A comparison of automorphic and Artin L-series of GL(2)-type agreeing at degree one primes

Kimball Martin and Dinakar Ramakrishnan

To James Cogdell, with friendship and admiration

Introduction

Let F be a number field and ρ an irreducible Galois representation of Artin type, i.e., ρ is a continuous $\mathbb{C}$-representation of the absolute Galois group Γ_F. Suppose π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that the L-functions $L(s, \rho)$ and $L(s, \pi)$ agree outside a set S of primes. (Here, these L-functions denote Euler products over just the finite primes, so that we may view them as Dirichlet series in a right half plane.) When S is finite, the argument in Theorem 4.6 of [**DS74**] implies these two L-functions in fact agree at all places (cf. Appendix A of [**Mar04**]). We investigate what happens when S is infinite of relative degree ≥ 2, hence density 0, for the test case $n = 2$. Needless to say, if we already knew how to attach a Galois representation ρ' to π with $L(s, \rho') = L(s, \pi)$ (up to a finite number of Euler factors), as is the case when F is totally real and π is generated by a Hilbert modular form of weight one ([**Wil88**], [**RT83**]), the desired result would follow immediately from Tchebotarev's theorem, as the Frobenius classes at degree one primes generate the Galois group. Equally, if we knew that ρ is modular attached to a cusp form π', whose existence is known for $F = \mathbb{Q}$ and ρ odd by Khare–Wintenberger ([**Kha10**]), then one can compare π and π' using [**Ram94**]. However, the situation is more complex if F is not totally real or (even for F totally real) if ρ is even. Hopefully, this points to a potential utility of our approach.

We prove the following

Theorem A. *Let F/k be a cyclic extension of number fields of prime degree p. Suppose ρ and π are as above with their L-functions agreeing at all but a finite number of primes P of F of degree one over k. Then $L(s, \rho) = L(s, \pi)$, and moreover, at each place v, π_v is associated to ρ_v according to the local Langlands correspondence. In particular, π is tempered, i.e., satisfies the Ramanujan conjecture everywhere, and $L(s, \rho)$ is entire.*

The curious aspect of our proof below is that we first prove the temperedness of π, and then the entireness of $L(s, \rho)$, before concluding the complete equality

---

2010 *Mathematics Subject Classification.* Primary 11R39; Secondary 11F70, 11F80.

The first author was partially supported by Simons Collaboration Grant 240605.
The second author was partially supported by NSF grant DMS-1001916.

of the global L-functions. Here are roughly the steps involved. As in the case when S is finite, consider the quotients $L(s,\pi)/L(s,\rho)$ and $L(s,\pi\times\overline{\pi})/L(s,\rho\otimes\overline{\rho})$, which reduce in $\mathrm{Re}(s)>1$ to a quotient of Euler factors involving (outside finitely many) those of degree ≥ 2. Then we work over $F'=F(\sqrt{-1})$ and, by applying [**Ram15**], we find a suitable p-power extension K of F' over which even the degree ≥ 2 Euler factors of the base changes π_K and ρ_K agree (outside finitely many), which furnishes the temperedness of π. Then we carefully analyze certain degree 8 extensions K of F over which we prove that $L(s,\rho_K)$ is necessarily entire, which is not *a priori* obvious. In fact we prove this for sufficiently many twists $\rho\otimes\chi$ for unitary characters χ of K. Applying the converse theorem over K, we then conclude the existence of an automorphic form ${}_K\Pi$ of $\mathrm{GL}(2)/K$ corresponding to ρ_K. We then identify π_K with ${}_K\Pi$. The final step is to descend this correspondence $\rho_K\leftrightarrow\pi_K$ down to F, first down to F' (by varying K/F') and then down to F.

The analytic estimate we use for the inverse roots α_P of Hecke (for π) is that it is bounded above by a constant times $N(P)^{1/4-\delta}$ for a uniform $\delta>0$, and in fact much stronger results are known for GL(2) ([**KS02**], [**Kim03**], [**BB11**]). We need the same estimate (of exponent $1/4-\delta$) for $\mathrm{GL}(n)$ in general to prove even a weaker analogue of our theorem. (There is no difficulty for $n=3$ if π is essentially selfdual.) There is a nice estimate for general n due to Luo, Rudnick and Sarnak ([**LRS99**]), giving the exponent $1-1/2n^2$, but this does not suffice for the problem at hand. We can still deduce that for any place v, $L(s,\pi_v)$ has no pole close to $s=0$ to its right, and similarly, we can rule out poles of (the global L-function) $L(s,\rho)$ in a thin region in the critical strip.

One can ask the same question more generally for ℓ-adic representations ρ_ℓ of Γ_F satisfying the Fontaine-Mazur conditions, namely that the ramification is confined to a finite number of primes and that ρ_ℓ is potentially semistable. For the argument of this Note to apply, one would need to know in addition that (i) the Frobenius eigenvalues are pure, i.e., of absolute value $N(P)^{w/2}$ for a fixed weight w for all but finitely many unramified primes P, and (ii) for any finite solvable Galois extension K/F, the L-function of the restrictions of ρ_ℓ and $\rho_\ell\otimes\rho_\ell^\vee$ to Γ_K admit a meromorphic continuation and functional equation of the expected type, which is known in the Artin case by Brauer. When F is totally real, we will need this for at least any compositum K of a totally real (solvable over F) K' with $F(\sqrt{-1})$. For F totally real with ρ_ℓ odd (2-dimensional), crystalline and sufficiently regular relative to ℓ, these conditions may follow from the potential modularity results of Taylor.

The first author thanks Jim Cogdell for being a wonderful teacher and for his encouragement. The second author would like to thank Jim Cogdell for many years of stimulating mathematical conversations and friendship. We also thank the referee for helpful comments which improved the exposition.

