RATIONALITY OF DARMON POINTS OVER GENUS FIELDS OF NON-MAXIMAL ORDERS

MATTEO LONGO, KIMBALL MARTIN AND YAN HU

Abstract. Stark–Heegner points, also known as *Darmon points*, were introduced by H. Darmon in [Dar01] as certain local points on rational elliptic curves, conjecturally defined over abelian extensions of real quadratic fields. The rationality conjecture for these points is only known in the unramified case, namely, when these points are specializations of global points defined over the strict Hilbert class field H_F^+ of the real quadratic field F and twisted by (unramified) quadratic characters of $\mathrm{Gal}(H_c^+/F)$. We extend these results to the situation of ramified quadratic characters; more precisely, we show that Darmon points of conductor $c \geq 1$ twisted by quadratic characters of $G_c^+ = \mathrm{Gal}(H_c^+/F)$, where H_c^+ is the strict ring class field of F of conductor c, come from rational points on the elliptic curve defined over H_c^+.

Contents

1. Introduction

The theory of *Stark–Heegner points*, also known as *Darmon points*, began with the foundational paper by H. Darmon [Dar01] in 2001. In this work, Darmon proposed a construction of local points on rational elliptic curves, the Stark–Heegner points, which, under appropriate arithmetic conditions, he conjectured to be global points defined over strict ring class fields of *real quadratic fields*, which are non-torsion when the central critical value of the first derivative of the complex L-function of the elliptic curve over the real quadratic field does not vanish. Note that the absence of a theory of complex multiplication in the real quadratic case, available in the imaginary quadratic case, makes the construction of global points on elliptic curves over real quadratic fields and their abelian extensions a rather challenging problem. The idea of Darmon was to define locally a family of candidates for their points, and conjecture that these come from global points. Following [Dar01], many authors proposed similar constructions in different situations, including the cases of modular and Shimura curves, and the higher weight analogue of Stark–Heegner, or Darmon, cycles; with no attempt to be complete, see for instance [Das05], [Gre09], [LRV12], [LRV13], [Tri06], [GS16], [RS12], [GMcS15], [GM15b], [GM15a], [GM14], and [GM13].

The arithmetic setting of the original construction in [Dar01] is the following. Fix a rational elliptic curve E of conductor $N = Mp$, with $p \nmid M$ an odd prime number and $M \geq 1$ and integer. Fix also a real quadratic field F satisfying the following *Stark–Heegner assumption*: all primes $\ell \mid M$ are split in F, while p is inert in F. Under these assumptions, the central

Date: December 29, 2018.

critical value $L(E/F,1)$ of the complex L-function of E over F vanishes. Darmon points are local points P_c for E defined of finite extension of F_p, the completion of F at the unique prime above p; their definition and the main properties are recalled in Section 2 below. The definition of these points depends on the choice of an auxiliary integer $c \geq 1$, called the *conductor* of a Darmon point P_c. The rationality conjecture predicts that these points P_c are localizations of global points $\mathbf{P}_c$ which are defined over the strict ring class field H_c^+ of F of conductor c.

Only partial results are known toward the rationality conjectures for Darmon points, or more generally cycles. The first result on the rationality of Darmon points is due to Bertolini and Darmon in the paper [BD09], where they show that a certain linear combination of these points with coefficients given by values of genus characters of the real quadratic field F comes from a global point defined over the Hilbert class field of F. The main idea behind the proof of these results is to use a factorization formula for p-adic L-functions to compare the localization of Heegner points and Darmon points. More precisely, the first step of the proof consists in relating Darmon points to the p-adic L-function interpolating central critical values of the complex L-functions over F attached to the arithmetic specializations of the Hida family passing through the modular form attached to E. The second step consists in expressing this p-adic L-function in terms of a product of two Mazur–Kitagawa p-adic L-functions, which are known to be related to Heegner points by the main result of [BD07]. A similar strategy has been adopted by [GSS16], [Sev12] [LV14], [LV16] obtaining similar results.

All known results in the direction of the conjectures in [Dar01] involve linear combination of Darmon points twisted by genus characters, which are quadratic unramified characters of $\mathrm{Gal}(H_F^+/F)$, where H_F^+ is the (strict) Hilbert class field of F. The goal of this paper is to prove a similar rationality result for more general quadratic characters, namely, quadratic characters of ring class fields of F, so we allow for ramification. In the remaining part of the introduction we briefly state our main result and the main differences with the case of genus characters treated up to now.

Let $E/\mathbb{Q}$ be an elliptic curve, and denote by N its conductor. Let $F/\mathbb{Q}$ be a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ of discriminant $D = D_F > 0$, prime to N. We assume one has a factorization $N = Mp$ with $p \nmid M$, such that all primes $\ell \mid M$ are split in F and p is inert in F.

Fix an integer c prime to $D \cdot N$ and a quadratic character

$$\chi : G_c^+ = \mathrm{Gal}(H_c^+/F) \longrightarrow \{\pm 1\},$$

where, as above, H_c^+ denotes the strict class field of F of conductor c. Let $\mathcal{O}_c$ be the order in F of conductor c. Recall that G_c^+ is isomorphic to the group of strict equivalence classes of projective $\mathcal{O}_c^+$-modules, which we denote $\mathrm{Pic}^+(\mathcal{O}_c)$, where two such modules are strictly equivalent if they are the same up to an element of F of positive norm. We assume that χ is *primitive*, meaning that it does not factor through any G_f^+ with f a proper divisor of c.

Fix embeddings $F \hookrightarrow \bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ throughout. Let $P_c \in E(F_p)$ be a Darmon point of conductor c (see Section 2 below for the precise definition of these points) where, as above, F_p is the completion of F at the unique prime of F above p. It follows from their construction that Darmon points of conductor c are in bijection with equivalence classes of quadratic forms of discriminant Dc^2, and this can be used to *define* a Galois action $P_c \mapsto P_c^\sigma$ on Darmon points, where P_c is a fixed Darmon point of conductor c and $\sigma \in G_c^+$. We may then form the point

$$P_\chi = \sum_{\sigma \in G_c^+} \chi^{-1}(\sigma) P_c^\sigma \tag{1}$$

which lives in $E(F_p)$. Finally, let $\log_E : E(\mathbb{C}_p) \to \mathbb{C}_p$ denote the formal group logarithm of E. Note that, since p is inert in F, it splits completely in H_c^+, and therefore for any point $Q \in E(H_c^+)$ the localization of Q at any of the primes in H_c^+ dividing p lives in $E(F_p)$. Our main result is the following:

Theorem 1.1. *Assume that c is is odd and coprime to DN. Let χ be a primitive quadratic character of G_c^+. Then there exists a point $\mathbf{P}_\chi$ in $E(H_c^+)$ and a rational number $n \in \mathbb{Q}^\times$ such that*

$$\log_E(P_\chi) = n \cdot \log_E(\mathbf{P}_\chi).$$

Moreover, the point $\mathbf{P}_\chi$ is of infinite order if and only if $L'(E/F, \chi, 1) \neq 0$.

If $c = 1$, this is essentially the main result of [BD09]. To be more precise, the work [BD09] needed to assume E had two primes of multiplicative reduction because of this assumption in [BD07]. However, this assumption has been removed by very recent work of Mok [Mok], which we also apply here.

The proof in the general case follows a similar line to that in [BD09]. However, some modifications are in order. The first difference is that the genus theory of non-maximal orders is more complicated than the usual genus theory, and the arguments need to be adapted accordingly. More importantly, one of the main ingredients in the proof of the rationality result in [BD09] is a formula of Popa [Pop06] for the central critical value of the L-function over F of the specializations at arithmetic points of the Hida family passing through the modular form associated with the elliptic curve E. However, this formula does not allow treat L-functions twisted by ramified characters. Instead, we recast an L-value formula from [MW09] which allows for ramification, expressed in terms of periods of Gross–Prasad test vectors, in a more classical framework to get our result.

Remark 1.2. When $(cD, N) \neq 1$, it may be that π and χ have joint ramification. In this case, we can instead use [FMP17] in lieu of [MW09], at least in the case that f has squarefree level.

Remark 1.3. The main result of this paper assumes that all primes dividing M are split and p is inert in F. More generally, the same results are expected to hold under the following *relaxed modified Heegner assumption*: p is inert in F and there is a factorization $M = M^+ \cdot M^-$ of M into coprime integers such that a prime number $\ell \mid M^+$ if and only if ℓ in split in F, and M^- is a product of an even number of distinct primes. If the conductor N of the elliptic curve E can be factorized as $N = M^+ \cdot M^- \cdot p$ with $p \nmid M$ and the discriminant D of the real quadratic field F, and χ is a primitive quadratic character of G_c^+ with c odd and coprime with ND, then one can show that there exists a point $\mathbf{P}_\chi$ in $E(H_c^+)$ and a rational number $n \in \mathbb{Q}^\times$ such that $\log_E(P_\chi) = n \cdot \log_E(\mathbf{P}_\chi)$; moreover, the point $\mathbf{P}_\chi$ is of infinite order if and only if $L'(E/F, \chi, 1) \neq 0$. The proof of this result can be obtained with the methods of this paper by replacing modular curves of level M with Shimura curves of level M^+ attached to quaternion algebras of discriminant M^-, following what is done in [LV14] in the case $c = 1$. However, since the notation in the quaternionic case is quite different from the notation in the case of modular curves, we prefer to only treat in detail the case when $M^- = 1$. Details in the case $M^- > 1$ are left to the reader.

