we see that

$$\sum_{i=1}^n w_i h_i \le \sum_{\varphi \in \Phi} \frac{|P_K(\varphi)|^2}{(\varphi, \varphi)} \le h_K \sum_{i=1}^n w_i h_i,$$

with the lower bound being equality if and only if we are in the semistable range, in which case (4.3) for $T = T_\mathfrak{o}$ is independent of χ. Now we simply subtract of the Eisenstein contribution as in Corollary 4.2. ☐

Note if $h_F^+ = 1$ and $\mathcal{O}$ is a special order of unramified quadratic type, $m^+(\mathcal{O}, \chi)$ is simply 1 if $\chi = 1$ and 0 otherwise. If in addition we are in the stable range, then we simply get

$$\sum_{\varphi \in \Phi_0} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K \left(u_K - \delta_{\chi,1} \frac{h_K}{m(\mathcal{O})} \right), \tag{4.5}$$

where $\delta_{\chi,\chi'}$ is the Kronecker delta. Note this is always nonnegative because being in the stable range implies $u_K m(\mathcal{O}) \ge h_K$.

In general, we can make the above bounds more elementary by observing that

$$h_K \le \sum_{i=1}^n w_i h_i \le w_B h_K, \tag{4.6}$$

where $w_B = \max[\mathcal{O}_B^\times : \mathfrak{o}^\times]$, where $\mathcal{O}_B$ runs over maximal orders of B. If $F = \mathbb{Q}$ and $D_B > 3$, then $w_B \le 3$ and can be determined by congruence conditions.

Finally, we remark the following inheritance properties for suborders.

Lemma 4.6. *Let $\mathcal{O} \subset B$ be an order, and $\mathcal{O}'$ be a suborder. If $\mathcal{O}$ is balanced, then so is $\mathcal{O}'$. Similarly if $\mathfrak{o}_K \subset \mathcal{O}'$, and $(\mathcal{O}, K)$ is in the (semi-)stable range then so is $(\mathcal{O}', K)$.*

Proof. We can write $\mathrm{Cl}(\mathcal{O}') = \{y_{ij}\}$, where as double cosets in $\hat{B}^\times$ we have $x_i = \bigsqcup_j y_{ij}$ for all $1 \le i \le n$. Then each $\mathcal{O}_\ell(y_{ij}) \subset \mathcal{O}_\ell(x_i)$, so each $[\mathcal{O}_\ell(y_{ij})^\times : \mathfrak{o}^\times] \le w_i$. In addition if $t \mapsto y(t)$ denotes the corresponding class map $\mathrm{Cl}(K) \to \mathrm{Cl}(\mathcal{O}')$, then we see $\sum_j h'_{ij} = h_i$, where $h'_{ij} = \#\{t \in \mathrm{Cl}(K) : y(t) = y_{ij}\}$. $\square$

4.3. **Integral embeddings.** Recall that a quadratic field extension K/F embeds in B if and only if K_v/F_v is non-split at each $v \in \mathrm{Ram}(B)$ and $v|\infty$. Here we recall some facts about embedding $\mathfrak{o}_K$ into a quaternionic order $\mathcal{O}$.

Consider an $\mathfrak{o}$-order $\mathcal{O} \subset B$. Let $\mathrm{Emb}(\mathfrak{o}_K, \mathcal{O})$ denote the set of embeddings of $\mathfrak{o}_K$ into $\mathcal{O}$ up to conjugation by $\mathcal{O}^\times$. Similarly, define $\mathrm{Emb}(\mathfrak{o}_{K,v}, \mathcal{O}_v)$ for $v < \infty$. We have the following formula (see [Brz89] or [Voi, Theorem 30.7.3]):

$$\sum_{i=1}^{n} \#\,\mathrm{Emb}(\mathfrak{o}_K, \mathcal{O}_\ell(\mathcal{I}_i)) = h_K \prod_{v|\mathfrak{N}} \#\,\mathrm{Emb}(\mathfrak{o}_{K,v}, \mathcal{O}_v). \tag{4.7}$$

On the left, $\mathcal{O}_\ell(\mathcal{I}_i)$ ranges over all isomorphism classes of orders in the genus of $\mathcal{O}$ (with multiplicity).

Now suppose $\mathcal{O}$ is special of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. In particular, we see that $\mathfrak{o}_K$ embeds into a special order in the genus of $\mathcal{O}$ if and only if $\mathfrak{o}_{K,v}$ locally embeds into $\mathcal{O}_v$ for all $v|\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$ and K/F is totally imaginary.

Local embedding numbers have been computed for Eichler orders in [Hij74] and for special orders in [HPS89] (see also [Voi, Chapter 30]). The precise description is complicated in general, and we will only state it under the following assumptions:

$$\mathrm{ord}_v(\mathfrak{N}_2) = 2 \text{ for all } v|\mathfrak{N}_2 \text{ and } \mathfrak{M} \text{ is squarefree.} \tag{4.8}$$

We recall this implies that all special orders of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ lie in the same genus.

Lemma 4.7. *Assume the level $\mathfrak{N}$ of $\mathcal{O}$ satisfies (4.8), and that K/F is a quadratic field which embeds in B.*

(1) For $v|\mathfrak{N}_1$,

$$\#\,\mathrm{Emb}(\mathfrak{o}_{K,v}, \mathcal{O}_v) = \begin{cases} 2 & K_v/F_v \textit{ unramified}, \\ 1 & K_v/F_v \textit{ ramified}, \mathrm{ord}_v(\mathfrak{N}_1) = 1, \\ 0 & K_v/F_v \textit{ ramified}, \mathrm{ord}_v(\mathfrak{N}_1) > 1. \end{cases}$$

(2) For $v|\mathfrak{N}_2$,

$$\#\,\mathrm{Emb}(\mathfrak{o}_{K,v}, \mathcal{O}_v) = \delta(q_v + 1),$$

where $\delta = 1$ if K_v/F_v is ramified, and $\delta = 0$ if K_v/F_v is unramified.

(3) For $v|\mathfrak{M}$,

$$\#\,\mathrm{Emb}(\mathfrak{o}_{K,v}, \mathcal{O}_v) = \begin{cases} 0 & K_v/F_v \textit{ unramified}, \\ 1 & K_v/F_v \textit{ ramified}, \\ 2 & K_v/F_v \textit{ split}. \end{cases}$$

Proof. The first two parts follow from [HPS89, Theorems 5.12 and 5.19] and Lemma 3.1(v). The third part follows from [Hij74, Section 2] (see also [Voi, Lemma 30.6.16]). $\square$

Now (4.7) and Lemma 4.7 immediately yield the following.

Corollary 4.8. *Let $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$ be a nonzero ideal in $\mathfrak{o}$ such that $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{M}$ are pairwise coprime, $\#\{v|\mathfrak{N}_1\mathfrak{N}_2\}+[F:\mathbb{Q}]$ is even, and $\mathrm{ord}_v(\mathfrak{N}_1)$ is odd for $v|\mathfrak{N}_1$. Further assume (4.8). Let K/F be a quadratic field extension. Then $\mathfrak{o}_K$ embeds in some special order $\mathcal{O}\subset B$ of level type $(\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{M})$ if and only if the following conditions holds:*

(i) K/F is totally imaginary;
(ii) K_v/F_v is non-split for each $v|\mathfrak{N}_1\mathfrak{N}_2$;
(iii) K_v/F_v is unramified for each $v|\mathfrak{N}_1$ such that $\mathrm{ord}_v(\mathfrak{N}_1)>1$;
(iv) K_v/F_v is ramified for each $v|\mathfrak{N}_2$;
(v) K_v/F_v is either ramified or split for each $v|\mathfrak{M}$.

4.4. **Balanced orders.** We can rephase the notion of balanced orders in terms of the existence of certain embeddings.