1. Notations and preliminaries

Let F/k be a cyclic extension of number fields of prime degree p, $\rho:\Gamma_F\to\mathrm{GL}_2(\mathbb{C})$ a continuous, irreducible representation, and π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω_π.

Suppose $L(s,\rho_v)=L(s,\pi_v)$ at almost all places v of F of degree 1 over k. Then, at such v, $\det(\rho_v)=\omega_{\pi_v}$. Consequently, $\det(\rho)=\omega_\pi$ globally. Hence ω_π is finite order, so π is unitary.

Now we will recall some basic results which we will use below.

LEMMA 1.1 (Landau). *Let $L(s)$ be a Dirichlet series with Euler product which converges in some right half-plane. Further suppose that $L(s)$ is of positive type, i.e., that $\log L(s)$ is a Dirichlet series with non-negative coefficients. Let s_0 be the infimum of all the real s_1 such that $L(s)$ is holomorphic and non-vanishing in $\mathrm{Re}(s) > s_1$. Then, if s_0 is finite, $L(s)$ has a pole at s_0, and has no zero to the right of s_0.*

In other words, for such an $L(s)$ of positive type, when we approach $s = 0$ on the real line from the right, from a real point of absolute convergence, we will not hit a zero of $L(s)$ until we hit a pole.

We will also need a suitable (weaker) bound towards the Generalized Ramanujan Conjecture for GL(2), which asserts temperedness of π_v everywhere, i.e., that $L(s, \pi_v)$ has no poles on $\mathrm{Re}(s) > 0$ for all v. We will need it for any finite solvable extension K of F.

THEOREM 1.2. *Let π be an isobaric automorphic representation of $GL_2(\mathbb{A}_K)$, for a number field K.*

(a) *If π is (unitary) cuspidal, there exists a $\delta < \frac{1}{4}$ such that, for any place v of K, $L(s, \pi_v)$ has no pole for $\mathrm{Re}(s) > \delta$.*
(b) *If π is an isobaric sum of unitary Hecke characters of K, then π is tempered, and so for any place v of K, $L(s, \pi_v)$ has no pole for $\mathrm{Re}(s) > 0$.*

The fact that $L(s, \pi_v)$ has no pole for $\mathrm{Re}(s) \geq \frac{1}{4}$ is originally due to Gelbart–Jacquet [**GJ78**] using Sym^2 L-functions. Subsequently, more precise bounds of $\delta = \frac{1}{9}$ were given by Kim–Shahidi [**KS02**], and later $\delta = \frac{7}{64}$ by Kim–Sarnak [**Kim03**] and Blomer–Brumley [**BB11**], using respectively the Sym^3 and Sym^4 L-functions of π.

REMARK 1. *Some of the estimates towards the Generalized Ramanujan Conjecture cited above are just stated for v such that π_v unramified. However, for unitary cuspidal π on $\mathrm{GL}(2)/F$, the general case easily reduces to the unramified situation. Indeed, if π_v is not tempered, we may write it as an irreducible principal series $\pi_v = \pi(\mu_1|\det|^t, \mu_2|\det|^{-t})$, where μ_1 and μ_2 are unitary, and t is real and non-zero. Since $\check{\pi}_v \simeq \bar{\pi}_v$ by unitarity, we see that $\{\mu_1^{-1}|\det|^{-t}, \mu_2^{-1}|\det|^t\} = \{\bar{\mu}_1|\det|^t, \bar{\mu}_2|\det|^{-t}\}$. Thus $\mu_1^{-1} = \bar{\mu}_2 = \mu_2^{-1}$, i.e., $\mu_1 = \mu_2$. Hence, $\pi_v = \pi(|\det|^t, |\det|^{-t}) \otimes \mu_1$ is just a unitary twist of an unramified principal series. Moreover, as μ_1 is a finite order character times $|\cdot|^{ix}$ for some $x \in \mathbb{R}$, we may choose a global unitary character λ of (the idele classes of) F such that $\lambda_v = \mu_1$, and so $\pi \otimes \lambda^{-1}$ is unitary cuspidal with its v-component unramified, resulting in a bound for t.*

The next result we need follows from Lemma 4.5 and the proof of Proposition 5.3 in [**Ram15**].

PROPOSITION 1.3. *Fix any $r \geq 1$. There exists a finite solvable extension K/k containing F such that any prime w of K lying over a degree p prime v of F/k has degree $\geq p^r$ over k.*

The three results above will be used to show π is tempered. In fact, a more explicit version of Proposition 1.3 for quadratic fields is used in the other two parts of the proof as well: showing $L(s,\rho)$ is entire, and showing ρ is modular. For deducing modularity, we will need the following version of the converse theorem à la Jacquet–Langlands for 2-dimensional Galois representations.

Denote by $L^*(s,\rho)$ and $L^*(s,\pi)$ the completed L-functions for ρ and χ.

THEOREM 1.4. [**JL70**, Theorem 12.2] *Suppose K/F is a finite Galois extension and ρ is a 2-dimensional representation of the Weil group $W_{K/F}$. If the L-functions $L^*(s,\rho\otimes\chi)$ are entire and bounded in vertical strips for all idele class characters χ of F, then ρ is modular.*

In our case of ρ being an Artin representation, it follows from a theorem of Brauer that each $L^*(s,\rho\otimes\chi)$ is a ratio of entire functions of finite order. Thus knowing $L^*(s,\rho\otimes\chi)$ is entire implies it is of finite order, whence bounded in vertical strips. Hence we will only need to check entireness to use this converse theorem.

We remark Booker and Krishnamurthy proved a converse theorem [**BK11**] requiring only a weaker hypothesis.