Remark 1.4. The main result of this paper plays a role in the forthcoming works by Darmon-Rotger [DR] and Bertolini-Seveso-Venerucci [BSV] proving the rationality of Darmon points (actually, cohomology classes closely related to Darmon points) in situations which go far beyond the case of quadratic characters considered in this paper. This application was one of the main motivations for this work.

Acknowledgements. We are grateful to H. Darmon for suggesting the problem and V. Rotger for many interesting discussions about the topics of this paper. We also thank C.P. Mok for providing us with a preliminary version of [Mok]. M.L. is supported by PRIN 2015 and BIRD 2017, Padova. K.M. was supported by a grant from the Simons Foundation/SFARI (512927, KM).

2. Darmon points

Let the notation be as in the introduction: $E/\mathbb{Q}$ is an elliptic curve of conductor $N = Mp$ with $p \nmid M$, and $F/\mathbb{Q}$ is a real quadratic field of discriminant $D = D_F$ such that all primes dividing M are split in F and p is inert in F. Finally, c is a positive integer prime to ND. The aim of this section is to review the definition of Darmon points and some results in [BD09] and [BD07].

We first set up some standard notation. For any field L, let $P_{k-2}(L)$ be the space of homogeneous polynomials in 2 variables of degree $k-2$, and let $V_{k-2}(L)$ be its L-linear dual. We let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(L)$ act from the right on $P(x,y) \in P_{k-2}(L)$ by the formula

$$(P|\gamma)(x,y) = P(ax+by, cx+dy)$$

and we equip $V_{k-2}(L)$ with the dual action. If G is any abelian group, let $\mathrm{MS}(G)$ be the group of G-valued modular symbols, i.e. the $\mathbb{Z}$-module of functions $I : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to G$ such that $I(x,y) + I(y,z) = I(x,z)$ for all $x, y, z \in \mathbb{P}^1(\mathbb{Q})$. The value $I(r,s)$ of $I \in \mathrm{MS}(G)$ on (x,y) will be usually denoted $I\{x \to y\}$. The group $\mathrm{GL}_2(\mathbb{Q})$ acts from the left by fractional linear transformations on $\mathbb{P}^1(\mathbb{Q})$, and if G is equipped with a left $\mathbb{P}^1(\mathbb{Q})$-action, then $\mathrm{MS}(G)$ inherits a right $\mathrm{GL}_2(\mathbb{Q})$-action by the rule $(I|\gamma)(x,y) = \gamma \cdot I(\gamma^{-1}x, \gamma^{-1}y)$. If Γ_0 is a subgroup of $\mathbb{P}^1(\mathbb{Q})$, we denote $\mathrm{MS}_{\Gamma_0}(G)$ the subgroup of those elements in $\mathrm{MS}(G)$ which are invariant under the action of γ for all $\gamma \in \Gamma_0$. If f is a cuspform of level $\Gamma_0(M)$ and weight k, we may attach to f the standard modular symbol $\tilde{I}_f \in \mathrm{MS}_{\Gamma_0(M)}(V_{k-2}(\mathbb{C}))$; explicitly, for $r, s \in \mathbb{P}^1(\mathbb{Q})$ and $P(x,y) \in P_{k-2}(\mathbb{C})$ an homogenous polynomial of degree $k-2$, put

$$\tilde{I}_f\{r \to s\}(P(x,y)) = 2\pi i \int_r^s f(z)P(z,1)\,dz.$$

The matrix $\omega_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts on the group of modular symbols $\mathrm{MS}_{\Gamma_0(M)}(V_{k-2}(\mathbb{C}))$, and we let $\tilde{I}_f^{\pm}$ denote the projections to the $\pm$-eigenspaces for this action. Suppose that f is a normalized eigenform and let K_f be the field generated over $\mathbb{Q}$ by the Fourier coefficients of f. Then there are complex periods $\Omega_f^{\pm}$ for each choice of sign $\pm$ such that their product equals the Petersson inner product $\langle f, f\rangle$, and $I_f^{\pm} := \tilde{I}_f^{\pm}/\Omega_f^{\pm}$ satisfies the condition that if $P(x,y) \in P_{k-2}(K_f)$ then $I_f\{r \to s\}(P(x,y))$ belongs to K_f.

2.1. Measure-valued modular symbols and Darmon points.

Let f be the newform of level N attached to E by modularity. Denote $B = \mathrm{M}_2(\mathbb{Q})$ the split quaternion algebra over $\mathbb{Q}$ and let R be the $\mathbb{Z}[1/p]$-order in B consisting of matrices in $\mathrm{M}_2(\mathbb{Z}[1/p])$ which are upper triangular modulo M. Define

$$\Gamma = \{\gamma \in R^\times \mid \det(\gamma) = 1\}.$$

Let $\mathrm{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})$ denote the $\mathbb{Z}$-module of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ with total measure equal to 0. By [BD09, Proposition 1.3], for each choice of sign $\pm$, there exists a unique function, which we call the *measure-valued modular symbol* attached to f,

$$\mu_f^{\pm} : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathrm{Meas}^0\left(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}\right)$$

denoted $(r,s) \mapsto \mu_f\{r \to s\}$, satisfying the following conditions:

(1) $\mu_f^{\pm}\{r \to s\}(\mathbb{Z}_p) = I_f^{\pm}\{r \to s\}$
(2) For all $\gamma \in \Gamma$ and all open compact subsets $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$,

$$\mu_f\{\gamma(r) \to \gamma(s)\}(U) = \mu_f\{r \to s\}(U),$$

where we let $\mathrm{GL}_2(\mathbb{Q}_p)$ act on $\mathbb{P}^1(\mathbb{Q}_p)$ by fractional linear transformations.

Let $\mathcal{H}_p = \mathbb{C}_p \setminus \mathbb{Q}_p$ denote the p-adic upper half plane. The system of measures μ_f can be used to define, for any $r, s \in \mathbb{P}^1(\mathbb{Q})$ and $\tau_1, \tau_2 \in \mathcal{H}_p$, a *double multiplicative integral*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{t-\tau_2}{t-\tau_1}\, d\mu_f\{r \to s\}(t).$$

(On the right, the notation $\times\!\!\!\!\!\int$ refers to the fact that the integration is relative to the multiplicative structure of $\mathbb{C}_p^\times$, and therefore is a limit of Riemann products instead of Riemann sums.) Let q be the Tate period of E at p, and let $\log_q$ be the branch of the p-adic logarithm satisfying $\log_q(q) = 0$. Define the additive version of the double multiplicative integral to be

$$\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \log_q \left(\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_r^s \omega_f \right).$$

We finally introduce the notion of *indefinite integral*. By [BD09, Proposition 1.5], there exists a unique function from $\mathcal{H}_p \times \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ to $\mathbb{C}$, denoted $(\tau, r, s) \to \int^\tau \int_r^s \omega_f$, satisfying the following conditions:

(1) The integral is Γ-invariant, in the sense that for all $\gamma \in \Gamma$, we have

$$\int^{\gamma(\tau)} \int_{\gamma(r)}^{\gamma(s)} \omega_f = \int^\tau \int_r^s \omega_f$$

(2) For any pair $\tau_1, \tau_2 \in \mathcal{H}_p$, we have

$$\int^{\tau_2} \int_r^s \omega_f - \int^{\tau_1} \int_r^s \omega_f = \int_{\tau_1}^{\tau_2} \int_r^s \omega_f;$$

(3) For all r, s, t in $\mathbb{P}^1(\mathbb{Q})$ we have

$$\int^\tau \int_r^s \omega_f + \int^\tau \int_s^t \omega_f = \int^\tau \int_r^t \omega_f.$$

We now define Darmon points using indefinite integrals above. Since p is inert in F, the set $\mathcal{H}_p \cap F$ is not empty and one may define the order $\mathcal{O}_\tau$ associated with $\tau \in \mathcal{H}_p \cap F$ as

$$\mathcal{O}_\tau = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R \mid a\tau + b = c\tau^2 + d\tau \right\}.$$

The map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to c\tau + d$ induces an embedding $\mathcal{O}_\tau \hookrightarrow F$, and thus $\mathcal{O}_\tau$ may be viewed as an order in F. For $\tau \in \mathcal{H}_p \cap F$, let $\gamma_\tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ denote the unique generator of the stabilizer of τ in Γ satisfying $c\tau + d > 1$ (with respect to the chosen embedding $F \subseteq \bar{\mathbb{Q}}$). Let $\Phi_{\mathrm{Tate}} : \mathbb{C}_p^\times / q^{\mathbb{Z}} \to E(\mathbb{C}_p)$ denote the Tate uniformization of E at p. Attached to τ, there is an indefinite integral

$$J_\tau^\times = \times\!\!\!\!\!\!\int^\tau \int_t^{\gamma_\tau(r)} \omega_f \in \mathbb{C}_p \tag{2}$$

where $r \in \mathbb{P}^1(\mathbb{Q}_p)$ is any base point, and one can show that $\Phi_{\mathrm{Tate}}(J_\tau^\times)$ is a well-defined point in $E(\mathbb{C}_p)$ independently of the choice of r, up to its torsion subgroup $E(\mathbb{C}_p)_{\mathrm{tors}}$.