Lemma 4.9. *Let $\varepsilon_1,\dots,\varepsilon_r$ be a set of generators for $\mathfrak{o}_+^\times$. Then an arbitrary order $\mathcal{O}$ is balanced if and only if no ring of the form $\mathfrak{o}[u]$ embeds the genus of $\mathcal{O}$, where either u is a root of unity of order ≥ 3 or $u=\sqrt{-\varepsilon}$ where $\varepsilon=\prod_S\varepsilon_i$ and $\emptyset\neq S\subset\{1,\dots,r\}$.*

Proof. Clearly $\mathcal{O}$ is unbalanced if and only if there exist an order $\mathcal{O}'=\mathcal{O}_\ell(\mathcal{I}_i)$ in the genus of $\mathcal{O}$ such that $(\mathcal{O}')^\times$ contains a unit $u\notin\mathfrak{o}^\times$. Suppose this is the case. Then $K=F(u)$ is a CM extension in B/F. Let μ_K denote the roots of unity of K. Hasse's unit index $Q_{K/F}=[\mathfrak{o}_K^\times:\mu_K\mathfrak{o}^\times]$ is either 1 or 2. If $Q_{K/F}=1$, then $u\in\mu_K$. Assume $Q_{K/F}=2$. Then $u^2\in\zeta\mathfrak{o}^\times$ for some $\zeta\in\mu_K$. Hence $\mathfrak{o}[\zeta]$ embeds in $\mathcal{O}'$, and we may restrict to the case $\zeta=\pm1$. Then $u^2=-\varepsilon_+$ for some $\varepsilon_+\in\mathfrak{o}_+^\times$. Write $\varepsilon_+=\prod\varepsilon_i^{d_i}$. Then, for an appropriate choice of $\eta\in\mathfrak{o}_+^\times$, one sees $\varepsilon:=\varepsilon_+\eta^2$ is of the form $\prod_S\varepsilon_i$ for some $\emptyset\neq S\subset\{1,\dots,r\}$ By replacing u with $u'=\sqrt{-\varepsilon}$, we see that $\mathfrak{o}[u]=\mathfrak{o}[u']=\mathfrak{o}[\sqrt{-\varepsilon}]$. This proves the "if" direction; the other direction is clear. ☐

Lemma 4.9 implies that we can guarantee an order $\mathcal{O}$ is balanced by checking a finite number of local conditions, which depend upon F. Namely, there are finitely many CM extensions $F(u)$ of F, with u as in Lemma 4.9: either $u=\zeta_m$ is a primitive m-th root of unity for some $m\geq 3$ such that the totally real subfield of $\mathbb{Q}(\zeta_m)$ is contained in F, or $u=\sqrt{-\varepsilon}$ for one of the finitely many $\varepsilon\in\mathfrak{o}_+^\times$ of the form above. For each such u, $\mathfrak{o}[u]$ does not embed in the genus of $\mathcal{O}$ if either (i) B/F is ramified at some prime where $F(u)/F$ splits, or (ii) if $\mathfrak{o}[u]=\mathfrak{o}_K$ where $K=F(u)$ and $\mathcal{O}$ is a special order of level type $(\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{M})$ satisfying the conditions of Corollary 4.8 but K does not. (When $\mathfrak{o}[u]\neq\mathfrak{o}_K$, one can also use results about optimal embeddings to get similar conditions.) In particular, if (i) is satisfied for all such u, then every order $\mathcal{O}\subset B$ is balanced.

We explicate our criteria in the case of $F=\mathbb{Q}$.

Corollary 4.10. *Suppose $F=\mathbb{Q}$ and $\mathcal{O}\subset B$ is a special order of level type (N_1,N_2,M) satisfying (4.8). Then $\mathcal{O}$ is balanced if and only if both (1) and (2) below hold.*

(1) One of the following holds:
  (a) there exists a prime $p\equiv 1 \bmod 4$ such that $p|N_1$; or
  (b) there exists a prime $p\equiv 3 \bmod 4$ such that $p|M$; or
  (c) $N_2\notin\{1,4\}$; or
  (d) $8|N_1$.
(2) And one of the following holds:
  (a) there exists a prime $p\equiv 1 \bmod 3$ such that $p|N_1$; or
  (b) there exists a prime $p\equiv 2 \bmod 3$ such that $p|M$; or

(c) $N_2 \notin \{1,9\}$; or
(d) $27|N_1$.

Proof. By Lemma 4.9, $\mathcal{O}$ is balanced if and only if neither $\mathbb{Z}[i]$ nor $\mathbb{Z}[\zeta_3]$ embed in the genus of $\mathcal{O}$. Now apply Corollary 4.8. ∎

In particular, in the setting of this corollary, we see that $\mathcal{O}$ is balanced if $N_2 > 9$, or if there exists $p \equiv 1 \bmod 12$ dividing N_1, or if there exists $p \equiv 11 \bmod 12$ dividing M.

Alternatively, one can treat special orders over $\mathbb{Q}$ by comparing the mass formula with the class number formula from [HPS89, Theorem 8.6].

Now we briefly discuss some sufficient conditions for balanced orders $\mathcal{O}$ when $F = \mathbb{Q}(\sqrt{d})$ is real quadratic. Assume $d > 1$ is squarefree and let ε be a generator for the rank 1 group $\mathfrak{o}_+^\times$. Then, assuming $\mathfrak{o}[\varepsilon]$ does not embed into the genus of $\mathcal{O}$, Lemma 4.9 tells us that $\mathcal{O}$ is balanced if and only if no cyclotomic ring $\mathbb{Z}[\zeta_m]$ with $m \geq 3$ embeds in the genus of $\mathcal{O}$.

If $\mathbb{Z}[\zeta_m]$ embeds in the genus of $\mathcal{O}$, then $K = K_m := F(\zeta_m)$ embeds in B. The only possibilities are $K = F\mathbb{Q}(i)$, $K = F\mathbb{Q}(\zeta_3)$ or that $K = \mathbb{Q}(\zeta_m)$ is cyclotomic of degree 4 containing F. The latter possibility implies that $m = 8$ and $d = 2$, $m = 12$ and $d = 3$, or $m = d = 5$. In these 3 cases, $\mathbb{Z}[\zeta_m]$ embeds in the genus of $\mathcal{O}$ if and only if $\mathfrak{o}_K$ does, so we can apply Corollary 4.8. When $m = 3, 4$, i.e., $K = F\mathbb{Q}(i)$ or $K = F\mathbb{Q}(\zeta_3)$, it is not necessarily true that $\mathfrak{o}_K = \mathfrak{o}_F[\zeta_m]$. However, if $d > 5$ and $d \equiv 1 \bmod 4$, then $\mathbb{Q}(i)$, $\mathbb{Q}(\zeta_3)$ and F have pairwise coprime discriminants, implying that they are pairwise linearly independent, and thus $\mathfrak{o}_K = \mathfrak{o}_F[\zeta_m]$ for $m = 3, 4$. In this situation, we can again use Corollary 4.8.

Example 4.11. Let $F = \mathbb{Q}(\sqrt{5})$. Then $\varepsilon = \frac{\sqrt{5}-1}{2}$ generates $\mathfrak{o}_+^\times$. Let $\mathfrak{p}_{11} = (\frac{1 \pm 3\sqrt{5}}{2})$ be a prime of F above 11. Both $F(\varepsilon)$ and $\mathbb{Q}(\zeta_5)$ split over $\mathfrak{p}_{11}$, whereas both $F\mathbb{Q}(i)$ and $F\mathbb{Q}(\zeta_3)$ split over the prime ideal (7). Hence if B is any definite quaternion algebra ramified at both (7) and $\mathfrak{p}_{11}$, then any order $\mathcal{O} \subset B$ is balanced.

Alternatively, $F(\varepsilon)$ and $F\mathbb{Q}(i)$ have rings of integers of the form $\mathfrak{o}[u]$ where $u = \varepsilon$ or $u = i$, and both are ramified above (2). Also $F\mathbb{Q}(\zeta_3)/F$ is split above (2). Hence if B is any definite quaternion algebra ramified at (2) and also at some prime $\mathfrak{p}$ lying above a prime p of $\mathbb{Q}$ that completely splits $\mathbb{Q}(\zeta_5)$ (i.e., $p \equiv 1 \bmod 5$), then any special order $\mathcal{O} \subset B$ of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ is balanced if $8|\mathfrak{N}_1$.

5. Twisted L-values

From now on, we assume that $h_F = 1$, K/F is quadratic, $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$ is a nonzero integral $\mathfrak{o}$-ideal, and that $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}$ and K/F satisfy the conditions in Corollary 4.8. From Section 5.4 onwards, we will further assume that K_v/F_v is split for each $v|\mathfrak{M}$. Let B/F be the definite quaternion algebra with discriminant $\mathfrak{D}_B = \prod_{\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2} \mathfrak{p}$. Then there exists a special order $\mathcal{O} \subset B$ of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ and an embedding of K into B such that $\mathfrak{o}_K \subset \mathcal{O}$. We fix such an $\mathcal{O}$ and such an embedding, which defines a class map $\mathrm{Cl}(K) \to \mathrm{Cl}(\mathcal{O})$, periods $P_{K,\chi}$, divisors $c_{K,\chi}$, and ideal class embedding numbers $h_1, \ldots, h_n$ as in the previous section.