We say the global representations ρ and π correspond if they do in the sense of the strong Artin conjecture, i.e., that their local L-factors agree almost everywhere. For GL(2), we show that this type of correspondence implies the stronger conclusion that we assert in Theorem A, that ρ_v and π_v are associated by the local Langlands correspondence at all v. The local Langlands correspondence for GL(2) was established by Kutzko [**Kut80**], and is characterized uniquely by matching of local L- and ϵ- factors of twists by finite-order characters of GL(1) (cf. [**JL70**, Corollary 2.19]).

PROPOSITION 1.5. *Suppose that $L(s,\rho_v)=L(s,\pi_v)$ for almost all v. Then ρ_v and π_v correspond in the sense of local Langlands at all v (finite and infinite).*

PROOF. This is a refinement of the argument in Theorem 4.6 of [**DS74**] and [**Mar04**, Appendix A].

At a finite place v where ρ_v and π_v are both unramified, this is immediate as ρ_v and π_v are determined by $L(s,\rho_v)$ and $L(s,\pi_v)$. So we only need to show this for $v|\infty$ and $v\in S$, where S is the set of nonarchimedean places v at which ρ or π is ramified or $L(s,\rho_v)\neq L(s,\pi_v)$.

Observe that, by class field theory, $\det\rho$ corresponds to an idele class character ω over F. From $L(s,\rho_v)=L(s,\pi_v)$, we know $\omega_v=\omega_{\pi_v}$ for all finite $v\notin S$, and therefore $\omega=\omega_\pi$ by Hecke's strong multiplicity one for GL(1). That is to say, $\det\rho$ and ω_π correspond via class field theory.

We will use the fact that if T is a finite set of places and μ_v, $v\in T$, are finite-order characters, there exists a finite-order idele class character χ globalizing the μ_v's, i.e., $\chi_v=\mu_v$ for $v\in T$. This is a standard application of the Grunwald–Wang theorem.

First we establish the local Langlands correspondence for $v|\infty$. Choose a finite-order idele class character χ which is highly ramified at each $u\in S$ and trivial at each infinite place. Then, for every $u<\infty$, we have $L(s,\rho_u\otimes\chi_u)=L(s,\pi_u\otimes\chi_u)$. Consequently, comparing the functional equations for $L(s,\rho\otimes\chi)$ and $L(s,\pi\otimes\chi)$ gives

$$\frac{L_\infty(1-s,\bar{\pi})}{L_\infty(1-s,\bar{\rho})}=\frac{\epsilon(s,\pi\otimes\chi)}{\epsilon(s,\rho\otimes\chi)}\frac{L_\infty(s,\pi)}{L_\infty(s,\rho)}. \tag{1}$$

Since the poles of the L-factors on the right hand side of (1) lie in $\mathrm{Re}(s) < \frac{1}{2}$, whereas the poles of the L-factors on the left hand side lie in $\mathrm{Re}(s) > \frac{1}{2}$, we can conclude $L_\infty(s,\pi) = L_\infty(s,\rho)$ as they must have the same poles (cf. [**Mar04**, Appendix A]).

Note that for $v|\infty$, ρ_v must be a direct sum of two characters with are 1 or sgn. If v is complex, of course $\rho_v = 1 \oplus 1$. Consequently

$$L_\infty(s,\pi) = L_\infty(s,\rho) = \Gamma_{\mathbb{R}}(s)^a \Gamma_{\mathbb{R}}(s+1)^b$$

for some non-negative integers a, b.

We claim that, at any archimedean place v, the local factors of π and ρ agree. Suppose not. Let v be an errant place, which cannot be complex since the only option is $\Gamma_{\mathbb{C}}(s)^2 = \Gamma_{\mathbb{R}}(s)^2\Gamma_{\mathbb{R}}(s+1)^2$ for either factor. So v is real. Since the local factors $L(s,\pi_v)$ are $L(s,\rho_v)$ are different by assumption, at least one of them, say $L(s,\pi_v)$, must be of the form $\Gamma_{\mathbb{R}}(s)^2$ or $\Gamma_{\mathbb{R}}(s+1)^2$. Call this local factor $G_1(s)$, and the other local factor, say $L(s,\rho_v)$, $G_2(s)$. Now twist by a finite-order character χ of F which is sufficiently ramified at the bad finite places in S, equals 1 at the archimedean places other than v, and at v equals sgn. Then $G_1(s)$ (say $L(s,\pi_v\otimes \mathrm{sgn})$) becomes $G_1(s+\delta)$, with $\delta \in \{1,-1\}$. Similarly we see that $G_2(s)$ (say $L(s,\rho_v \otimes \mathrm{sgn})$) becomes $G_2(s+\delta')$ with $\delta' \in \{1,0,-1\}$, where $\delta'=0$ if and only if $G_2(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$. The total archimedean identity $L_\infty(s,\pi\otimes\chi) = L_\infty(s,\rho\otimes\chi)$ persists in this case, and comparing with $L_\infty(s,\pi) = L_\infty(s,\rho)$ gives $G_1(s+\delta)/G_1(s) = G_2(s+\delta')/G_2(s)$. The only way this can happen is if $G_1(s+\delta) = G_2(s+\delta')$, which forces $G_1(s) = G_2(s)$, contradicting the assumption. Hence the claim.