Definition 2.1. Let $\tau \in \mathcal{H}_p \cap F$. The point $P_\tau = \Phi_{\mathrm{Tate}}(J_\tau^\times) \in E(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$, with $J_\tau^\times$ defined in (2), is the *Darmon point* attached to τ.

2.2. **Shimura reciprocity law.** Fix an integer c prime to $D\cdot N$, and let $\mathcal{O}_c$ be the order of F of conductor c. Denote $\mathcal{Q}_{Dc^2}$ the set of primitive binary quadratic forms of discriminant Dc^2. Let $\mathrm{SL}_2(\mathbb{Z})$ act from the right on the set $\mathcal{Q}_{Dc^2}$ via the formula

$$(Q|\gamma)(x,y)=Q(ax+by,cx+dy) \tag{3}$$

for $Q\in\mathcal{Q}_{Dc^2}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. The set of equivalence classes $\mathcal{Q}_{Dc^2}/\mathrm{SL}_2(\mathbb{Z})$ is equipped with a group structure given by the Gaussian composition law. If H_c^+ is the strict ring class field of F of conductor c, then its Galois group $G_c^+=\mathrm{Gal}(H_c^+/F)$ is isomorphic to the group $\mathcal{Q}_{Dc^2}/\mathrm{SL}_2(\mathbb{Z})$ via global class field theory (see [Coh78, Theorem 14.19]).

Fix $\delta\in\mathbb{Z}$ such that $\delta^2\equiv D \pmod{4M}$. Let $\mathcal{F}_{Dc^2}$ denote the subset of $\mathcal{Q}_{Dc^2}$ consisting of forms $Q(x,y)=Ax^2+Bxy+Cy^2$ such that $M\mid A$ and $B\equiv\delta \pmod{2M}$. The group $\Gamma_0(M)$ acts on $\mathcal{F}_{Dc^2}$ by the formula (3). Since $(M,D)=1$, we also have $(\delta,M)=1$, and therefore, by [GKZ87, Proposition, p. 505], the map $Q\mapsto Q$ sets up a bijection between $\mathcal{F}_{Dc^2}/\Gamma_0(M)$ and $\mathcal{Q}_{Dc^2}/\mathrm{SL}_2(\mathbb{Z})$. In particular, the set $\mathcal{F}_{Dc^2}/\Gamma_0(M)$ is equipped with a structure of principal homogeneous space under G_c^+. If $Q\in\mathcal{F}_{Dc^2}/\Gamma_0(M)$ and $\sigma\in G_c^+$, we denote Q^σ for the image of Q by σ.

Define

$$\mathcal{H}_p^{(Dc^2)}=\{\tau\in\mathcal{H}_p\cap F\mid\mathcal{O}_\tau=\mathcal{O}_c\}.$$

Given $Q(x,y)=Ax^2+Bxy+Cy^2$ a quadratic form in $\mathcal{F}_{Dc^2}$, let $\tau_Q=\frac{-B+c\sqrt{D}}{2A}$ be a fixed root of the quadratic polynomial $Q(x,1)$. Then τ_Q belongs to $\mathcal{H}_p^{(Dc^2)}$ (via the fixed p-adic embedding of F into $\bar{\mathbb{Q}}_p$) and its image in $\Gamma\backslash\mathcal{H}_p^{(Dc^2)}$ does not depend on the $\Gamma_0(M)$-equivalence class of Q. Given $\sigma\in G_c^+$, we will sometimes write τ_Q^σ for τ_{Q^σ}.

Let P be a point in $E(H_c^+)$. Since p is inert in K, it splits completely in H_c^+, and therefore, after fixing a prime of H_c^+ above p, the point P localizes to a point in $E(F_p)$, where F_p is the completion of F at unique prime above p.

Conjecture 2.2. The Darmon point P_{τ_Q} is the localization of a global point P_c, defined over H_c^+, and the Galois action on this point is described by the following Shimura reciprocity law: if $P_c\in E(H_c^+)$ localizes to $P_{\tau_Q}\in E(F_p)$ then P_c^σ localizes to $P_{\tau_q^\sigma}$.

2.3. **Real conjugation.** Denote by $\tau_p\in\mathrm{Gal}(H_c^+/\mathbb{Q})$ the Frobenius element at p, well defined only up to conjugation. As recalled above, since p is inert in F, it splits completely in H_c^+ and (after fixing as above a prime of H_c^+ above p), τ_p corresponds to an involution of $\mathrm{Gal}(F_p/\mathbb{Q}_p)$. By [Dar01, Proposition 5.10], it is known that there exists an element $\sigma_\tau\in G_c^+$ such that

$$\tau_p(J_\tau)=-w_M J_{\tau^{\sigma_\tau}} \tag{4}$$

and $\tau_p(P_\tau)=w_N P_{\tau^{\sigma_\tau}}$ where w_M and w_N are the signs of the Atkin-Lehner involution W_M and W_N, respectively, acting on f.

2.4. **Families of measure-valued modular symbols and Darmon points.** Let

$$\mathcal{X}=\mathrm{Hom}_{\mathbb{Z}_p}^{\mathrm{cont}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)$$

and embed $\mathbb{Z}$ into $\mathcal{X}$ via $k\mapsto[x\mapsto x^{k-2}]$. There are rigid analytic functions $\kappa\mapsto a_n(\kappa)$ for integers $n\geq 1$, simultaneously defined on a suitable neighborhood $\mathcal{U}$ of 2 (which we may assume containing only integers k with $k\equiv 2 \mod p-1$) such that the formal power series expansion

$$f_\infty(\kappa)=\sum_{n\geq 1}a_n(\kappa)q^n$$

when evaluated at $\kappa=k\in\mathbb{Z}$ is the q-expansion of a normalized eigenform on $\Gamma_0(N)$ of weight k, and such that $f_2=f$, where recall that f is the modular form attached to E by modularity.

If $k \neq 2$, f_k is necessarily old, and we let $f_k^\sharp$ be the form of level $\Gamma_0(M)$ and weight k whose p-stabilization coincides with f; so f_k and $f_k^\sharp$ are related by the formula:

$$f_k(z) = f_k^\sharp(z) - p^{k-1} a_p(k)^{-1} f_k^\sharp(pz).$$

For $k = 2$ we simply put $f_2^\sharp = f_2 = f$.

Let $\mathcal{W} = \mathbb{Q}_p^2 - \{(0,0)\}$ and let $\mathbb{D}$ denote the $\mathbb{Q}_p$-vector space of compactly supported $\mathbb{Q}_p$-valued measures on $\mathcal{W}$. Let $L_* = \mathbb{Z}_p^2$. Say that $(x, y) \in L_*$ is *primitive* if p does not divide both x and y and let $L'_* = (\mathbb{Z}_p^2)'$ denote the subset of L_* consisting of of primitive vectors. Let $\mathbb{D}_*$ be the subspace consisting of measures which are supported on the L'_*. Let $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ be the Iwasawa algebra of $\mathbb{Z}_p^\times$, identified with a subring of the ring of analytic functions on $\mathcal{X}$. The $\mathbb{Q}_p$-vector space $\mathbb{D}$ is equipped with a structure of Λ-algebra arising form the action of $\mathbb{Z}_p^\times$ on $\mathcal{W}$ and $\mathbb{Z}_p^2$ given by $(x, y) \mapsto (\lambda x, \lambda y)$ for $\lambda \in \Lambda$. Also, $\mathrm{GL}_2(\mathbb{Q}_p)$ acts from the left on $\mathbb{D}_*$ by translations, and $\mathrm{MS}_{\Gamma_0(M)}(\mathbb{D}_*)$ is naturally equipped with an action of Hecke operators. In particular, we have a U_p-operator acting on $\mathrm{MS}_{\Gamma_0(M)}(\mathbb{D}_*)$ by the formula

$$\int_X \phi \, d(U_p\mu)\{r \to s\} = \sum_{a=0}^{p-1} \int_{p^{-1}\gamma_a X} (\phi | p\gamma_a^{-1}) \, d\mu\{\gamma_a(r) \to \gamma_a(s)\}$$

for any locally constant function ϕ on $\mathcal{W}$. Here $\gamma_a = \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$, and for any open compact subset $X \subseteq \mathcal{W}$ and any locally constant function ϕ on $\mathcal{W}$, we put $\int_X \phi \, d\mu = \int_{L'_*} \phi(x) \mathrm{char}_X(x) \, d\mu(x)$, where char_X is the characteristic function of X. In particular, we may define $\mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbb{D}_*)$ to be the maximal submodule of $\mathrm{MS}_{\Gamma_0(M)}(\mathbb{D}_*)$ on which U_p acts invertibly. For each $k \in \mathcal{U}$ there is a specialization map

$$\rho_k : \mathbb{D}_*^\dagger \longrightarrow V_{k-2}(\mathbb{C}_p)$$

defined by

$$\rho_k(\mu)(P(x, y)) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y) \, d\mu(x, y).$$