For an (irreducible) cuspidal representation π of $B^\times(\mathbb{A}_F)$ or $\mathrm{GL}_2(\mathbb{A}_F)$, denote by $c(\pi)$ the conductor of π. We mean that $c(\pi)$ is the ideal in $\mathfrak{o}_F$ corresponding to the exact level of a newform $\varphi \in \pi$. For $v < \infty$, the local conductor is $c(\pi_v) := \mathrm{ord}_v(c(\pi))$.

5.1. **Local vanishing criteria.** Let $\varphi \in M(\mathcal{O})$ be a eigenform and π the associated representation of $B^\times(\mathbb{A}_F)$ (necessarily of trivial central character since $h_F = 1$). Let χ be a character of $\mathrm{Cl}(K)$. It is easy to see that $P_{K,\chi}(\varphi) = 0$ unless the restriction of χ to $\mathbb{A}_F^\times$ is trivial, so let us assume this.

Our definition (4.1) of the period $P_{K,\chi}$ can also be expressed as an integral

$$P_{K,\chi}(\varphi) = \int_{K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times} \varphi(t)\chi^{-1}(t)\, dt,$$

for a suitable choice of Haar measure on $\mathbb{A}_K^\times$. Via this integral, $P_{K,\chi}$ extends to a χ-equivariant linear functional on π. In other words, $P_{K,\chi}$ is an element of $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$. Thus the following gives sufficient criteria for the vanishing of $P_{K,\chi}(\varphi)$, where $\varphi \in \pi$.

Denote by St_v the Steinberg representation of $\mathrm{GL}_2(F_v)$ for $v < \infty$.

Proposition 5.1. *Let π be a cuspidal representation of $B^\times(\mathbb{A}_F)$ occurring in $S(\mathcal{O})$. Let χ be a character of $\mathrm{Cl}(K)$ which is trivial on $\mathbb{A}_F^\times$. The global Hom space $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$ is nonzero if and only if both of the following hold for all $v|\mathfrak{N}$ such that K_v/F_v is ramified:*

(i) if $v|\mathfrak{N}_1\mathfrak{N}_2$ and π_v is 1-dimensional, then $\pi_v(\varpi_{K,v}) = \chi_v(\varpi_{K,v})$;
(ii) if $v|\mathfrak{M}$ and $\pi_v \simeq \mathrm{St}_v \otimes \mu_v$, then $\mu_v(\varpi_v) = -\chi_v(\varpi_{K,v})$.

Proof. The global Hom space $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$ is nonzero if and only if each local Hom space $\mathrm{Hom}_{K_v^\times}(\pi_v, \chi_v)$ is. For $v|\infty$, π_v and χ_v are both trivial, so this local Hom space is clearly nonzero. Assume from now on v is finite. Necessary and sufficient criteria for each local Hom space to be nonzero were given by Tunnell [Tun83] and Saito [Sai93]. In particular, if π_v is an (unramified or ramified) principal series or K_v/F_v is split (so necessarily B_v is split), then $\mathrm{Hom}_{K_v^\times}(\pi_v, \chi_v) \neq 0$ for all $\chi_v : K_v^\times/F_v^\times \to \mathbb{C}^\times$.

In all other cases, $v|\mathfrak{N}$ and either π_v is a representation of the unit group $D_v^\times$ of the local quaternion division algebra or π_v is a discrete series representation of $\mathrm{GL}_2(F_v)$. In either case, let π'_v be the Jacquet–Langlands correspondent on the other group. Furthermore, K_v/F_v is a field extension which embeds into D_v, and we have the dichotomy relation:

$$\dim \mathrm{Hom}_{K_v^\times}(\pi_v, \chi_v) + \dim \mathrm{Hom}_{K_v^\times}(\pi'_v, \chi_v) = 1.$$

Let $\pi_{D,v}$ be the element of $\{\pi_v, \pi'_v\}$ which is a representation of $D_v^\times$. Then $\mathrm{Hom}_{K_v^\times}(\pi_{D,v}, \chi_v) \neq 0$ if and only if $\pi_{D,v}|_{K_v^\times}$ contains χ_v as a subrepresentation. We note that the Tunnell–Saito criterion says that this is the case if and only if $\varepsilon(\frac{1}{2}, \pi_{K,v} \otimes \chi_v) = -1$, though we do not directly use this in our proof.

First suppose K_v/F_v is unramified. By our embedding conditions, this means $v|\mathfrak{N}_1$, so in particular $\pi_v = \pi_{D,v}$. Then $K_v^\times = F_v^\times \mathfrak{o}_{K,v}^\times$, so χ_v occurs in π_v if and only if π_v contains an $\mathfrak{o}_{K,v}^\times$-fixed vector. In fact $\pi_v^{\mathcal{O}_v^\times}$ contains an $\mathfrak{o}_{K,v}^\times$-fixed vector since $\mathfrak{o}_{K,v} \subset \mathcal{O}_v$. Thus in this case we always have $\dim \mathrm{Hom}_{K_v^\times}(\pi_v, \chi_v) \neq 0$.

Now assume K_v/F_v is ramified. By our embedding criteria, either $\pi_{D,v}$ is 1-dimensional or $v|\mathfrak{N}_2$ and $\pi_v = \pi_{D,v}$ has dimension > 1. In addition, there are 2 possibilities for χ_v according to $\chi_v(\varpi_{K,v}) = \pm 1$, since $K_v^\times/\mathfrak{o}_{K,v}^\times F_v^\times \simeq \langle \varpi_{K,v} \rangle / \langle \varpi_{K,v}^2 \rangle$.

If $v|\mathfrak{N}_2$ and $\pi_v = \pi_{D,v}$ has dimension > 1, then π'_v is minimal supercuspidal representation of even conductor (in fact depth 0), so by [Mar, Theorem 3.6] $\pi_v|_{K_v^\times}$ contains both unramified characters of $K_v^\times/F_v^\times$. In particular, $\mathrm{Hom}_{K_v^\times}(\pi_v, \chi_v) \neq 0$.

Finally we are reduced to the case that $\pi_{D,v} = \mu_v \circ N$ is 1-dimensional, so $\pi'_v \simeq \mathrm{St}_v \otimes \mu_v$. It is clear that $\mathrm{Hom}_{K_v^\times}(\pi_{D,v}, \chi_v) \neq 0$ if and only if $\mu_v(\varpi_v) = \chi_v(\varpi_{K,v})$. The proposition now follows from the dichotomy relation by observing that $\pi_v = \pi_{D,v}$ if $v|\mathfrak{N}_1\mathfrak{N}_2$ and $\pi'_v = \pi_{D,v}$ if $v|\mathfrak{M}$. $\square$

5.2. **Local L-factors.** In this section, we describe local L-factors in cases that are relevant for us. Here v denotes a finite place of F, K_v/F_v is a quadratic field extension, π_v is a representation of $\mathrm{PGL}_2(F_v)$, and χ_v an unramified character of $K_v^\times/F_v^\times$. We say a supercuspidal π_v is unramified dihedral if it is induced from a regular character of the unramified quadratic extension of F_v. A supercuspidal π_v is unramified dihedral if and only if π_v is minimal of even conductor [Tun78, Proposition 3.5].

Lemma 5.2. *We have the following local L-factor ratios.*

(1) Suppose π_v is supercuspidal. Then

$$\frac{L(\frac{1}{2}, \pi_{K,v} \otimes \chi_v)}{L(1, \pi_v, \mathrm{Ad})} = \begin{cases} 1 + q_v^{-1} & \pi_v \textit{ unramified dihedral} \\ 1 & \pi_v \textit{ else.} \end{cases}$$

(2) Suppose K_v/F_v is ramified and $\pi_v \simeq \mathrm{St}_v \otimes \mu_v$ is a twisted Steinberg representation, where μ_v is a quadratic character of $F_v^\times$ such that $\mu_v \circ N_{K_v/F_v}$ is an unramified character of $K_v^\times$ and $\mu_v(\varpi_{F,v}) = \chi_v(\varpi_{K,v})$. Then

$$\frac{L(\frac{1}{2}, \pi_{K,v} \otimes \chi_v)}{L(1, \pi_v, \mathrm{Ad})} = 1 + q_v^{-1}.$$

Proof. First suppose π is supercuspidal. Then π_{K_v} is either supercuspidal or a ramified principal series, but in any case has conductor at least 2, so $L(s, \pi_{K,v} \otimes \chi_v) = 1$. From the calculations in [NPS14], $L(s, \pi_v, \mathrm{Ad})$ is either $(1 + q_v^{-s})^{-1}$ or 1 according to whether π_v is unramified dihedral or not. This yields (1).