This in fact implies that π_v is associated to ρ_v by the local Langlands correspondence for v archimedean. Namely, for complex v, $L(s,\pi_v) = \Gamma_{\mathbb{C}}(s)$ and π_v unitary (or $\omega_{\pi_v} = \det\rho_v = 1$) implies $\pi_v = \pi(1,1)$. For real v, if $\pi_v = \pi(\mu_1,\mu_2)$ is a unitarizable principal series then $\mu_i = \mathrm{sgn}^{m_i}|\cdot|^{s_i}$ with $|s_i| < \frac{1}{2}$. Hence $L(s,\mu_i) = \Gamma_{\mathbb{R}}(s)$ implies $\mu_i = 1$ and $L(s,\mu_i) = \Gamma_{\mathbb{R}}(s+1)$ implies $\mu_i = \mathrm{sgn}$. Consequently, if $L(s,\pi_v) = \Gamma_{\mathbb{R}}(s)^c\Gamma_{\mathbb{R}}(s+1)^d$ with $c+d=2$, then π_v is an isobaric sum of c copies of 1 and d copies of sgn, and therefore matches ρ_v in the sense of Langlands.

Finally, consider a finite place $v \in S$. Let μ be a finite-order character of $F_v^\times$. Let χ be an idele class character which is highly ramified at all $u \in S-\{v\}$ such that $\chi_v = \mu$. For all $u \notin S$, we have $L(s,\pi_u\otimes\chi_u) = L(s,\rho_u\otimes\chi_u)$ and $\epsilon(s,\pi_u\otimes\chi_u,\psi_u) = \epsilon(s,\rho_u\otimes\chi_u,\psi_u)$ by the local Langlands correspondence. But the same is also true for $u \in S-\{v\}$ by a result of Jacquet and Shalika [**JS85**] which is often called stability of γ-factors: for twists by sufficiently ramified characters, the L-factors are 1 and the ϵ-factors are equal since $\omega_{\pi_u} = \det_{\rho_u}$. Hence the same comparison of functional equations that led to (1) in the archimedean case gives us

$$\frac{L(1-s,\bar\pi_v\otimes\bar\mu)}{L(1-s,\bar\rho_v\otimes\bar\mu)} = \frac{\epsilon(s,\pi_v\otimes\mu,\psi_v)}{\epsilon(s,\rho_v\otimes\mu,\psi_v)}\frac{L(s,\pi_v\otimes\mu)}{L(s,\rho_v\otimes\mu)}. \tag{2}$$

Again, a comparison of the poles implies $L(s,\pi_v\otimes\mu) = L(s,\rho_v\otimes\mu)$, and similarly that $L(1-s,\bar\pi_v\otimes\bar\mu) = L(1-s,\bar\rho_v\otimes\bar\mu)$. Consequently, $\epsilon(s,\pi_v\otimes\mu,\psi_v) = \epsilon(s,\rho_v\otimes\mu,\psi_v)$. Since this is true for all μ, we conclude that π_v and ρ_v must correspond in the sense of local Langlands. □

2. Temperedness

To show π is tempered, we will make use of solvable base change.

Let K be a solvable extension of F. Denote by ρ_K the restriction of ρ to Γ_K. Denote by π_K the base change of π to $\mathrm{GL}_2(\mathbb{A}_K)$, whose automorphy we know by Langlands ([**Lan80**]). More precisely, π_K is either cuspidal or else an isobaric sum of two unitary Hecke characters of K.

Put

$$\Lambda_K(s) = \frac{L^*(s, \pi_K \times \bar{\pi}_K)}{L^*(s, \rho_K \times \bar{\rho}_K)}. \tag{3}$$

If there is no confusion, we will write Λ instead of Λ_K. This has a factorization,

$$\Lambda(s) = \prod_v \Lambda_v(s), \quad \text{where } \Lambda_v(s) = \frac{L(s, \pi_{K,v} \times \bar{\pi}_{K,v})}{L(s, \rho_{K,v} \times \bar{\rho}_{K,v})},$$

for any place v of K, and is *a priori* analytic with no zeroes for $\mathrm{Re}(s) > 1$. What is crucial for us is that the logarithms of the numerator and denominator of the non-archimedean part of Λ are Dirichlet series with positive coefficients, so we will be able to apply Landau's lemma.

For an arbitrary set S of places of K, we write $\Lambda_S(s) = \prod_{v \in S} \Lambda_v(s)$. Denote by S_j the set of finite places v of K of degree j over k for which ρ_K, π_K and K are unramified, but $\rho_{K,v}$ and $\pi_{K,v}$ do not correspond. Then we can write

$$\Lambda(s) = \Lambda_\Sigma(s) \prod_{j \geq 2} \Lambda_{S_j}(s), \tag{4}$$

where Σ is a finite set containing S_1, the archimedean places, and the set of finite places where π, ρ or F is ramified.

Then $\Lambda(s)$ satisfies a functional equation of the form

$$\Lambda(s) = \epsilon(s)\Lambda(1-s), \tag{5}$$

where $\epsilon(s)$ is an invertible holomorphic function on $\mathbb{C}$.

To show π is tempered, we will use the following lemma in two different places.

LEMMA 2.1. *Let δ be as in Theorem 1.2, and S_j as above for $K = F$. Then*
(i) $L_{S_j}(s, \pi \times \bar{\pi})$ has no poles or zeroes on $\mathrm{Re}(s) > \frac{1}{j} + 2\delta$*; and*
(ii) $L_{S_j}(s, \rho \times \bar{\rho})$ has no poles or zeroes on $\mathrm{Re}(s) > \frac{1}{j}$.

PROOF. Let us prove (i).

Suppose $v \in S_j$. Let $\alpha_{1,v}$ and $\alpha_{2,v}$ be the Satake parameters for π_v and $\beta_v = \max\{|\alpha_{1,v}|, |\alpha_{2,v}|\}$. Then

$$\log L(s, \pi_v \times \bar{\pi}_v) \leq \sum_{i=1}^{2} \sum_{l=1}^{2} \sum_{m \geq 1} \frac{(\alpha_{i,v}\bar{\alpha}_{l,v})^m}{m q_v^{ms}} \leq 4 \sum_{m \geq 1} \left(\frac{\beta_v^2}{q_v^s} \right)^m \leq 4 \sum_{m \geq 1} \frac{1}{q_v^{(s-2\delta)m}}, \tag{6}$$

where the last step follows from the bound in Theorem 1.2, which is equivalent to $\beta_v < q_v^\delta$. In particular, the local factor $L(s, \pi_v \times \bar{\pi}_v)$ (which is never zero) is holomorphic for $\mathrm{Re}(s) > 2\delta$.