Let $\Lambda^\dagger$ denote the ring of $\mathbb{C}_p$-valued functions on $\mathcal{X}$ which can be represented by a convergent power series expansion in some neighborhood of $2 \in \mathcal{X}$ and define $\mathbb{D}_*^\dagger = \mathbb{D}_* \otimes_\Lambda \Lambda^\dagger$. For any $\mu = \sum_i \lambda_i \mu_i$ with $\lambda_i \in \Lambda^\dagger$ and $\mu_i \in \mathbb{D}_*$, we call a *neighborhood of regularity* for μ any neighborhood U_μ of 2 in $\mathcal{U}$ such that all λ_i converge in U_μ. The module $\mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbb{D}_*)$ inherits a Λ-action from the Λ-module structure of $\mathbb{D}_*$, and we may define $\mathrm{MS}^{\mathrm{ord},\dagger}_{\Gamma_0(M)}(\mathbb{D}_*) = \mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbb{D}_*) \otimes_\Lambda \Lambda^\dagger$. This $\Lambda^\dagger$-module is free of finite rank, and given $\mu \in \mathrm{MS}^{\mathrm{ord},\dagger}_{\Gamma_0(M)}(\mathbb{D}_*)$ it is possible to find a common neighborhood of regularity for all the measures $\mu\{r \to s\}$, which we denote U_μ. The specialization map ρ_k induces a map, denoted by the same symbol,

$$\rho_k : \mathrm{MS}^{\mathrm{ord},\dagger}_{\Gamma_0(M)}(\mathbb{D}_*) \longrightarrow \mathrm{MS}_{\Gamma_0(N)}(V_{k-2}(\mathbb{C}_p)),$$

and [BD07, Theorem 1.5] shows that for each choice of sign $\pm$ there exists a neighborhood U of 2 in $\mathcal{X}$ and $\mu_*^\pm \in \mathrm{MS}^{\mathrm{ord},\dagger}_{\Gamma_0(M)}(\mathbb{D}_*)$ such that $\rho_2(\mu_*^\pm) = I_f^\pm$ and for all integers $k \in U$, $k \geq 2$, there is $\lambda^\pm(k) \in \mathbb{C}_p$ such that $\rho_k(\mu_*^\pm) = \lambda^\pm(k) I_{f_k}^\pm$; also, U can be chosen so that $\lambda^\pm(k) \neq 0$ for all $k \in U$.

Theorem 2.3. *If $Q \in \mathcal{F}_{Dc^2}$, then*

$$\log_q(P_{\tau_Q}) = \int_{(\mathbb{Z}_p^2)'} \log_q(x - \tau_Q y) \, d\mu_*^\pm\{r \to s\}(x, y)$$

Proof. This follows from [BD09, Theorem 2.5] as in [BD09, Corollary 2.6] noticing that the set

$$\{(x,y)\in\mathbb{Q}_p^2 \mid x-\tau_Q y\in\mathcal{O}_K\otimes\mathbb{Z}_p\}$$

coincides with $\mathbb{Z}_p^2$. □

3. Complex L-functions of real quadratic fields

Here we recast the special value formula of the second author and Whitehouse [MW09], restricted to the setting of this paper, in a form convenient for our purposes.

Let $f\in S_k(\Gamma_0(M))$ be a even weight $k\geq 2$ newform for $\Gamma_0(M)$. Let $F/\mathbb{Q}$ be a real quadratic field of discriminant $D>0$, prime to M, and let χ_D be the associated quadratic Dirichlet character; with a slight abuse of notation, we will denote by the same symbol $\chi_D:\mathbb{A}_\mathbb{Q}^\times\to\mathbb{C}^\times$ the associated Hecke character, where $\mathbb{A}_\mathbb{Q}$ is the adele ring of $\mathbb{Q}$. We assume that all primes $\ell\mid M$ are split in F.

Let c be an integer prime to DM and let H_c^+ be the strict ring class field of F of conductor c. Let $G_c^+=\mathrm{Gal}(H_c^+/F)$. Let $\chi:G_c^+\to\mathbb{C}^\times$ be a primitive character, namely, a character which does not factor through G_f^+ for any proper divisor $f\mid c$; with a slight abuse of notation, we will denote by the same symbol $\chi:\mathbb{A}_F^\times\to\mathbb{C}^\times$ the associated Hecke character, where $\mathbb{A}_F$ is the adele ring of F.

3.1. **Optimal embedding theory.** We set up the theory of optimal embeddings, and its relation to the strict, or narrow, class group of $\mathcal{O}_c$ and quadratic forms of discriminant Dc^2. For more details, see [LV14, §4.3] or [LRV13, §4.1].

Let us denote by $\mathcal{B}=\mathrm{M}_2(\mathbb{Q})$ the split quaternion algebra over $\mathbb{Q}$ and denote by R_0 the order in $\mathcal{B}$ consisting of matrices in $\mathrm{M}_2(\mathbb{Z})$ which are upper triangular modulo M. Let $\mathcal{O}_c=\mathbb{Z}+c\cdot\mathcal{O}_F$ be the order of F of conductor c, where $\mathcal{O}_F$ is the ring of integers of F. Let $\mathrm{Emb}(\mathcal{O}_c,R_0)$ be the set of optimal embeddings $\psi:F\to\mathcal{B}$ of $\mathcal{O}_c$ into R_0 (so $\psi(\mathcal{O}_c)=R_0\cap\psi(F)$). For every prime $\ell\mid M$ fix orientations of R_0 and $\mathcal{O}_c$ at ℓ, i.e., ring homomorphisms $\mathfrak{O}_\ell:R_0\to\mathbb{F}_\ell$ and $\mathfrak{o}_\ell:\mathcal{O}_c\to\mathbb{F}_\ell$. Two embeddings $\psi,\psi'\in\mathrm{Emb}\big(\mathcal{O}_c,R_0\big)$ are said to have the same orientation at a prime $\ell\mid M$ if $\mathfrak{O}_\ell\circ(\psi|_{\mathcal{O}_c})=\mathfrak{O}_\ell\circ(\psi'|_{\mathcal{O}_c})$ and are said to have opposite orientations at ℓ otherwise. An embedding $\psi\in\mathrm{Emb}\big(\mathcal{O}_c,R_0\big)$ is said to be *oriented* if $\mathfrak{O}_\ell\circ(\psi|_{\mathcal{O}_c})=\mathfrak{o}_\ell$ for all primes $\ell\mid M$. We denote the set of oriented optimal embeddings of $\mathcal{O}_c$ into R_0 by $\mathcal{E}(\mathcal{O}_c,R_0)$. The action of $\Gamma_0(M)$ on $\mathrm{Emb}(\mathcal{O}_c,R_0)$ from the right by conjugation restricts to an action on $\mathcal{E}(\mathcal{O}_c,R_0)$. If $\psi\in\mathcal{E}(\mathcal{O}_c,R_0)$ then $\psi^*:=\omega_\infty\psi\omega_\infty^{-1}$ belongs to $\mathcal{E}(\mathcal{O}_c,R_0)$ as well, where recall that $\omega_\infty=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, and ψ and ψ^* have opposite orientations at all $\ell\mid M$. If ℓ is a prime dividing M then ψ and $\omega_\ell\psi\omega_\ell^{-1}$, where $\omega_\ell=\left(\begin{smallmatrix}0&-1\\\ell&0\end{smallmatrix}\right)$, have opposite orientations at ℓ and the same orientation at all primes dividing M/ℓ.

Let $\mathfrak{a}\subset\mathcal{O}_c$ be an ideal representing a class $[\mathfrak{a}]\in\mathrm{Pic}^+(\mathcal{O}_c)$ and let $\psi\in\mathrm{Emb}(\mathcal{O}_c,R_0)$. The left R_0-ideal $R_0\psi(\mathfrak{a})$ is principal; let $a\in R_0$ be a generator of this ideal with positive reduced norm, which is unique up to elements in $\Gamma_0(M)$. The right action of $\psi(\mathcal{O}_c)$ on $R_0\psi(\mathfrak{a})$ shows that $\psi(\mathcal{O}_c)$ is contained in the right order of $R_0\psi(\mathfrak{a})$, which is equal to $a^{-1}R_0a$. This defines an action of $\mathrm{Pic}^+(\mathcal{O}_c)$ on conjugacy classes of embeddings given by $[\mathfrak{a}]\cdot[\psi]=\big[a\psi a^{-1}\big]$ in $\mathrm{Emb}(\mathcal{O}_c,R_0)/\Gamma_0(M)$. The principal ideal $(\sqrt{D})$ is a proper $\mathcal{O}_c$-ideal of F; denote $\mathfrak{D}$ its class in $\mathrm{Pic}^+(\mathcal{O}_c)$ and define $\sigma_F:=\mathrm{rec}(\mathfrak{D})\in G_c^+$, where rec is the arithmetically normalized reciprocity map of class field theory. If $\mathfrak{a}=(\sqrt{D})$ then we can take $a=\omega_\infty\cdot\psi(\sqrt{D})$ in the above discussion, which shows that $\mathfrak{D}\cdot[\psi]=\big[\omega_\infty\psi\omega_\infty^{-1}\big]=[\psi^*]$. Using the reciprocity map of class field theory, for all $\sigma\in G_c^+$ and $[\psi]\in\mathrm{Emb}(\mathcal{O}_c,R_0)/\Gamma_0(M)$ define $\sigma\cdot[\psi]:=\mathrm{rec}^{-1}(\sigma)\cdot[\psi]$. In particular, $\sigma_F\cdot[\psi]=[\psi^*]$ for all $\psi\in\mathrm{Emb}(\mathcal{O}_c,R_0)$.