Next assume the hypotheses of (2). Then $\pi_{K,v} \simeq \mathrm{St}_{K,v} \otimes (\mu_v \circ N)$, so $L(s, \pi_{K,v} \otimes \chi_v) = L(s, (\mu_v \circ N)\chi_v) = L(s, 1_{F_v})$. Also $L(s, \pi_v, \mathrm{Ad}) = (1 - q_v^{-s-1})^{-1}$. This yields (2). $\square$

5.3. **Local spectral distributions.** Here we will compute relevant local spectral distributions at a nonarchimedean place v which splits K/F. To ease notation, for this section only, all objects are local and we drop the subscripts v. In particular, F is now a $\mathfrak{p}$-adic field, and $K = F \times F$.

Let $\pi = \pi(\mu, \mu^{-1})$ be an unramified principal series representation of $\mathrm{GL}_2(F)$ with trivial central character. Consider a character χ of $K^\times = F^\times \times F^\times$ given by $\chi(a, b) = \chi_1(a)\chi_2(b)$, where χ_1, χ_2 are 2 unramified characters of $F^\times$ such that $\chi_2 = \chi_1^{-1}$. Let ψ be an additive character of order 0, and $\mathcal{W}$ be the ψ-Whittaker model for π. For $W \in \mathcal{W}$, we abbrieviate $W \begin{pmatrix} a & \\ & 1 \end{pmatrix}$ by $W(a)$. (Alternatively, one can think of W as an element of the Kirillov model.) Let $d^\times a$ denote the Haar measure on $F^\times$ which gives $\mathfrak{o}_F^\times$ volume 1, i.e., $d^\times a$ is self-dual with respect to ψ.

Put

$$\ell(W) = \int_{F^\times} W(a)\chi_1^{-1}(a)\, d^\times a.$$

As in [MW09, Section 2.1], we define the local distribution

$$\tilde{J}_\pi(f) = \sum_W \ell(\pi(f)W)\overline{\ell(W)}, \tag{5.1}$$

where $f \in C_c^\infty(\mathrm{GL}_2(F))$ and W runs over an orthonormal basis for $\mathcal{W}$.

If f_v is the characteristic function of $\mathrm{GL}_2(\mathfrak{o})$ divided by its volume, then

$$\tilde{J}_\pi(f) = \frac{L(2, 1_F)}{L(1, 1_F)} \cdot \frac{L(\frac{1}{2}, \pi \otimes \chi_1) L(\frac{1}{2}, \pi \otimes \chi_2)}{L(1, \pi, \mathrm{Ad})}. \tag{5.2}$$

Lemma 5.3. *Consider the Eichler order* $R = \begin{pmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix}$ *of level* $\mathfrak{p}$ *in* $M_2(\mathfrak{o})$*. When* $f = \mathrm{vol}(R^\times)^{-1} 1_{R^\times}$*, we have*

$$\tilde{J}_\pi(f) = \frac{L(2, 1_F)}{L(1, 1_F)} \left(\frac{L(\frac{1}{2}, \pi \otimes \chi_1) L(\frac{1}{2}, \pi \otimes \chi_2)}{L(1, \pi, \mathrm{Ad})} + \frac{1}{L(1, 1_F)} \right).$$

Proof. Let W_0 be the Whittaker new vector in $\mathcal{W}$ invariant under $\mathrm{GL}_2(\mathfrak{o})$ of norm 1. Then $\pi(f)$ acts as orthogonal projection onto the subspace V of $\mathcal{W}$ spanned by W_0 and $W_1 = \pi \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} W_0$. From [FMP17, Section 8B], an orthonormal basis for V is given by W_1 and W_2, where

$$W_2 = \left(\frac{L(1, \pi, \mathrm{Ad}) L(1, 1_F)}{L(2, 1_F)^2} \right)^{\frac{1}{2}} \left(W_0 - \frac{\mu(\varpi) + \mu(\varpi)^{-1}}{q^{\frac{1}{2}}(1 + q^{-1})} W_1 \right).$$

Hence $\tilde{J}_\pi(f) = |\ell(W_1)|^2 + |\ell(W_2)|^2$. It is well known that

$$\ell(W_0) = \left(\frac{L(2, 1_F)}{L(1, \pi, \mathrm{Ad}) L(1, 1_F)} \right)^{\frac{1}{2}} L(\frac{1}{2}, \pi \otimes \chi_1),$$

and we see that $\ell(W_1) = \chi_1(\varpi)\ell(W_0)$ and

$$|\ell(W_2)|^2 = |1 - \chi_1(\varpi)(\mu(\varpi) + \mu(\varpi)^{-1})q^{-\frac{1}{2}} + q^{-1}|^2 \frac{L(1, \pi, \mathrm{Ad})}{L(1, 1_F)} |\ell(W_0)|^2.$$

This yields the lemma. □

5.4. **L-values and periods.** For the rest of this section, assume that K_v/F_v is split for each $v|\mathfrak{M}$.

Suppose π is a cuspidal representation of $B^\times(\mathbb{A}_F)$ or $\mathrm{GL}_2(\mathbb{A}_F)$. Denote by $S(\pi)$ the set of finite places v at which π is ramified and $S(\pi, K)$ the subset of $v \in S(\pi)$ such that K_v/F_v is ramified. Denote by $\eta = \eta_{K/F}$ the quadratic character of $\mathbb{A}_F^\times$ associated to K/F by class field theory. Recall $u_K = [\mathfrak{o}_K^\times : \mathfrak{o}_F^\times]$. Put

$$C(K; \mathfrak{N}) = 2^{\#S(\pi,K)-1} h_F D_F^{-2} u_K^2 |D_K|^{1/2} \prod_{v|\mathfrak{N}_2} \frac{1}{1 + q_v^{-1}}. \tag{5.3}$$

By our assumption that K_v/F_v is split at all $v|\mathfrak{M}$, we see that $S(\pi, K)$ and thus $C(K; \mathfrak{N})$ does not actually depend upon π.

We also set $\Lambda_{\mathfrak{M}}(\pi, \chi) = \prod_{v|\mathfrak{M}} \Lambda_v(\pi, \chi)$, where

$$\Lambda_v(\pi, \chi) = \begin{cases} \frac{1}{1+q_v^{-1}} \left(1 + (1 - q_v^{-1}) \frac{L(1, \pi_v, \mathrm{Ad})}{L(\frac{1}{2}, \pi_{K_v} \otimes \chi)} \right) & v|\mathfrak{M}, v \notin S(\pi) \\ 1 & v|\mathfrak{M}, v \in S(\pi). \end{cases} \tag{5.4}$$

Proposition 5.4. *Let π be a cuspidal representation of $B^\times(\mathbb{A}_F)$ occurring in $S(\mathcal{O})$, and Φ_π be an orthogonal basis for $\pi^{\hat{\mathcal{O}}^\times} \subset S(\mathcal{O})$. Let $\chi \in \widehat{\mathrm{Cl}}(K)$ be such that $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi, \chi) \neq 0$. Then*

$$N(\mathfrak{N}) \sum_{\varphi \in \Phi_\pi} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = C(K; \mathfrak{N}) \Lambda_{\mathfrak{M}}(\pi, \chi) \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \mathrm{Ad})}. \tag{5.5}$$

Proof. Let $\mathfrak{N}' = c(\pi)$, and let $\mathcal{O}'$ be a special order of level type $(\mathfrak{N}_1', \mathfrak{N}_2', \mathfrak{M}')$ such that $\mathfrak{N}' = \mathfrak{N}_1'\mathfrak{N}_2'\mathfrak{M}'$ and $\mathcal{O}' \supset \mathcal{O}$.

Take for measures on $B^\times(\mathbb{A}_F)$, $\mathbb{A}_K^\times$ and $\mathbb{A}_F^\times$ the product of the local Tamagawa measures. Consider a test function $f = \prod f_v \in C_c^\infty(B^\times(\mathbb{A}_F))$ where f_v is the characteristic function of $(\mathcal{O}_v')^\times$ divided by its volume for $v < \infty$. For $v|\infty$, choose f_v so that $\int_{B^\times(F_v)} f_v(g)\, dg = 1$. Then $\pi(f)$ acts as orthogonal projection onto $S(\mathcal{O}') \cap \pi$. For $\varphi, \varphi' \in S(\mathcal{O})$, put

$$(\varphi, \varphi')_{\mathrm{Tam}} = \int_{\mathbb{A}_F^\times B^\times \backslash B^\times(\mathbb{A}_F)} \varphi(x)\overline{\varphi'(x)}\, dx.$$

Since $\mathbb{A}_F^\times B^\times \backslash B^\times(\mathbb{A}_F)$ has Tamagawa volume 2, we see $(\varphi, \varphi')_{\mathrm{Tam}} = \frac{2}{m(\mathcal{O})}(\varphi, \varphi')$.