Let $p_v^{f_v}$ denote the norm of the prime of k below v, where p_v is a rational prime. Then from (6), we see

$$\log L_{S_j}(s, \pi \times \bar{\pi}) \leq 4 \sum_{v \in S_j} \sum_{m \geq 1} \frac{1}{p_v^{f_v j(s-2/9)m}} \leq 4 \sum_{v \in S_j} \sum_{m \geq 1} \frac{1}{p_v^{j(s-2/9)m}}$$
$$\leq 4[k : \mathbb{Q}] \sum_{p_i} \sum_{m \geq 1} \frac{1}{(p_i^m)^{j(s-2/9)}} \leq 4 \sum_{n \geq 1} \frac{1}{n^{j(s-2/9)}},$$

where p_i runs over all primes in the penultimate inequality. This series converges absolutely, and uniformly in compact subsets of the region of $s \in \mathbb{C}$ with

$$\operatorname{Re}(s) > \frac{1}{j} + 2\delta. \tag{7}$$

Since $L_{S_j}(s, \pi \times \bar{\pi})$ is of positive type, we may apply Landau's lemma (Lemma 1.1) to conclude that it is also *non-zero* and holomorphic for $\operatorname{Re}(s) > \frac{1}{j} + 2\delta$, which implies (i). (Since this incomplete L-function has infinitely many Euler factors, it is not obvious that it is non-zero in this region without applying Landau.)

The argument for (ii) is the same, except that one uses the fact that the Frobenius eigenvalues for ρ_v lie on $\operatorname{Re}(s) = 0$ in place of Theorem 1.2. □

Proposition 2.2. *For each place v of F, π_v is tempered.*

Proof. By Theorem 1.2, for any finite set Σ, Λ_Σ has no poles for $\operatorname{Re}(s) \geq \frac{1}{2}$.

Fix N such that $N > (\frac{1}{2} - 2\delta)^{-1}$. Then Lemma 2.1 implies that $\Lambda_{S_j}(s)$ has no poles (or zeroes) in the region $\operatorname{Re}(s) \geq \frac{1}{2}$ for $j \geq N$. Fix r such that $p^r \geq N$. By Proposition 1.3, there is a solvable K/F such that each prime v of degree p in F/k splits into primes all of degree $\geq p^r$ in K/k. Then for all but finitely many primes v of degree $< N$ in K/k, we have $\Lambda_{K,v}(s) = 1$, so in fact we can rewrite (4) as

$$\Lambda_K(s) = \Lambda_{K,\Sigma}(s) \prod_{j \geq N} \Lambda_{K,S_j}(s), \tag{8}$$

for a finite set Σ.

Hence Lemma 2.1 implies Λ_K has no zeroes or poles for $\operatorname{Re}(s) \geq \frac{1}{2}$. Therefore, by the functional equation (5), Λ_K is entire and nowhere zero.

Now suppose π_v is non-tempered for some place v of F. Then π_v is an irreducible principal series $\pi_v = \pi(\mu_1, \mu_2)$ and $L(s, \pi_v) = L(s, \mu_1)L(s, \mu_2)$, so $L(s, \pi_v \times \bar{\pi}_v) = \prod_{i=1}^2 \prod_{j=1}^2 L(s, \mu_i \bar{\mu}_j)$.

Consider first $v < \infty$. Write $L(s, \mu_i) = (1 - \alpha_{i,v} q_v^{-s})^{-1}$ for $i = 1, 2$. Interchanging $\alpha_{1,2}$ and $\alpha_{2,v}$ if necessary, we may assume $|\alpha_{1,v}| > 1$. Then for any place w of K above v, $L(s, \pi_{K,w})^{-1}$ has a factor of the form $(1 - \alpha_{1,v}^{f_w} q_w^{-s})$ where $f_w = [K_w : F_v]$. Hence $L(s, \pi_{K,w} \times \bar{\pi}_{K,w})^{-1}$ has $(1 - |\alpha_{1,v}|^{2f_w} q_w^{-s})$ as a factor. Looking at the Dirichlet series $\log L(s, \pi_{K,w} \times \bar{\pi}_{K,w})$, Landau's lemma tells us $L(s, \pi_{K,w} \times \bar{\pi}_{K,w})$ has a pole at $s_0 = \frac{2 \log |\alpha_{1,v}|}{\log q_v}$.

Now consider $v | \infty$. We may assume $L(s, \mu_1)$ has a pole to the right of $\operatorname{Re}(s) = 0$. Again suppose w is a place of K with $w | v$. Either $L(s, \pi_{K,w} \times \bar{\pi}_{K,w})$ is $L(s, \mu_1 \bar{\mu}_1)$ (when v splits in K) or it equals $L(s, \mu_{1,\mathbb{C}} \bar{\mu}_{1,\mathbb{C}})$, where $\mu_{1,\mathbb{C}}(z) = \mu_1(z\bar{z})$. If we choose $s_0 > 0$ so that both $L(s, \mu_1 \bar{\mu}_1)$ and $L(s, \mu_{1,\mathbb{C}} \bar{\mu}_{1,\mathbb{C}})$ have poles in $\operatorname{Re}(s) \geq s_0$, then the same is true for $L(s, \pi_{K,w} \times \bar{\pi}_{K,w})$.