If ψ is an oriented optimal embedding then the Eichler order $a^{-1}R_0a$ inherits an orientation from the one of R_0 and it can be checked that we get an induced action of $\mathrm{Pic}^+(\mathcal{O}_c)$ (and G_c^+) on the set $\mathcal{E}(\mathcal{O}_c,R_0)/\Gamma_0(M)$, and this action is free and transitive. To describe a (non-canonical) bijection between $\mathcal{E}(\mathcal{O}_c, R_0)/\Gamma_0(M)$ and G_c^+, fix once and for all an auxiliary embedding $\psi_0 \in \mathcal{E}(\mathcal{O}_c, R_0)$; then $\sigma \mapsto \sigma \cdot [\psi_0]$ defines a bijection $E : G_c^+ \to \mathcal{E}(\mathcal{O}_c, R_0)/\Gamma_0(M)$ whose inverse, $G = E^{-1} : \mathcal{E}(\mathcal{O}_c, R_0)/\Gamma_0(M) \longrightarrow G_c^+$ satisfies the relation $G([\psi^*]) = \sigma_F \cdot G([\psi])$ for all $\psi \in \mathcal{E}(\mathcal{O}_c, R_0)$. Choose for every $\sigma \in G_c^+$ an embedding $\psi_\sigma \in E(\sigma)$, so that the family $\{\psi_\sigma\}_{\sigma \in G_c^+}$ is a set of representatives of the $\Gamma_0(M)$-conjugacy classes of oriented optimal embeddings of $\mathcal{O}_c$ into R_0. If $\gamma, \gamma' \in R_0$ write $\gamma \sim \gamma'$ to indicate that γ and γ' are in the same $\Gamma_0(M)$-conjugacy class, and adopt a similar notation for (oriented) optimal embeddings of $\mathcal{O}_c$ into R_0. For all $\sigma, \sigma' \in G_c^+$ one has $\sigma \cdot \psi_{\sigma'} \sim \psi_{\sigma\sigma'}$ and $\psi_\sigma^* \sim \psi_{\sigma_F \sigma}$ for all $\sigma \in G_c^+$.

Finally, note that the set $\mathcal{E}(\mathcal{O}_c, R_0)/\Gamma_0(M)$ is in bijection with $\mathcal{F}_{Dc^2}/\Gamma_0(M)$, since both sets are in bijection with G_c^+; explicitly, to the class of the oriented optimal embedding ψ corresponds the class of the quadratic form

$$Q_\psi(x, y) = Cx^2 - 2Axy - By^2$$

with $\psi(\sqrt{D}c) = \left(\begin{smallmatrix} A & B \\ C & -A \end{smallmatrix}\right)$.

3.2. **Adelic ring class groups.** Below we will want to view the ring class group G_c^+ adelically. Since this is omitted from the literature on class field theory that we are aware of (adelic treatments usually explain ray class fields but not ring class fields, and expositions of ring class groups which treat real quadratic extensions, e.g., [Coh78], tend to not use adelic language), we explain briefly the passage from classical ring class groups to adelic ring class groups here. For a point of reference, we also describe the relation with ray class groups. As it causes little extra difficulty, in this subsection only, we allow F to be an arbitrary (real or imaginary) quadratic field of discriminant D and do not require c to be coprime to D.

Let $c \in \mathbb{N}$ and $\mathfrak{m} = \mathfrak{m}_f \mathfrak{m}_\infty$ where $\mathfrak{m}_f = c\mathcal{O}_F$ and $\mathfrak{m}_\infty$ is a subset of the real places of F. For a real place v, let σ_v be the associated embedding of F into $\mathbb{R}$. Let $J_\mathfrak{m}$ be the group of fractional ideals of $\mathcal{O}_F$ which are prime to $\mathfrak{m}_f$. Let $F_\mathfrak{m}^1$ be the subset of $F^\times$ consisting of $x \in F^\times$ such that $\sigma_v(x) > 0$ for each $v \in \mathfrak{m}_\infty$ and $v_\mathfrak{p}(x-1) \geq v_\mathfrak{p}(c)$ for $\mathfrak{p} | \mathfrak{m}_f$. Let $P_\mathfrak{m}^1$ denote the set of principal ideals generated by elements of $F_\mathfrak{m}^1$. Then the ray class group mod $\mathfrak{m}$ of F is $\mathrm{Cl}_\mathfrak{m}(F) = J_\mathfrak{m}/P_\mathfrak{m}^1$.

Let $F_\mathfrak{m}^\mathbb{Z}$ be the set of $x \in F^\times$ such that $\sigma_v(x) > 0$ for each $v \in \mathfrak{m}_\infty$ and for each $\mathfrak{p} | \mathfrak{m}_f$ there exists $a \in \mathbb{Z}$ coprime to c such that $v_\mathfrak{p}(x-a) \geq v_\mathfrak{p}(c)$. Let $P_\mathfrak{m}^\mathbb{Z}$ be the set of principal ideals in F generated by elements of $F_\mathfrak{m}^\mathbb{Z}$. Then the ring class group mod $\mathfrak{m}$ of F is $G_\mathfrak{m}(F) = J_\mathfrak{m}/P_\mathfrak{m}^\mathbb{Z}$. Note we can write $F_\mathfrak{m}^\mathbb{Z} = \bigcup_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} aF_\mathfrak{m}^1$. Hence $\mathrm{Cl}_\mathfrak{m}(F)/\mathrm{Im}(\mathbb{Z}/c\mathbb{Z})^\times \simeq G_\mathfrak{m}(F)$, where $\mathrm{Im}(\mathbb{Z}/c\mathbb{Z})^\times$ denotes the image of the natural map from $(\mathbb{Z}/c\mathbb{Z})^\times$ to $\mathrm{Cl}_\mathfrak{m}(F)$, which is not in general injective.

Via the usual correspondence between ideals and ideles, $J_\mathfrak{m}$ is identified with $\hat{F}_\mathfrak{m}^\times / \hat{\mathcal{O}}_F^\times$, where $\hat{F}_\mathfrak{m}^\times$ consists of finite ideles (α_v) such that $\alpha_v \in \mathcal{O}_{F,v}^\times$ for all $v | \mathfrak{m}_f$ and $\hat{\mathcal{O}}_F^\times = \prod_{v<\infty} \mathcal{O}_{F,v}^\times$. For $v < \infty$, we put $W_v = \mathcal{O}_{F,v}^\times$ unless $v \mid \mathfrak{m}_f$, in which case $W_v = 1 + \mathfrak{m}_f \mathcal{O}_{F,v}$. For $v \mid \infty$, we put $W_v = F_v^\times$ unless $v \mid \mathfrak{m}_\infty$, in which case $W_v = \mathbb{R}_{>0}$. Now define $W = \prod W_v$ and $\mathbb{A}_{F,\mathfrak{m}}^1 = \prod'_{v \nmid \mathfrak{m}} F_v^\times \times \prod_{v | \mathfrak{m}} W_v$. Then we have $F_\mathfrak{m}^1 = F^\times \cap \mathbb{A}_{F,\mathfrak{m}}^1$ and $J_\mathfrak{m} \simeq \mathbb{A}_{F,\mathfrak{m}}^1/W$, so $\mathrm{Cl}_\mathfrak{m}(F) = F_\mathfrak{m}^1 \backslash \mathbb{A}_{F,\mathfrak{m}}^1/W = F^\times \backslash \mathbb{A}_F^\times/W$.

For the ring class group, again we can realize it as a quotient of the idele class group $F^\times \backslash \mathbb{A}_F^\times$, but now it will be a quotient by a subgroup $U = \prod U_\ell \times U_\infty$ which is a product over rational primes, rather than primes of F. As usual, for a rational prime $\ell < \infty$, write $\mathcal{O}_{F,\ell}$ for $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_\ell$, which is isomorphic to $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$ if ℓ splits in F and otherwise is $\mathcal{O}_{F,v}$ if v is the unique prime of F above ℓ. Now set $U_\ell = \mathcal{O}_{F,\ell}^\times$ if $\ell \nmid c$ and $U_\ell = (\mathbb{Z}_\ell + c\mathcal{O}_{F,\ell})^\times$ if $\ell \mid c$. We can uniformly write $U_\ell = \mathcal{O}_{c,\ell}^\times$ for $\ell < \infty$, where $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_F$ and $\mathcal{O}_{c,\ell} = \mathcal{O}_c \otimes_\mathbb{Z} \mathbb{Z}_\ell$. For later use, we will write $\hat{\mathcal{O}}_c^\times = \prod U_\ell$. Note that this is different from the product $\prod_{v<\infty} \mathcal{O}_{c,v}^\times$ for v running over primes of F if c is divisible by primes which split in F. Put $U_\infty = W_\infty = \prod_{v|\infty} W_\infty$ and

$\mathbb{A}_{F,\mathfrak{m}}^{\mathbb{Z}} = \prod'_{v\nmid\mathfrak{m}} F_v^\times \times \prod_{v|\mathfrak{m}} U_v$. Then $F_{\mathfrak{m}}^{\mathbb{Z}} = \mathbb{A}_{F,\mathfrak{m}}^{\mathbb{Z}} \cap F^\times$ and we see the ring class group is

$$G_{\mathfrak{m}}(F) = F_{\mathfrak{m}}^{\mathbb{Z}} \backslash \mathbb{A}_{F,\mathfrak{m}}^{\mathbb{Z}}/U = F^\times \backslash \mathbb{A}_F^\times/U.$$