Let $S_0(\pi) = S(\pi, K) \cup \{v \in S(\pi) : c(\pi_v) \geq 2\}$. Then by the proof of [MW09, Theorem 4.1] (see [FMP17, Theorem 1.1] for a formulation consistent with our present choice of measures), we have the formula

$$J_\pi(f) = \frac{2^{\#S(\pi,K)-1}}{\pi^{[F:\mathbb{Q}]}} \sqrt{\frac{D_F}{|D_K|}} L_{S(\pi)}(1, \eta) L^{S(\pi)}(2, 1_F) \frac{L^{S_0(\pi)}(\frac{1}{2}, \pi_K \otimes \chi)}{L^{S_0(\pi)}(1, \pi, \mathrm{Ad})},$$

where $J_\pi(f)$ is a certain spectral distribution, which for our choice of test function f is given by,

$$J_\pi(f) = \frac{m(\mathcal{O})}{2} \left(\frac{\mathrm{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} \right)^2 \sum_\varphi \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)}, \tag{5.6}$$

and φ runs over an orthogonal basis Φ' for $\pi^{(\hat{\mathcal{O}}')^\times}$. Note

$$\frac{\mathrm{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} = \frac{2L(1, \eta)}{h_K} = \frac{2^{[F:\mathbb{Q}]}\sqrt{D_F}}{u_K h_F \sqrt{|D_K|}}.$$

Putting everything together and using the identity

$$\frac{L(2, 1_F)}{|\zeta_F(-1)|} = \frac{(2\pi)^{[F:\mathbb{Q}]}}{D_F^{3/2}}$$

shows that $N(\mathfrak{N}) \sum_{\varphi \in \Phi'} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi,\varphi)}$ equals

$$C(K; \mathfrak{N}) \cdot \prod_{v \in S(\pi,K)} (1 + q_v^{-1}) \cdot \prod_{v | \mathfrak{M}(\mathfrak{M}')^{-1}} \frac{1}{1 + q_v^{-1}} \cdot \frac{L^{S_0(\pi)}(\frac{1}{2}, \pi_K \otimes \chi)}{L^{S_0(\pi)}(1, \pi, \mathrm{Ad})}.$$

By Lemma 5.2, we may rewrite this as

$$N(\mathfrak{N}) \sum_{\varphi \in \Phi'} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = C(K; \mathfrak{N}) \cdot \prod_{v | \mathfrak{M}(\mathfrak{M}')^{-1}} \frac{1}{1 + q_v^{-1}} \cdot \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \mathrm{Ad})}. \tag{5.7}$$

Now we need to relate the sum of square periods over Φ' to one over Φ_π.

By [Mar], we have $\pi^{\hat{\mathcal{O}}^\times} = \pi^{(\hat{\mathcal{O}}'')^\times}$, where $\mathcal{O}''$ is a special order of level type $(\mathfrak{N}_1', \mathfrak{N}_2', \mathfrak{M})$ lying between $\mathcal{O}'$ and $\mathcal{O}$. Now we take a test function $f' = \prod f_v' \in C_c^\infty(B^\times(\mathbb{A}_F))$ such that $f_v' = f_v$ for $v \nmid \mathfrak{M}(\mathfrak{M}')^{-1}$ and f_v' is the characteristic function of $(\mathcal{O}_v'')^\times = \mathcal{O}_v^\times$ divided by its volume for $v|\mathfrak{M}(\mathfrak{M}')^{-1}$. Then, as in [FMP17, Section 8B], we can write

$$J_\pi(f') = J_\pi(f) \cdot \prod_{v|\mathfrak{M}(\mathfrak{M}')^{-1}} \frac{\tilde{J}_{\pi_v}(f_v')}{\tilde{J}_{\pi_v}(f_v)},$$

where $J_\pi(f')$ is given by (5.6) but now with φ running over Φ_π. Now comparing (5.2) with Lemma 5.3 shows that for $v|\mathfrak{M}(\mathfrak{M}')^{-1}$,

$$\frac{\tilde{J}_{\pi_v}(f_v')}{\tilde{J}_{\pi_v}(f_v)} = 1 + (1 - q_v^{-1}) \frac{L(1, \pi_v, \mathrm{Ad})}{L(\frac{1}{2}, \pi_{K_v} \otimes \chi)}.$$

Combining this with (5.7) gives the proposition. $\square$

5.5. **Average L-values.** Let $\mathcal{F}(\mathfrak{N})$ denote the set of holomorphic parallel weight 2 cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character such that (i) $c(\pi)|\mathfrak{N}$; (ii) $c(\pi_v)$ is odd for $v|\mathfrak{N}_1$; and (iii) π_v is a discrete series representation for $v|\mathfrak{N}_2$. For $\pi \in \mathcal{F}(\mathfrak{N})$, let π_B be the corresponding automorphic representation of $B^\times(\mathbb{A}_F)$. For a character χ of $\mathrm{Cl}(K)$, denote by $\mathcal{F}(\mathfrak{N};\chi)$ the subset of $\pi \in \mathcal{F}(\mathfrak{N})$ such that $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi_B, \chi) \neq 0$. Thus by Proposition 5.1, $\pi \in \mathcal{F}(\mathfrak{N};\chi)$ if and only if, for any place $v|c(\pi)$ such that K_v/F_v is ramified (so $v|\mathfrak{N}_1\mathfrak{N}_2$) and $\pi_v \simeq \mathrm{St}_v \otimes \mu_v$, we have $\mu_v(\varpi_{F,v}) = \chi_v(\varpi_{K,v})$.

Theorem 5.5. *Let $\mathfrak{m}$ be a nonzero integral ideal of $\mathfrak{o}_F$ which is coprime to $\mathfrak{N}\mathfrak{D}_B^{-1}$. Then*

$$C(K;\mathfrak{N}) \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{\pi \in \mathcal{F}(\mathfrak{N};\chi)} \lambda_{\mathfrak{m}}(\pi) \Lambda_{\mathfrak{M}}(\pi,\chi) \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \mathrm{Ad})} =$$

$$N(\mathfrak{N}) h_K \left(\langle c_K, a(\mathfrak{m}) \rangle - \delta^+(\mathcal{O}, \mathfrak{m}) \deg T_{\mathfrak{m}} \frac{h_K |\mathrm{Cl}^+(N(\hat{\mathcal{O}}))|}{m(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})} \right).$$

Proof. Since the periods $P_{K,\chi}$ automatically vanish on π_B if $\mathrm{Hom}_{\mathbb{A}_K^\times}(\pi_B, \chi) = 0$, this theorem follows directly from Proposition 5.4, Corollary 4.2 and Theorem 3.5. $\square$

Remark 5.6. By the ε-factor criteria of Tunnell and Saito discussed above, note that $\pi \in \mathcal{F}(\mathfrak{N};\chi)$ implies that $\pi_K \otimes \chi$ has root number $+1$. (The converse is not typically true.) Thus the restriction to such π precludes one trivial reason for $L(\frac{1}{2}, \pi_K \otimes \chi)$ to vanish, the root number over K being -1. There is also a trivial way for this central value to vanish if the root number of K is $+1$—if $L(s, \pi_K \otimes \chi) = L(s, \tau_1) L(s, \tau_2)$ factors as two degree 2 L-functions over F with root number -1. For instance, if χ is trivial, then $L(s, \pi_K) = L(s, \pi) L(s, \pi \otimes \eta_{K/F})$ so $L(\frac{1}{2}, \pi_K)$ is forced to vanish if π has root number -1. See also [MW, Remark 6.10] regarding a factorization when χ is quadratic.

As in Section 4, this double average formula simplifies in a variety of situations, can be used to obtain bounds not involving $\mathcal{O}$, and yields exact bounds on single averages. Under suitable conditions, we can also pick out the average restricted to newforms. For simplicity, we only carry this out over $\mathbb{Q}$ below.

5.6. **Consequences over $\mathbb{Q}$.** We now assume $F = \mathbb{Q}$, and explain how to deduce the double average value formula Theorem 1.3 from Theorem 5.5. Write N, N_*, M, D_B in place of $\mathfrak{N}, \mathfrak{N}_*, \mathfrak{M}, \mathfrak{D}_B$, where the roman types now represent positive generators of their fraktur counterparts. If π is an automorphic representation corresponding to a weight 2 newform f, we also write the factors $\Lambda_*(\pi, \chi)$ as $\Lambda_*(f, \chi)$.