Hence, in either case $v < \infty$ or $v | \infty$, there exists $s_0 > 0$, independent of K, such that $L(s, \pi_{K,w} \times \bar{\pi}_{K,w})$ has a pole in $\operatorname{Re}(s) \geq s_0$. Another application of Landau's lemma tells us $L_\Sigma(s,\pi_K \times \bar\pi_K)\prod_{j\geq N} L_{S_j}(s,\pi_K\times\bar\pi_K)$ must also have a pole in $\mathrm{Re}(s)\geq s_0$.

Since Λ_K is entire, it must be that the denominator $L_\Sigma(s,\rho_K\times\bar\rho_K)\prod_{j\geq N} L_{S_j}(s,\rho_K\times\bar\rho_K)$ of $\Lambda_K(s)$ also has a pole at s_0. However, $L_\Sigma(s,\rho_K\times\bar\rho_K)$ has no poles to the right of $\mathrm{Re}(s)=0$, and by Lemma 2.1, $L_{S_j}(s,\rho_K\times\bar\rho_K)$ has no poles to the right of $\mathrm{Re}(s)=\frac{1}{j}$. Hence $s_0\leq\frac{1}{j}$.

Take $r_0\geq r$ such that $p^{r_0}>\frac{1}{s_0}$. By Proposition 1.3, we may replace K by a larger solvable extension so that every prime of degree p in F splits into primes of degree at least p^{r_0} in K. But then the denominator of $\Lambda_K(s)$ has no poles to the right of $p^{-r_0}<s_0$, a contradiction. □

3. Entireness

Now we would like to deduce that $L(s,\rho)$ is entire. However, we cannot directly do this when F/k is quadratic, but only over a quadratic or biquadratic extension K. Consequently we deduce that ρ and π correspond over K. In the final section, we will deduce that ρ and π correspond over F, which will imply the entireness of $L(s,\rho)$. From here on, i will denote a primitive fourth root of 1 in $\overline{F}$.

Proposition 3.1. *There exists a solvable extension K/k containing F, depending only on F/k, such that $L^*(s,\rho_K)$ is entire. In fact, if $p=[F:k]>2$, we can take $K=F$. If $p=2$ and $i\in F$, then we can take K/k to be cyclic of degree 4. If $p=2$ and $i\notin F$, then we can take K/k so that K/F is biquadratic.*

We will actually construct the field K in the proof, and the explicit construction will be used in the next section to prove our main theorem.

Proof. Consider the ratio of L-functions

$$\mathcal{L}_K(s)=\frac{L^*(s,\rho_K)}{L^*(s,\pi_K)}, \tag{9}$$

which is analytic for $\mathrm{Re}(s)>1$. Since $L(s,\pi_K)$ has no poles, to show $L^*(s,\rho_K)$ is entire, it suffices to show $\mathcal{L}_K(s)$ has no poles.

As with Λ, we may write $\mathcal{L}_K=\prod_v\mathcal{L}_{K,v}$ and define $\mathcal{L}_{K,S}=\prod_{v\in S}\mathcal{L}_{K,v}$. Then note that we can write

$$\mathcal{L}_K(s)=\mathcal{L}_{K,\Sigma}(s)\prod_{j>1}\frac{L_{S_j}(s,\rho_K)}{L_{S_j}(s,\pi_K)}, \tag{10}$$

for some finite set Σ. As with Λ, it satisfies a functional equation of the form

$$\mathcal{L}_K(s)=\epsilon_{\mathcal{L},K}(s)\mathcal{L}_K(1-s), \tag{11}$$

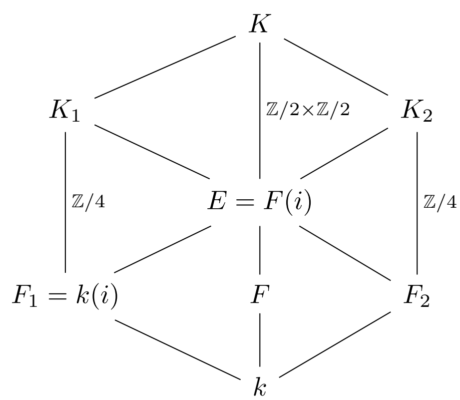
with $\epsilon_{\mathcal{L},K}(s)$ everywhere invertible. Hence it suffices to show $\mathcal{L}_K(s)$ is analytic in $\mathrm{Re}(s)\geq\frac{1}{2}$.

Note $\mathcal{L}_{K,\Sigma}(s)$ is analytic for $\mathrm{Re}(s)\geq\frac{1}{2}$ (in fact for $\mathrm{Re}(s)>0$, since π must be tempered). The same argument as in Lemma 2.1 shows that $L_{S_j}(s,\rho_K)$ and $L_{S_j}(s,\pi_K)$ are both holomorphic and never zero in $\mathrm{Re}(s)>\frac{1}{j}$, using the fact that now we know π is tempered (Proposition 2.2). Hence if $p>2$, then, already for $K=F$, (10) implies that $\mathcal{L}_K(s)$ is analytic in $\mathrm{Re}(s)>\frac{1}{p}$, and we are done.

Now suppose $p=2$. Here we use a more explicit version of Proposition 1.3 for cyclic p^2-extensions: if $K\supset F\supset k$ is a chain of cyclic p-extensions with K/k cyclic, then every unramified inert prime v in F/k lies under a (unique) degree p^2 prime w in K/k [**Ram15**, Lemma 4.4].

Write $F = k(\sqrt{\alpha})$ with $\alpha \in k$. If $i \in F$, then $K = k(\alpha^{1/4})$ is a cyclic extension of degree 4. Hence the lemma just quoted means that $\mathcal{L}_K(s)$ is analytic in $\mathrm{Re}(s) > \frac{1}{4}$ and we are done.