In our case of interest, namely F is real quadratic and $\mathfrak{m}_\infty$ contains both real places of F, we write $U_\infty = F_\infty^+$. Thus we can write our strict ring class group as

$$G_c^+ = F^\times \backslash \mathbb{A}_F^\times / \hat{\mathcal{O}}_c^\times F_\infty^+. \tag{5}$$

3.3. **Special value formulas.** We return to our case of interest where $F/\mathbb{Q}$ is real quadratic of discriminant D, f is a weight k newform for $\Gamma_0(M)$, c is an integer coprime with DM, and $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_F$. Let H_c be the corresponding ring class field and h_c be the degree of H_c/F, which coincides with the cardinality of $\mathrm{Pic}(\mathcal{O}_c)$. Denote by h_c^+ the cardinality of G_c^+, so h_c^+/h_c is equal to 1 or 2. Fix ideals $\mathfrak{a}_\sigma$ for all $\sigma \in G_c = \mathrm{Gal}(H_c/F)$ in such a way that $\Sigma_c = \{\mathfrak{a}_\sigma \mid \sigma \in G_c\}$ is a complete system of representatives for $\mathrm{Pic}(\mathcal{O}_c)$. Clearly $\Sigma_c^+ = \Sigma_c$ is also a system of representatives for $\mathrm{Pic}^+(\mathcal{O}_c)$ if $h_c^+ = h_c$, while if $h_c^+ \neq h_c$ the set Σ_c^+ of representatives of $\mathrm{Pic}^+(\mathcal{O}_c)$ can be written as $\Sigma_c \cup \Sigma_c'$ with $\Sigma_c' = \{\mathfrak{d}a_\sigma \mid \sigma \in G_c\}$ and $\mathfrak{d} = (\sqrt{D})$. Let $\epsilon_c > 1$ be the smallest totally positive power of a fundamental unit in $\mathcal{O}_c^\times$, and for all $\sigma \in G_c^+$ define $\gamma_\sigma = \psi_\sigma(\epsilon_c)$. Finally, define

$$\alpha = \prod_{\ell | c,\, \left(\frac{D}{\ell}\right) = -1} \ell, \tag{6}$$

where ℓ runs over all rational primes dividing c which are inert in F.

Denote by π_f and π_χ the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to f and χ, respectively.

Theorem 3.1. *Let c be an integer such that $(c, DM) = 1$. Let χ be a character of G_c^+ such that the absolute norm of the conductor of χ is $c(\chi) = c^2$. For any choice of the base point $\tau_0 \in \mathcal{H}$, we have*

$$L(\pi_f \otimes \pi_\chi, 1/2) = \frac{4}{\alpha^2 \cdot (Dc^2)^{(k-1)/2}} \left| \sum_{\sigma \in G_c^+} \chi^{-1}(\sigma) \int_{\tau_0}^{\gamma_\sigma(\tau_0)} f(z) Q_{\psi_\sigma}(z,1)^{(k-2)/2}\, dz \right|^2.$$

When $c = 1$, this is the positive weight case of [Pop06, Theorem 6.3.1], which also treats weight 0 Maass forms. If desired, one could similarly extend the above result to weight 0 Maass forms.

Proof. Write $\pi := \pi_f = \otimes'_v \pi_v$, where v runs over all places of $\mathbb{Q}$, and let $n_\ell(\pi)$ be the conductor of π_ℓ for each prime number ℓ. Define

$$U_f(M) = \prod_\ell U_\ell(M), \quad U_\ell(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_\ell) : c \equiv 0 \mod M \right\}.$$

We associate to f the automorphic form $\varphi_\pi = \varphi_f$ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by

$$\varphi_\pi : Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/U_f(M) \longrightarrow \mathbb{C}$$

$$\varphi_\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2 j(g; i)^k f\left(\frac{ai+b}{ci+d}\right),$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$, where we write $j(g; i) = \det(g)^{1/2}(ci+d)^{-1}$ for the automorphy factor. Then φ_π is $R_{0,\ell}$-invariant for each finite prime ℓ. The scaling factor of 2 is present so that the archimedean zeta integral of φ_π gives the archimedean L-factor.

For $\phi \in \pi$, let

$$(\phi, \phi) = \int_{Z(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})} \phi(t)\overline{\phi(t)}\, dt$$

be the Petersson norm of ϕ, where we take as measures on the groups $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and $Z(\mathbb{A}_\mathbb{Q})$ the products of the local Tamagawa measures. Here, as usual, we take the quotient measure on the quotient, giving $\mathrm{GL}_2(\mathbb{Q}) \subset \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ the counting measure.

Denote by π_F the base change of π to F. Let $L(\pi_F \otimes \chi, s)$ be the L-function of π_F twisted by χ, which equals the Rankin–Selberg L-function $L(\pi_f \otimes \pi_\chi, s)$. Since F splits at each prime ℓ where π is ramified and at ∞, $\epsilon(\pi_{F,v}\otimes, \chi_v, 1/2) = +1$ for all places v of $\mathbb{Q}$. Then, in our setting, the main result in [MW09, Theorem 4.2] states that

$$\frac{|P_\chi(\varphi)|^2}{(\varphi,\varphi)} = \frac{L(\pi_F \otimes \chi, 1/2)}{(\varphi'_\pi, \varphi'_\pi)} \cdot \frac{4}{\sqrt{Dc(\chi)}} \cdot \prod_{\ell | c} \left(\frac{\ell}{\ell - \chi_D(\ell)} \right)^2, \tag{7}$$

where $\varphi \in \pi$ is a suitable test vector,

$$P_\chi(\varphi) = \int_{F^\times \mathbb{A}_\mathbb{Q}^\times \backslash \mathbb{A}_F^\times} \varphi(t)\chi^{-1}(t)\, dt,$$

and φ'_π (denoted φ_π in *loc. cit.*) is a vector in π differing from φ_π only at ∞. We describe φ and φ'_π precisely below. Similar to before, we take the products of local Tamagawa measures on $\mathbb{A}_F^\times$ and $\mathbb{A}_\mathbb{Q}^\times$, and give $F^\times$ the counting measure.

First we describe the choice of the test vector φ, which we only need to specify up to scalars, as the left-hand side above is invariant under scalar multiplication. We will take $\varphi = \otimes'_v \varphi_v$, where v runs over all places of $\mathbb{Q}$. For ℓ a finite prime of $\mathbb{Q}$, let $c(\chi_\ell)$ denote the smallest n such that χ_ℓ is trivial on $(\mathbb{Z}_\ell + \ell^n \mathcal{O}_{F,\ell})^\times$. Since χ is a character of G_c^+, we have $c(\chi_\ell) \le v_\ell(c)$ for all ℓ. In particular, χ_ℓ is trivial on $\mathbb{Z}_\ell^\times$, so $c(\chi_\ell)$ is the smallest n such that χ_ℓ is trivial on $(1 + \ell^n \mathcal{O}_{F,\ell})^\times$, and thus agrees with the usual definition of the conductor of χ_ℓ when ℓ is inert in F. Similarly, if ℓ is ramified in F, say $\ell = \mathfrak{p}^2$, then $c(\chi_\ell)$ is twice the conductor of $\chi_\ell = \chi_\mathfrak{p} : F_\mathfrak{p}^\times \to \mathbb{C}^\times$, though this case does not occur by our assumption $(c, D) = 1$. If $\ell = \mathfrak{p}_1\mathfrak{p}_2$ is split in F, then we can write $\chi_\ell = \chi_{\mathfrak{p}_1} \otimes \chi_{\mathfrak{p}_2}$ with $\chi_{\mathfrak{p}_1}$, $\chi_{\mathfrak{p}_2}$ characters of $\mathbb{Q}_\ell^\times$, which are inverses of each other on $\mathbb{Z}_\ell^\times$ as χ_ℓ is trivial on $\mathbb{Z}_\ell^\times$. Hence $\chi_{\mathfrak{p}_1}$ and $\chi_{\mathfrak{p}_2}$ have the same conductor, which is $c(\chi_\ell)$. Consequently, $c(\chi)$, the absolute norm of the conductor of χ, is

$$c(\chi) = \mathrm{Norm}\left(\prod_\ell \ell^{c(\chi_\ell)} \right) = \prod_\ell \ell^{2c(\chi_\ell)}. \tag{8}$$

Note that since $(c, M) = 1$, we have $c(\chi_\ell) = 0$ whenever π_ℓ is ramified, i.e., the conductor $c(\pi_\ell) > 0$. If $c(\chi_\ell) = 0$, let $R_{\chi,\ell}$ be an Eichler order of reduced discriminant $\ell^{c(\pi_\ell)}$ in $M_2(\mathbb{Q}_\ell)$ containing $\mathcal{O}_{F,\ell}$. If $c(\chi_\ell) > 0$, so π_ℓ is unramified, let $R_{\chi,\ell}$ be a maximal order of $M_2(\mathbb{Q}_\ell)$ which optimally contains $\mathbb{Z}_\ell + \ell^{c(\chi_\ell)}\mathcal{O}_{F,\ell}$. In either case, $R_{\chi,\ell}$ is unique up to conjugacy and pointwise fixes a 1-dimensional subspace of π_ℓ. For $\ell < \infty$, take $\varphi_\ell \in \pi_\ell^{R_{\chi,\ell}}$ nonzero, normalized in such a way that $\otimes' \varphi_\ell$ converges. For instance, we can take $\varphi_\ell = \varphi_{\pi,\ell}$ at almost all ℓ. Each φ_ℓ is a local Gross–Prasad test vector, and our assumptions imply that the local Gross–Prasad test vectors φ_ℓ are (up to scalars) translates of the new vectors $\varphi_{\pi,\ell}$. (Gross–Prasad test vectors are not translates of new vectors in general.)