Proof of Theorem 1.3. Let $f \in \mathcal{F}(N; \chi)$ be a newform level $N_f \mid N$ and π be the associated cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. To translate our automorphic L-values to classical quantities first note that by [Hid81, Theorem 5.1], we have

$$L_{\mathrm{fin}}^{S_2(\pi)}(1, \pi, \mathrm{Ad}) = \frac{8\pi^3}{N_f}(f, f),$$

where $(f, f) = (f, f)_{N_f} = \int_{\Gamma_0(N_f)\backslash\mathfrak{H}} |f(x+iy)|^2 \, dx \, dy$ denotes the usual Petersson norm on $X_0(N_f)$, and $S_2(\pi)$ denotes the set of finite primes p where $c(\pi_p) \geq 2$. From the calculations in the proof of Lemma 5.2, we have

$$L_{\mathrm{fin}}(1, \pi, \mathrm{Ad}) = \Lambda_{N_2}(f)^{-1} L_{\mathrm{fin}}^{S_2(\pi)}(1, \pi, \mathrm{Ad}),$$

where $\Lambda_{N_2}(f) = \prod_{p|N_2} \Lambda_p(f)$ and

$$\Lambda_p(f) = \Lambda_p(f, \chi) = \begin{cases} 1 + p^{-1} & p|N_2, \ \pi_p \text{ supercuspidal}, \\ 1 - p^{-2} & p|N_2, \ \pi_p \text{ ramified twist of } \mathrm{St}_p, \\ 1 & p|N_2, \ \pi_p \text{ unramified twist of } \mathrm{St}_p. \end{cases} \tag{5.8}$$

Also note that for $v|\infty$,

$$\frac{L(\frac{1}{2}, \pi_{K,v} \otimes \chi_v)}{L(1, \pi_v, \mathrm{Ad})} = 2\pi.$$

Thus

$$\frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \mathrm{Ad})} = \frac{N_f}{4\pi^2} \Lambda_{N_2}(f) \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f, f)}. \tag{5.9}$$

Now Theorem 1.3 follows from Theorem 5.5 together with Lemma 3.3. $\square$

In the above proof, (5.9) means we can rewrite (5.5) as

$$\sum_{\varphi \in \Phi_\pi} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = \frac{C(K; N)}{4\pi^2} \Lambda_N(f, \chi) \frac{N_f}{N} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f, f)}. \tag{5.10}$$

For some of our remaining results on L-value averages, we will use this together with the results of Section 4.

For explicit calculations for later use, we note that we can express the factors $\Lambda_p(f, \chi)$ for $p \mid M$, $p \nmid N_f$ from (5.4) in terms of Fourier coefficients $a_p(f)$ as

$$\Lambda_p(f, \chi) = \frac{1}{1 + p^{-1}} \left(1 + \frac{|1 + \chi(\mathfrak{p})(\chi(\mathfrak{p}) - a_p(f))p^{-1}|^2}{1 + 2p^{-1} + (1 - a_p(f)^2)p^{-2}} \right) \qquad (\text{for } p \mid M, \ p \nmid N_f), \tag{5.11}$$

where $\mathfrak{p}$ denotes a prime of K above p.

Lemma 5.7. *Let $\mathcal{O} \subset B$ be a special order of level type (N_1, N_2, M), and set $N = N_1 N_2 M$. Then $\#\mathcal{O}^\times \leq 6$ unless $N \leq 4$.*

Proof. Suppose $\#\mathcal{O} > 6$. Note that if $\mathcal{O}' \subset \mathcal{O}$, then $\#(\mathcal{O}')^\times \geq \#\mathcal{O}^\times$. Moreover $\#\mathcal{O}_B^\times \leq 6$ for every maximal order $\mathcal{O}_B \subset B$ unless $D_B = 2, 3$, in which case $\mathcal{O}_B$ is unique up to isomorphism. When $D_B = 2$, $\#\mathcal{O}_B^\times = 24$, and when $D_B = 3$, $\#\mathcal{O}_B^\times = 12$. In both cases, the $\mathbb{Z}$-order in B generated by $\mathcal{O}_B^\times$ is all of $\mathcal{O}_B$. Hence the only possibility with $N > 3$ is that $D_B = 2$ and $\mathcal{O}$ is a non-maximal order in B. Assume this.

Then $\mathcal{O}^\times$ is a proper subgroup of $\mathcal{O}_B^\times \simeq \mathrm{SL}_2(\mathbb{F}_3)$ of order greater than 6, and the only possibility is $\#\mathcal{O}^\times = 8$. Necessarily $\mathbb{Z}[i]$ embeds in the genus of $\mathcal{O}$, so by Lemma 4.7 we must have that N_1N_2 is 2 or 4. If $N = 4$, then there is a unique order up to isomorphism and $\#\mathcal{O}^\times = 8$ does occur.

Suppose $N > 4$. Then $M > 1$, and by enlarging $\mathcal{O}$ if necessary we may suppose $M = p$ and $N = 2p$. Again by Lemma 4.7, we must have $p \equiv 1 \bmod 4$. From the mass formula (Lemma 3.2), $m(\mathcal{O}) = \frac{p+1}{12}$. By the class number formula, $h(\mathcal{O}) = m(\mathcal{O}) + \frac{7}{6}$ or $h(\mathcal{O}) = m(\mathcal{O}) + \frac{1}{2}$, depending on whether $p \equiv 1 \bmod 3$ or $p \equiv 2 \bmod 3$. Correspondingly $\sum(1 - \frac{1}{w_i})$ is either $\frac{7}{6}$ or $\frac{1}{2}$. In the latter case, there must be exactly one $w_i > 1$, and it must be 2, which would imply $\#\mathcal{O}^\times \leq 4$. So consider the former case. Then there must either be 2 or 3 i's such that $w_i > 1$, and each such w_i is 2, 3 or 4. In any case, we see that $\sum(1 - \frac{1}{w_i}) = \frac{7}{6}$ with some $w_i = 4$ is impossible. □

Lemma 5.8. *Suppose $D_B > |D_K|$ with $\gcd(D_B, D_K) = 1$. Then for any order $\mathcal{O} \subset B$ containing $\mathfrak{o}_K$, the pair $(\mathcal{O}, K)$ lies in the stable range.*

Proof. The stable average value formula [FW09, Theorem 6.10] together with Corollary 4.5 implies that $(\mathcal{O}_B, K)$ is in the stable range for any maximal order $\mathcal{O}_B$ containing $\mathfrak{o}_K$. Now apply Lemma 4.6. □

Proof of Theorem 1.4. We want to relate double averages over $\mathcal{F}(N)$ to double averages of $\mathcal{F}_0(N)$. We do this by subtracting away the contribution of the N_1N_2-oldforms using inclusion-exclusion.

For a subset Σ of primes dividing $D_B^{-1}N_1N_2$ at which we want to lower the level, define $N^\Sigma = N_1^\Sigma N_2^\Sigma M$ as follows. First put $N_2^\Sigma = \prod p^2$, where p runs over primes dividing N_2 such that $p \notin \Sigma$. Then set $N_1^\Sigma = \prod p^{\mathrm{ord}_p(N_1)} \prod q^{\mathrm{ord}_p(N_1)-2} \prod r$, where p runs over the primes dividing N_1 not in Σ, and q (resp. r) runs over the primes dividing N_1 (resp. N_2) which are in Σ.