We may therefore assume $i \notin F$. Put $F_1 = k(i)$ and $F_2 = k(\sqrt{-\alpha})$. Let $E = k(i, \sqrt{\alpha}) = F(i)$ be the compositum of these fields, which is biquadratic over k. For $i = 1, 2$, let $\alpha_i \in F_i$ such that $E = F_i(\sqrt{\alpha_i})$, and put $K_i = F_i(\alpha_i^{1/4})$. Then K_i/F_i is cyclic of degree 4 with with E as the intermediate subfield ($i = 1, 2$). Denote by K the compositum K_1K_2. (This construction of K is the $p = 2$ case of a construction given in [**Ram15**, Section 5].) Here is a diagram for the case $i \notin F$.



Fix any prime v of degree 2 in F/k. We claim any prime w of K above v has degree ≥ 4 over k. If not, v splits into 2 primes v_1, v_2 in E. Say $w|v_1$. By [**Ram15**, Lemma 5.8], v_1 is degree 2 either over F_1 or over F_2. Let $i \in \{1, 2\}$ be such that v_1 is degree 2 over F_i, and let u be the prime of F_i under v_i. Then u is inert in K_i/F_i, and so v_1 is inert in K_i/E. Hence w has degree ≥ 2 in K/E, and therefore degree ≥ 4 in K/k, as claimed.

Thus, for such K, $\mathcal{L}_K(s)$ is analytic in $\mathrm{Re}(s) > \frac{1}{4}$. □

REMARK 2. *The above argument for $p > 2$ in fact shows the following: if F/k is any extension, π is tempered, and ρ corresponds to π at all but finitely many places of degree ≤ 2, then $L^*(s, \rho)$ is entire.*

COROLLARY 3.2. *With K as in the previous proposition, and χ and idele class character of K. Then $L^*(s, \rho_K \otimes \chi)$ is entire.*

PROOF. First suppose $p > 2$, so we can take $K = F$. Then if χ is an idele class character of F, $\rho \otimes \chi$ and $\pi \otimes \chi$ locally correspond at almost all degree 1 places, so the previous proposition again applies to say $L^*(s, \rho \otimes \chi)$ is entire.

Now suppose $p = 2$, and let K be an extension as constructed in the proof of the above proposition. Then, for all but finitely many places of degree ≤ 2, we have that ρ_K and π_K locally correspond. Hence the same is true for $\rho_K \otimes \chi$ and $\pi_K \otimes \chi$ for an idele class character χ of K. The above remark then gives the corollary. □

COROLLARY 3.3. *With K as in proposition above, the Artin representation ρ_K is automorphic, and corresponds to the base change π_K.*

Proof. The automorphy of ρ_K now follows from the previous corollary and the converse theorem (Theorem 1.4). Furthermore, since ρ_K corresponds to an automorphic representation which agrees with π_K at a set of places of density 1, it must agree with π_K everywhere by [**Ram94**]. □

4. Descent

In this section, we will finish the proof of Theorem A, that ρ corresponds to π globally. The last corollary (together with Proposition 1.5) already tells us this is the case if $p > 2$, so we may assume $p = 2$ in what follows. In the previous section, we showed ρ_K corresponds to π_K over a suitable quadratic (when $i \in F$) or biquadratic (when $i \notin F$) extension K/F. Here we will show that by varying our choice of K, we can descend this correspondence to F.

As above, let S_2 be the set of finite unramified primes v of degree 2 in F/k at which ρ_v and π_v are both unramified but do not correspond. If S_2 is finite, then an argument of Deligne–Serre [**DS74**] (cf. [**Mar04**, Appendix A] or Proposition 1.5) already tells us ρ and π correspond everywhere (over F at finite places and at the archimedean place over $\mathbb{Q}$), so we may assume S_2 is infinite. As before, write F as $k(\sqrt{\alpha})$ for some $\alpha \in k$.

First suppose that F contains i. Then, as in the proof of Proposition 3.1, $K = k(\alpha^{1/4})$ is a quadratic extension of F, with K cyclic over k, and π_K corresponds to ρ_K globally. Let T_2 denote the subset of S_2 consisting of finite places v of F which are unramified in K. Clearly, the complement of T_2 in S_2 is finite.

We claim that T_2 is empty, so that ρ and π correspond *everywhere*, already over F. Suppose not. Pick any element v_0 of T_2. Since $\det(\rho)$ corresponds to ω_π at all places, and since ρ_K and π_K correspond exactly, if χ_{v_0} is the quadratic character of F_{v_0} attached to the quadratic extension $K_{\bar{v}_0}$, ρ_{v_0} must correspond to $\pi_{v_0} \otimes \chi_{v_0}$. As $K_{\bar{v}_0}/F_{v_0}$ is unramified, so is χ_{v_0}.

Now we may modify the choice of K as follows. Pick a β in k which is a square but not a fourth power, and put $\tilde{K} = k((\alpha\beta)^{1/4})$. Then $\tilde{K}$ contains F and is a cyclic quartic extension of k (just like K), so that all but a finite number of places in S_2 are inert in $\tilde{K}$. But now we may choose β such that v_0 ramifies in $\tilde{K}$. It follows (as above) that over $\tilde{K}$, the base changes of of π and ρ correspond everywhere, and that ρ_{v_0} corresponds to $\pi_{v_0} \otimes \tilde{\chi}_{v_0}$, where $\tilde{\chi}_{v_0}$ is now the ramified local character at v_0 (attached to $\tilde{K}/F$). This gives a contradiction as we would then need $\pi_{v_0} \otimes \chi_{v_0}$ to be isomorphic to $\pi_{v_0} \otimes \tilde{\chi}_{v_0}$ (only the latter twist is ramified). So the only way to resolve this is to have ρ_{v_0} correspond to π_{v_0}. Then v_0 cannot lie in T_2. As it was taken to be a general element of T_2, the whole set T_2 must be empty, proving the claim. We are now done if i belongs to F.