Embed F into $\mathrm{M}_2(\mathbb{Q})$ as follows. Consider a quadratic form

$$Q(x, y) = -\frac{C}{2}x^2 + Axy + \frac{B}{2}y^2 \in \mathcal{F}_{Dc^2}.$$

This means Q is primitive of discriminant $Dc^2 = A^2 + BC$, $2 \mid B$ and $2M \mid C$, which implies $A^2 \equiv Dc^2 \mod 4$. Take the embedding of F into $\mathrm{M}_2(\mathbb{Q})$ induced by $\sqrt{D}c \mapsto \left(\begin{smallmatrix} A & B \\ C & -A \end{smallmatrix}\right)$. Then $\mathcal{O}_c = R_0 \cap F$, and

$$F_\infty^\times = \left\{ g(x, y) := \begin{pmatrix} x + Ay & By \\ Cy & x - Ay \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}.$$

For a prime $\ell \nmid c$, we have $\mathcal{O}_{c,\ell} = \mathcal{O}_{F,\ell} \subset R_{0,\ell}$. Thus we may take $R_{\chi,\ell} = R_{0,\ell}$ for $\ell \nmid c$ such that χ_ℓ is unramified—in particular, for $\ell \nmid cD$. When χ_ℓ is ramified, we may take $R_{\chi,\ell} = R_{0,\ell}$ if and only if $c(\chi_\ell) = v_\ell(c)$. By assumption, $c(\chi) = \prod_\ell \ell^{2c(\chi_\ell)} = c^2$, so we may take $R_{\chi,\ell} = R_{0,\ell}$ at each finite place ℓ. Thus we may and will take φ_ℓ to be $\varphi_{\pi,\ell}$ at each ℓ.

Now we describe φ_∞. Note we can identify $F_\infty^\times/\mathbb{Q}_\infty^\times$ with $F_\infty^1/\{\pm 1\}$, where

$$F_\infty^1 = F_\infty^{1,+} \cup F_\infty^{1,-}, \quad F_\infty^{1,\pm} = \{g(x,y) \in F_\infty^\times : \det g(x,y) = x^2 - Dc^2y^2 = \pm 1\}.$$

Let

$$\gamma_\infty = \begin{pmatrix} A + \sqrt{D}c & A - \sqrt{D}c \\ C & C \end{pmatrix}.$$

Then

$$\gamma_\infty^{-1} \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \gamma_\infty = \begin{pmatrix} \sqrt{D}c & 0 \\ 0 & -\sqrt{D}c \end{pmatrix}.$$

So

$$\gamma_\infty^{-1} F_\infty^1 \gamma_\infty = \left\{ \begin{pmatrix} x + y\sqrt{D}c & 0 \\ 0 & x - y\sqrt{D}c \end{pmatrix} : x^2 - Dc^2y^2 = \pm 1 \right\}.$$

The maximal compact subgroup of F_∞^1 is

$$\Gamma_F = \gamma_\infty \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \gamma_\infty^{-1} = \{\pm I, \pm g(0, -(\sqrt{D}c)^{-1})\},$$

where one reads the $\pm$ signs independently. Let $U_\infty = \gamma_\infty \mathrm{O}(2) \gamma_\infty^{-1}$, where $\mathrm{O}(2)$ denotes the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$. Then $U_\infty \supset \Gamma_F$, and the archimedean test vector in [MW09] is the unique up to scalars nonzero vector φ_∞ lying in the minimal U_∞-type such that Γ_F acts by χ_∞ on φ_∞. Specifically, we can take

$$\varphi_\infty = \pi_\infty(\gamma_\infty)(\varphi_{\infty,k} \pm \varphi_{\infty,-k}), \tag{9}$$

where $\varphi_{\infty,\pm k} = \frac{1}{2}\pi_\infty\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\varphi_\pi$ is a vector of weight $\pm k$ in π_∞, and the $\pm$ sign in (9) matches the sign of $\chi_\infty\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This completely describes the test vector φ chosen in [MW09].

For our purposes, we would like to work with a different archimedean component than φ_∞, corresponding to (a translate of) φ_π. Let φ^- be the pure tensor in π which agrees with φ at all finite places, and is defined like φ_∞ at infinity except using the opposite sign in the sum (9). Then necessarily any χ_∞-equivariant linear function on π_∞ kills φ_∞^-, so $P_\chi(\varphi^-) = 0$. Hence $P_\chi(\varphi) = P_\chi(\varphi')$ where $\varphi' = \varphi + \varphi^-$, and we can write $\varphi' = \otimes \varphi'_v$, where $\varphi'_\ell = \varphi_\ell$ for finite primes ℓ and $\varphi'_\infty = \pi_\infty(\gamma_\infty)\varphi_\pi$, i.e., $\varphi'(x) = \varphi_\pi(x\gamma_\infty)$.

Finally, we describe the vector φ'_π appearing in (7). It is defined to a factorizable function in π whose associated local Whittaker functions are new vectors whose zeta functions are the local L-factors of π at finite places, and at infinity is the vector in the minimal $\mathrm{O}(2)$-type that transforms by χ_∞ under $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$ such that the associated Whittaker function (restricted to first diagonal component) at infinity is $W_\infty(t) = 2\chi_\infty\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)|t|^{k/2}e^{-2\pi|t|}$. (This normalization gives $L_\infty(s,\pi) = \int_0^\infty W_\infty(t)|t|^{s-1/2}\, d^\times t$.) Thus φ'_π agrees with φ_π at all finite places and $\varphi'_{\pi,\infty} = 2(\varphi_{\infty,k} \pm \varphi_{\infty,-k})$, where the $\pm$ sign matches that in (9).

Hence $\varphi = \frac{1}{2}\pi(\gamma_\infty)\varphi'_\pi$, so by invariance of the inner product we have $(\varphi,\varphi) = \frac{1}{4}(\varphi'_\pi, \varphi'_\pi)$, and (7) becomes

$$\left|P_\chi(\varphi')\right|^2 = |P_\chi(\varphi)|^2 = L(\pi_F \otimes \chi, 1/2) \cdot \frac{1}{\sqrt{D}c} \cdot \prod_{\ell | c} \left(\frac{\ell}{\ell - \chi_D(\ell)} \right)^2. \tag{10}$$

Now we want to rewrite $P_\chi(\varphi')$. Recall that $\epsilon_c > 1$ is the smallest totally positive power of a fundamental unit in $\mathcal{O}_c^\times$. From (5), we obtain the isomorphism

$$\mathbb{A}_\mathbb{Q}^\times F^\times \backslash \mathbb{A}_F^\times / \hat{\mathcal{O}}_c^\times \simeq G_c^+ \cdot (F_\infty^+ / \langle \epsilon_c \rangle \mathbb{Q}_\infty^+) \simeq G_c^+ \cdot (F_\infty^{1,+} / \langle \pm \epsilon_c \rangle).$$

We may identify

$$F_\infty^{1,+}/\langle \pm\epsilon_c\rangle = \left\{ \begin{pmatrix} x+Ay & By \\ Cy & x-Ay \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : 1 \le x + y\sqrt{D}c < \epsilon_c \right\},$$

and the orbit of $\gamma_\infty i$ in the upper half plane by this set is the geodesic segment connecting $\gamma_\infty i$ to $\epsilon_c \gamma_\infty i$, i.e., the image under γ_∞ of $\{iy : 1 \le y \le \epsilon_c^2\} \subset \mathcal{H}$. Let us call this arc Υ.

Since $\mathbb{A}_\mathbb{Q}^\times \subset F^\times \hat{\mathcal{O}}_c^\times F_\infty^+$ and $G_c^+ \simeq F^\times \backslash \mathbb{A}_F^\times / \hat{\mathcal{O}}_c^\times F_\infty^+$ where $F_\infty^+ = (\mathbb{R}_{>0})^2$, we see that χ is trivial on $\mathbb{A}_\mathbb{Q}^\times \hat{\mathcal{O}}_c^\times F_\infty^+$. The Tamagawa measure gives $\mathrm{vol}(F^\times \mathbb{A}_\mathbb{Q}^\times \backslash \mathbb{A}_F^\times) = 2L(1,\eta) = 4h_F \log \epsilon_F \mathrm{vol}(\hat{\mathcal{O}}^\times)$, where η is the quadratic character of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ attached to $F/\mathbb{Q}$ and ϵ_F is the fundamental unit of F. This implies $\mathrm{vol}(\mathbb{A}_\mathbb{Q}^\times F^\times \backslash \mathbb{A}_F^\times / \hat{\mathcal{O}}_c^\times) = 2h_c^+ \mathrm{len}(\Upsilon)$, where $\mathrm{len}(\Upsilon) = 2\log \epsilon_c$ is the length of Υ with respect to the usual hyperbolic distance. Thus we compute

$$\begin{aligned} P_\chi(\varphi') &= 2\,\mathrm{vol}(\hat{\mathcal{O}}_c^\times) \sum_{t \in G_c^+} \chi^{-1}(t) \int_{F_\infty^{1,+}/\langle \pm\epsilon_c\rangle} \varphi_\pi(tg\gamma_\infty)\, dg \\ &= 4\,\mathrm{vol}(\hat{\mathcal{O}}_c^\times) \sum_{t \in G_c^+} \chi^{-1}(t) \int_1^{\epsilon_c} j\left(t\gamma_\infty \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right); i\right)^k f\left(t\gamma_\infty \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \cdot i\right) d^\times u \\ &= 2\,\mathrm{vol}(\hat{\mathcal{O}}_c^\times) \sum_{t \in G_c^+} \chi^{-1}(t) \int_1^{\epsilon_c^2} j\left(t\gamma_\infty \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right); i\right)^k f\left(t\gamma_\infty \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot i\right) d^\times y, \end{aligned}$$

where we use that f has trivial central character and the substitution $y = u^2$ at the last step.