Let

$$A_0(N_1, N_2) = \frac{C(K;N)}{4\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \sum_{f \in \mathcal{F}_0(N;\chi)} \Lambda_N(f,\chi) N_f \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f,f)},$$

and $A(N_1, N_2)$ denote the corresponding expression where the sum over $f \in \mathcal{F}_0(N;\chi)$ is replaced by the sum over $f \in \mathcal{F}(N;\chi)$. Then

$$A_0(N_1, N_2) = \sum_\Sigma (-1)^{\#\Sigma} \left(\prod_{p | N_2, p \in \Sigma} \frac{1}{1 + p^{-1}} \right) A(N_1^\Sigma, N_2^\Sigma),$$

where Σ runs over all subsets of $\{p : p | D_B^{-1} N_1 N_2\}$. It follows from Lemmas 4.6, 5.7 and 5.8, that any of the conditions (1)–(3) in the initial assumption of Theorem 1.4 imply that the equality $\sum h_i = u_K h_K$ for any order $\mathcal{O} \supset \mathfrak{o}_K$ of special level type $(N_1^\Sigma, N_2^\Sigma, M)$. Now from Theorem 1.3, we have

$$A(N_1^\Sigma, N_2^\Sigma) = N^\Sigma h_K^2 (u_K - 2^{\omega'(N_2^\Sigma)} c(N_1^\Sigma, N_2^\Sigma, M)).$$

Then the theorem follows from the facts that

$$\sum_{\Sigma} (-1)^{\#\Sigma} N^{\Sigma} \left(\prod_{p|N_2, p\in\Sigma} \frac{1}{1+p^{-1}} \right) = N \prod_{p|N_1'} \left(1 - \frac{1}{p^2}\right) \cdot \prod_{p|N_2} \frac{p}{p+1},$$

and

$$\sum_{\Sigma} (-1)^{\#\Sigma} N^{\Sigma} \left(\prod_{p|N_2, p\in\Sigma} \frac{1}{1+p^{-1}} \right) 2^{\omega'(N_2^{\Sigma})} c(N_1^{\Sigma}, N_2^{\Sigma}, M)$$

is 0 unless $N_1' = 1$ and N_2 is odd, in which case it is

$$Nc(N_1, N_2, M) = 12 \prod_{p|N_1N_2} \frac{1}{1-p^{-1}} \prod_{p|N_2M} \frac{1}{1+p^{-1}}.$$

□

Proof of Corollary 1.5. Keeping the notation of the previous proof, define $A^{\chi}(N_1)$ and $A_0^{\chi}(N_1)$ to be the "χ-parts" of $A(N_1, 1)$ and $A_0(N_1, 1)$—i.e., the analogous expressions with only a single sum over f in $\mathcal{F}(N; \chi)$ or $\mathcal{F}_0(N; \chi)$ for a fixed χ. Then

$$A_0^{\chi}(N_1) = \sum_{\Sigma} (-1)^{\#\Sigma} A^{\chi}(N_1^{\Sigma})$$

By (5.10) and Corollary 4.5, in the stable range we have

$$A^{\chi}(N_1^{\Sigma}) = N^{\Sigma}(u_K h_K - \delta_{\chi,1} h_K^2 c(N_1^{\Sigma}, 1, M)).$$

Then the formula follows by the calculations from the previous proof. □

5.7. **Lower bounds and effective nonvanishing.** We keep to the situation of $F = \mathbb{Q}$, and obtain exact lower bounds for single averages of L-values in special situations. In particular, this will give the effective nonvanishing results stated in the introduction.

We first remark that, when $N = N_1 > 4$ is squarefree, (5.10) and Corollary 4.5 imply

$$\frac{C(K;N)}{4\pi^2} \sum_{f\in\mathcal{F}^{\mathrm{new}}(N)} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f\otimes\eta_K)}{(f,f)} = \sum_{i=1}^{n} w_i h_i - h_K^2 c(N_1, 1, 1) \tag{5.12}$$

$$\geq h_K(u_K - \frac{12h_K}{\varphi(N)}).$$

Here $\varphi(N)$ is the Euler totient function. In particular, we get that $L(\frac{1}{2}, f)L(\frac{1}{2}, f\otimes\eta_K) \neq 0$ for some $f \in S_2^{\mathrm{new}}(N)$ as soon as $\varphi(N) > \frac{12h_K}{u_K}$.

Proposition 5.9. *Fix a prime p and let $r > 1$ be odd or 2. Let $N = N_0 p^r \geq 11$ where N_0 is a squarefree product of an even number of primes (possibly $N_0 = 1$) coprime to p. If r is odd and $N \neq 27$, we have*

$$\frac{C(K;N)}{4\pi^2} \sum_{f\in\mathcal{F}^{\mathrm{new}}(N)} \frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f\otimes\eta_K)}{(f,f)} \geq h_K(u_K - \frac{3h_K}{p^2}).$$

If $r = 2$, we have

$$\frac{C(K;N)}{4\pi^2} \sum_{f\in\mathcal{F}^{\mathrm{new}}(N)} \Lambda_p(f) \frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f\otimes\eta_K)}{(f,f)} \geq h_K(u_K - \frac{3h_K}{p+1}(1 + \frac{4}{(p-1)\varphi(N_0)})).$$

Proof. We realize $N = N_1N_2$ with our running conventions, so $M = 1$ and $D_B = N_0p$. Let

$$A_0(N) = \frac{C(K;N)}{4\pi^2} \sum_{f\in\mathcal{F}_0(N;1_K)} \Lambda_{N_2}(f) N_f \frac{L_{\mathrm{fin}}(\frac{1}{2}, f, 1_K)}{(f,f)},$$

and $A(N)$ denote the corresponding sum over $f \in \mathcal{F}(N;1_K)$. (These are the $\chi = 1_K$ parts of the sums denoted $A_0(N_1,N_2)$ and $A(N_1,N_2)$ in the previous section.) Note that $\mathcal{F}_0(N;1_K) \subset \mathcal{F}^{\mathrm{new}}(N)$, so by non-negativity of central L-values, the left hand sides above are at least $\frac{A_0(N)}{N}$. Hence it suffices to show the right hand sides are lower bounds for $\frac{A_0(N)}{N}$.

We have $A_0(N) = A(N) - \frac{1}{1+p^{-1}}A(N/p)$ or $A_0(N) = A(N) - A(N/p^2)$, according to whether $r = 2$ or r is odd. As in the proof of Theorem 1.4, our restrictions on N imply that, for a special order of level N/p or N/p^2, the quantity $\sum w_ih_i \le 3h_K$. Now the asserted upper bound on $\frac{A_0(N)}{N}$ follows from using Corollary 4.5 for a lower bound on $A(N)$ and upper bounds on $A(N/p)$ and $A(N/p^2)$. □

The $N_0 = 1$ case of this proposition, combined with (5.12), yields Corollary 1.2, and an analogue for levels $N = p^2$. E.g., if K is ramified at $p \ge 7$, then one gets a non-vanishing result in level p^2 provided $p + 1 > \frac{5h_K}{u_K}$.

Proposition 5.10. *Suppose $N_1 > 3$ is squarefree, $N_2 = 1$ and $M = p > 3$. Then*

$$\frac{C(K;N)}{4\pi^2} \sum_{f\in\mathcal{F}^{\mathrm{new}}(N)} \Lambda_p(f,1_K) \frac{L_{\mathrm{fin}}(\frac{1}{2}, f)L_{\mathrm{fin}}(\frac{1}{2}, f\otimes\eta_K)}{(f,f)} \ge h_K(u_K - \frac{3h_K\Xi(p)}{p}),$$

where $\Xi(p) := 2\left(\frac{1+p^{-1/2}}{1-p^{-1}}\right)^2$.

Proof. Define quantities $A_0(N)$ and $A(N)$ as in the previous proof. By Deligne's bound $a_p(f) \le 2\sqrt{p}$ and (5.11), we have

$$\frac{1}{1+p^{-1}} \le \Lambda_p(f,1_K) \le \Xi(p)$$

for $f \in \mathcal{F}(N_1;1_K)$. Then use $A_0(N) \ge A(N) - \Xi(p)A(N/p)$ and proceed as before. □

Note $\Xi(p)$ is a decreasing function in p. In particular, $\Xi(p) < 4$ if $p \ge 13$ and $\Xi(p) < 3$ if $p \ge 31$. Hence one of the L-values appearing on the left of Proposition 5.10 must be nonzero if $p \ge \max\{\frac{12h_K}{u_K}, 13\}$ or if $p \ge \max\{\frac{9h_K}{u_K}, 31\}$.

6. Examples

In this section, we illustrate our formulas with a few examples when $F = \mathbb{Q}$. In each of examples, we numerically computed the relevant L-values to serve as a quality check for our formulas. We performed our calculations using a combination of Magma [BCP97] and Sage [Sage]. We also used Dan Collins' Sage code to compute Petersson norms (see [Col]).

Example 6.1. Let $N = D_B = 11$, so $\mathcal{O}$ is a maximal order. Then $n = 2$ and there are 2 maximal orders up to isomorphism, one with 4 units and one with 6 units. Order the ideal classes $\mathcal{I}_1, \mathcal{I}_2$ so that $w_1 = 2$ and $w_2 = 3$. We may write $S(\mathcal{O}) = \langle\varphi\rangle$, where $\varphi(x_1) = 2$, $\varphi(x_2) = -3$. Then φ corresponds to the unique newform $f \in S_2(11)$.