Thus we may assume from here on that $i \notin F$. Then $E = F(i)$ is a biquadratic extension of k, and every element v of S_2 which is unramified in E must split there, say into v_1 and v_2. So it suffices to prove that ρ and π correspond exactly over E. Suppose not. Without loss of generality, say ρ_{E,v_1} and π_{E,v_1} do not correspond.

Consider the construction of K, biquadratic over F, with subfields K_1, K_2, E, F_1, F_2 as in the proof of Proposition 3.1. Interchanging indices on K_i's and F_i's if necessary, we may assume that v_1 has degree 2 in E/F_1, so it lies under a unique prime w in $K_1 = F_1(\alpha_1^{1/4})$. Since w splits in K, $\rho_{K_1,w}$ must correspond to $\pi_{K_1,w} \otimes \chi$, where χ is the unramified quadratic character attached to $K_{1,w}/E_{v_1}$.

As in the previous case, we may modify K_1 to $\tilde{K}_1 = F_1((\alpha_1\beta_1)^{1/4})$ for some $\beta_1 \in F_1$ which is a square but not a fourth power. We may choose β_1 so that $\tilde{K}_1/E$ is ramified. Then $\tilde{K}_1/F_1$ is a cyclic extension of degree 4 containing E, and Proposition 3.1 is still valid with K replaced by $\tilde{K} = \tilde{K}_1 K_2$. Let $\tilde{w}$ be the prime of $\tilde{K}_1$ over v_1. Since $\tilde{w}$ splits over $\tilde{K}$, we also have that $\rho_{\tilde{K}_1,w}$ corresponds to $\pi_{\tilde{K}_1,w} \otimes \tilde{\chi}$, where $\tilde{\chi}$ is the ramified quadratic character associated to $\tilde{K}_{1,w}/E_{v_1}$. Again, this implies $\pi_{K_1,w} \otimes \chi \simeq \pi_{\tilde{K}_1,w} \otimes \tilde{\chi}$, which is a contradiction. □

References

[BB11] Valentin Blomer and Farrell Brumley, *On the Ramanujan conjecture over number fields*, Ann. of Math. (2) **174** (2011), no. 1, 581–605, DOI 10.4007/annals.2011.174.1.18. MR2811610

[BK11] Andrew R. Booker and M. Krishnamurthy, *A strengthening of the* GL(2) *converse theorem*, Compos. Math. **147** (2011), no. 3, 669–715, DOI 10.1112/S0010437X10005087. MR2801397 (2012f:11090)

[DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975) (French). MR0379379 (52 #284)

[GJ78] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. MR533066 (81e:10025)

[JL70] H Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Mathematics, vol. 114, Springer-Verlag, Berlin-New York, 1970. MR0401654 (70)

[JS85] Hervé Jacquet and Joseph Shalika, *A lemma on highly ramified ϵ-factors*, Math. Ann. **271** (1985), no. 3, 319–332, DOI 10.1007/BF01456070. MR787183 (87i:22048)

[Kha10] Chandrashekhar Khare, *Serre's conjecture and its consequences*, Jpn. J. Math. **5** (2010), no. 1, 103–125, DOI 10.1007/s11537-010-0946-5. MR2609324 (2011d:11121)

[Kim03] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, DOI 10.1090/S0894-0347-02-00410-1. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203 (2003k:11083)

[KS02] Henry H. Kim and Freydoon Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197, DOI 10.1215/S0012-9074-02-11215-0. MR1890650 (2003a:11057)

[Kut80] Philip Kutzko, *The Langlands conjecture for* Gl_2 *of a local field*, Ann. of Math. (2) **112** (1980), no. 2, 381–412, DOI 10.2307/1971151. MR592296 (82e:12019)

[Lan80] Robert P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980. MR574808 (82a:10032)

[LRS99] Wenzhi Luo, Zeév Rudnick, and Peter Sarnak, *On the generalized Ramanujan conjecture for* GL(n), Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math., vol. 66, Amer. Math. Soc., Providence, RI, 1999, pp. 301–310. MR1703764 (2000e:11072)

[Mar04] Kimball Martin, *Four-dimensional Galois representations of solvable type and automorphic forms*, ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–California Institute of Technology. MR2706615

[Ram15] Dinakar Ramakrishnan, *A mild Tchebotarev theorem for* GL(n), J. Number Theory **146** (2015), 519–533, DOI 10.1016/j.jnt.2014.08.002. MR3267122

[Ram94] ———, *A refinement of the strong multiplicity one theorem for* GL(2). *Appendix to: "l-adic representations associated to modular forms over imaginary quadratic fields. II" [Invent. Math.* **116** *(1994), no. 1-3, 619–643; MR1253207 (95h:11050a)] by R. Taylor*, Invent. Math. **116** (1994), no. 1-3, 645–649, DOI 10.1007/BF01231576. MR1253208 (95h:11050b)

[RT83] J. D. Rogawski and J. B. Tunnell, *On Artin L-functions associated to Hilbert modular forms of weight one*, Invent. Math. **74** (1983), no. 1, 1–42, DOI 10.1007/BF01388529. MR722724 (85i:11044)

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573, DOI 10.1007/BF01394275. MR969243 (89j:11051)

(Kimball Martin) Department of Mathematics, University of Oklahoma, Norman, OK 73019 USA

(Dinakar Ramakrishnan) Department of Mathematics, Caltech, Pasadena, CA 91125