Quadratic forms associated to $\mathcal{I}_1$ and $\mathcal{I}_2$ are $Q_1(x,y,z,w) = \frac{(2x+z)^2+(2y+w)^2+11z^2+11w^2}{4}$ and $Q_2(x,y,z,w) = \frac{(4x+z+2w)^2+(4y+2z-w)^2+11z^2+11w^2}{4}$. By (3.7), the diagonal Brandt matrix entries are $a_{11}(n) = \frac{r_{Q_1}(n)}{4}$ and $a_{22}(n) = \frac{r_{Q_2}(n)}{6}$. Eichler's work on the basis problem shows $a_n(f) = 3a_{22}(n) - 2a_{11}(n) = \frac{r_{Q_2}(n) - r_{Q_1}(n)}{2}$ for all n. Comparing this with the trace of the p-th Brandt matrix shows, for $p \neq 11$, we have $a_p(f) = a_{11}(p) + a_{22}(p) - p - 1 = \frac{5r_{Q_1}(p) - 12p - 12}{8}$.

First suppose $K = \mathbb{Q}(i)$. Then necessarily $h_1 = 1$ and $h_2 = 0$ so $\frac{|P_K(\varphi)|^2}{(\varphi,\varphi)} = \frac{4}{5}$. From (5.10) or Theorem 1.3, we see

$$\frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_K)}{(f,f)} = \frac{4\pi^2}{5}.$$

Consider a prime $p \neq 11$. Then by Theorem 1.3 with $m = p$, we get

$$a_p(f) = \frac{5}{4\pi^2} \frac{a_p(f) L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_K)}{(f,f)} = \frac{5r_{Q_1}(p) - 12p - 12}{8}.$$

This independently recovers the above formula for $a_p(f)$ which arose from linear relations of theta series.

Next suppose $K = \mathbb{Q}(\sqrt{-11})$ which also has class number 1. One can check that $\mathfrak{o}_K$ embeds in $\mathcal{O}_\ell(\mathcal{I}_1)$, so $h_1 = 1$ and $h_2 = 0$. Again we have $\frac{|P_K(\varphi)|^2}{(\varphi,\varphi)} = \frac{4}{5}$. Then (5.10) or Theorem 1.3 implies that

$$C(\mathbb{Q}(\sqrt{-11}), 11) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_{\mathbb{Q}(\sqrt{-11})}) = C(\mathbb{Q}(i), 11) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_{\mathbb{Q}(i)}).$$

Hence

$$\frac{L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_{\mathbb{Q}(\sqrt{-11})})}{L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_{\mathbb{Q}(i)})} = \frac{4}{\sqrt{11}},$$

which agrees with numerical calculations. This provides a check on our local factors when K is ramified at a prime $p | N_1$.

Example 6.2. Let $D_B = 11$ and $N = 22$. Then $n = 3$ and we may order the ideal classes so that $w_1 = 2, w_2 = w_3 = 1$. We can take an orthogonal basis of $S(\mathcal{O})$ to be $\{\varphi_1, \varphi_2\}$ where $\varphi_1(x_1) = \varphi_1(x_2) = 2$, $\varphi_1(x_3) = -3$ and $\varphi_2(x_1) = 2$, $\varphi_2(x_2) = -1$, $\varphi_2(x_3) = 0$. (There is a maximal order $\mathcal{O}_B \supset \mathcal{O}$ such that as double cosets $\mathrm{Cl}(\mathcal{O}_B) = \{x_1 \sqcup x_2, x_3\}$ and φ_1 the old eigenform in $S(\mathcal{O}_B)$ denoted φ in the previous example.) Here $S(\mathcal{O})$ is the $\hat{\mathcal{O}}^\times$-invariant subspace of the cuspidal representation π of $B^\times(\mathbb{A}_\mathbb{Q})$ corresponding to the newform $f \in S_2(11)$.

Suppose $K = \mathbb{Q}(\sqrt{-15})$ so $h_K = 2$. We compute $h_1 = 0$ and $h_2 = h_3 = 1$. Hence

$$\frac{|P_K(\varphi_1)|^2}{(\varphi_1,\varphi_1)} + \frac{|P_K(\varphi_2)|^2}{(\varphi_2,\varphi_2)} = \frac{1}{15} + \frac{1}{3} = \frac{2}{5}.$$

Since $a_2(f) = -2$, we see $\Lambda_2(f, 1_K) = 4$ from (5.11). Then (5.10) tells us that

$$\frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_K)}{(f,f)} = \frac{8\pi^2}{5\sqrt{15}}.$$

If χ is the nontrivial character of $\mathrm{Cl}(K)$, we see

$$\frac{|P_{K,\chi}(\varphi_1)|^2}{(\varphi_1,\varphi_1)} + \frac{|P_{K,\chi}(\varphi_2)|^2}{(\varphi_2,\varphi_2)} = \frac{25}{15} + \frac{1}{3} = 2.$$

Since the Hilbert class field H_K of K is unramified at a prime $\mathfrak{p}_2$ of K lying above 2, we see $\chi(\mathfrak{p}_2) = -1$, and get $\Lambda_2(f, \chi) = \frac{4}{5}$. Thus (5.10) tells us that

$$\frac{L_{\mathrm{fin}}(\frac{1}{2}, f, \chi)}{(f, f)} = \frac{8\sqrt{15}\pi^2}{3}.$$

We numerically verified both of these L-values in Magma in terms of elliptic curve base change L-values. Namely, if E is an elliptic curve of conductor 11 associated to f. Then $L(\frac{1}{2}, f, 1_K) = L(\frac{1}{2}, E_K)$, and $L_{\mathrm{fin}}(\frac{1}{2}, f, \chi) = L(\frac{1}{2}, E_{H_K})/L(\frac{1}{2}, E_K)$.

While we could have obtained these L-values using the maximal order $\mathcal{O}_B$, this example provides a check on our formulas (5.4), (5.11) for the local factors $\Lambda_p(f, \chi)$ when $p|M$.

Example 6.3. Let $N = 27$, so $D_B = 3$. Since $\dim S_2(27) = 1$, we have $n = 2$. We also have $m(\mathcal{O}) = \frac{3}{2}$, so we can order $\mathrm{Cl}(\mathcal{O})$ such that $w_1 = 2, w_2 = 1$. Thus $S(\mathcal{O})$ is spanned by the function φ given by $\varphi(x_1) = 2$, $\varphi(x_2) = -1$, and φ corresponds to the unique newform $f \in S_2(27)$.

Suppose $K = \mathbb{Q}(i)$. Then $h_1 = 1, h_2 = 0$, so $\frac{|P_K(\varphi)|^2}{(\varphi,\varphi)} = \frac{4}{3}$, and thus by Theorem 5.5 (or (5.10)) we have

$$\frac{L_{\mathrm{fin}}(\frac{1}{2}, f) L_{\mathrm{fin}}(\frac{1}{2}, f \otimes \eta_K)}{(f, f)} = \frac{4\pi^2}{3},$$

which matches numerical computations. Note $N = 27$ is excluded from Theorem 1.1 and this example shows (1.2) does not hold when $N = 27$.

Example 6.4. Let $N = 75$ with $D_B = 5$. This corresponds to the triple $(1, 25, 3)$, which is balanced so each $w_i = 1$. Here $n = 8$ with $\dim \mathrm{Eis}(\mathcal{O}) = 2$. Let g be the unique newform in $S_2(15)$. Note $S_2^{\mathrm{new}}(75)$ is generated by 3 newforms, say f_1, f_2, f_3. One of them, say f_1, is a quadratic twist of g, and thus locally a ramified twist of Steinberg at 5. The other two, f_2 and f_3, are quadratic twists of each other and minimal at 5, whence locally supercuspidal at 5. We remark that Theorem 3.5 tells us that $S(\mathcal{O}) \simeq \mathbb{C}g \oplus \mathbb{C}f_1 \oplus 2\mathbb{C}f_2 \oplus 2\mathbb{C}f_3$.

Let $K = \mathbb{Q}(\sqrt{-5})$, so $h_K = 2$ and K is split at 3 and ramified at 5. Then Theorem 1.3 says

$$\frac{5\sqrt{5}}{12\pi^2} \sum_{\chi \in \widehat{\mathrm{Cl}}(K)} \left(\sum_{i=1}^{3} \Lambda_5(f_i) \frac{L_{\mathrm{fin}}(\frac{1}{2}, f_i, \chi)}{(f_i, f_i)} + \frac{1}{5} \frac{L_{\mathrm{fin}}(\frac{1}{2}, g, \chi)}{(g, g)} \right) = 3.$$

From (5.8), we see that $\Lambda_5(f_1) = \frac{24}{25}$ and $\Lambda_5(f_2) = \Lambda_5(f_3) = \frac{6}{5}$. This agrees with numerical calculations of base change elliptic curve L-values.

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