For ℓ a rational prime dividing c, note that $\mathcal{O}_{F,\ell}^\times / \mathcal{O}_{c,\ell}^\times \simeq \mathbb{Z}_\ell^\times/(1 + c\mathbb{Z}_\ell)$ when ℓ splits in F and $\mathcal{O}_{F,\ell}^\times/\mathcal{O}_{c,\ell}^\times \simeq (\mathcal{O}_{F,\ell}^\times/(1 + c\mathcal{O}_{F,\ell}))/(\mathbb{Z}_\ell^\times/(1+c\mathbb{Z}_\ell))$ when ℓ is inert in F. Hence, with our choice of measures,

$$\mathrm{vol}(\hat{\mathcal{O}}_c^\times) = \mathrm{vol}(\hat{\mathcal{O}}_F^\times) \prod_{\ell | c} [\mathcal{O}_{F,\ell}^\times : \mathcal{O}_{c,\ell}^\times]^{-1} = \frac{1}{\sqrt{D}} \prod_{\ell | c,\, (\frac{D}{\ell}) = 1} \frac{1}{(\ell - 1)\ell^{v_\ell(c) - 1}} \cdot \prod_{\ell | c,\, (\frac{D}{\ell}) = -1} \frac{1}{(\ell+1)\ell^{v_\ell(c)}},$$

where ℓ runs over rational primes.

The next task is then to rewrite the integral appearing the in right hand side of the above formula. Let $z = \gamma_\infty iy$. Then

$$z = \frac{A}{C} + \frac{\sqrt{D}c}{C}\left(1 - \frac{2}{1+iy}\right).$$

Since

$$\frac{2iy}{(1+iy)^2} = \frac{2}{1+iy} - \frac{1}{2}\left(\frac{2}{1+iy}\right)^2 = \frac{BC + 2ACz - C^2z^2}{2Dc^2},$$

we have

$$j\left(\gamma_\infty \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}; i\right)^{-2} = \frac{C(1+iy)^2}{2\sqrt{D}cy} = \frac{2i\sqrt{D}c}{-Cz^2 + 2Az + B} = \frac{i\sqrt{D}c}{Q(z,1)},$$

and

$$dz = \frac{2iy\sqrt{D}c}{C(1+iy)^2}\, d^\times y, \quad \text{i.e.} \quad d^\times y = \frac{2\sqrt{D}c}{-Cz^2 + 2Az + B}\, dz = \frac{\sqrt{D}c}{Q(z,1)}\, dz.$$

Making the change of variable $z = \gamma_\infty iy$, the above expression can be rewritten as

$$P_\chi(\varphi') = \frac{2\,\mathrm{vol}(\hat{\mathcal{O}}_c^\times)}{i^{k/2} \cdot (\sqrt{D}c)^{(k-2)/2}} \cdot \sum_{t \in G_c^+} \chi^{-1}(t) \int_\Upsilon f(tz) \cdot Q(z,1)^{(k-2)/2}\, dz.$$

After another change of variable $z' = tz$, the above integral becomes

$$\begin{aligned}\int_{t\Upsilon} f(z')\cdot Q(t^{-1}z',1)^{(k-2)/2}\,dz' &= \int_{t\Upsilon} f(z')\cdot (Q|t^{-1})(z',1)^{(k-2)/2}\,dz' \\ &= \int_{\tau_t}^{(t\epsilon_c t^{-1})\tau_t} f(z')\cdot (Q|t^{-1})(z',1)^{(k-2)/2}\,dz',\end{aligned}$$

where $\tau_t = t\gamma_\infty i$. Now, as long as t varies in G_c^+, the quadratic forms $Q|t^{-1}$ are representatives for the classes in $\mathcal{F}_{Dc^2}/\Gamma_0(M)$, as discussed in §3.1. Moreover, since Υ is closed in $\mathcal{H}/\Gamma_0(N)$, this integral does not depend on the choice of base point. Plugging this into (10) gives the asserted formula. □

3.4. **Genus fields attached to orders.** Assume from now on that c is odd. The *genus field* attached to the order $\mathcal{O}_c$ of discriminant Dc^2 is the finite abelian extension of $\mathbb{Q}$, with Galois group isomorphic to copies of $\mathbb{Z}/2\mathbb{Z}$, contained in the strict class field H_c^+ of F of conductor c and generated by the quadratic extensions $\mathbb{Q}(\sqrt{D_i})$ and $\mathbb{Q}(\sqrt{\ell^*})$ where $D = \prod_i D_i$ is any possible factorization of D into primary discriminants, $\ell \mid c$ is a prime number and $\ell^* = (-1)^{(\ell-1)/2}\ell$. See [Coh78, pp. 242-244] for details.

Fix a quadratic character $\chi : G_c^+ \to \{\pm 1\}$.

Definition 3.2. We say that χ is *primitive* if it does not factor through H_f^+ for a proper divisor $f \mid c$.

We assume that $\chi : G_c^+ \to \{\pm 1\}$ is primitive. By (8), this means $c(\chi) = c^2$. Then χ cuts out a quadratic extension H_χ/F which, by genus theory for $\mathcal{O}_c$, is biquadratic over $\mathbb{Q}$. Each quadratic extension of $\mathbb{Q}$ contained in the genus field of the order $\mathcal{O}_c$ is of the form $\mathbb{Q}(\sqrt{\Delta})$ for some $\Delta = D'\cdot\prod_{j=1}^s \ell_j^*$, with $\ell_j \mid c$ and D' a fundamental discriminant dividing D. Write $H_\chi = \mathbb{Q}(\sqrt{\Delta_1},\sqrt{\Delta_2})$, with $\Delta_i = D_i\cdot\prod_{j=1}^{s_i}\ell_{i,j}^*$ for $i=1,2$ as above (so $\ell_{i,j}$ are primes dividing c), and let $K_1 = \mathbb{Q}(\sqrt{\Delta_1})$ and $K_2 = \mathbb{Q}(\sqrt{\Delta_2})$. Since the third quadratic extension contained in H_χ is the quadratic extension is $\mathbb{Q}(\sqrt{D})$, we have $\Delta_1\cdot\Delta_2 \equiv D\cdot x^2$ for some $x \in \mathbb{Q}^\times$. We can write $\Delta_1 = D_1 d$ and $\Delta_2 = D_2 d$ for some $d = \prod_{j=1}^s \ell_j^*$ with $\ell_i \mid c$ primes and $D = D_1\cdot D_2$ a factorization into fundamental discriminants, allowing $D_1 = D$ or $D_2 = D$. If $d \neq \pm c$, then χ factors through the extension $H_d^+ \neq H_c^+$ by the genus theory of the order of conductor Dd^2, and therefore χ is not a primitive character of H_c^+. So $d = \pm c$. Thus we conclude that the quadratic fields $K_1 = \mathbb{Q}(\sqrt{D_1 d})$ and $K_2 = \mathbb{Q}(\sqrt{D_2 d})$ satisfy the following properties:

- $D_1\cdot D_2 = D$, where D_1 and D_2 are two coprime fundamental discriminants (possibly equal to 1).
- $d = \pm c$ and d is a fundamental discriminant.

Let $\chi_{D_1 d}$ and $\chi_{D_2 d}$ be the quadratic characters attached to the extensions K_1 and K_2 respectively; thus $\chi_{D_1 d}(x) = \left(\frac{D_1 d}{x}\right)$ and $\chi_{D_2 d}(x) = \left(\frac{D_2 d}{x}\right)$. Similarly, let χ_D be the quadratic character attached to the extension $F/\mathbb{Q}$, i.e., $\chi_D(x) = \left(\frac{D}{x}\right)$. In particular, for all $\ell \nmid c$ we have

$$\chi_D(\ell) = \chi_{D_1 d}(\ell)\cdot\chi_{D_2 d}(\ell). \tag{11}$$

Say that χ has sign $+1$ if H_χ/F is totally real, and sign -1 otherwise. If χ has sign $w_\infty \in \{\pm 1\}$, put $I_f = I_f^{w_\infty}$ and $\Omega_f = \Omega_f^{w_\infty}$. Define

$$\mathbb{L}(f,\chi) := \sum_{\sigma\in G_c^+} \chi^{-1}(\sigma) I_f\{\tau_0 \to \gamma_{\psi_\sigma}(\tau_0)\}\big(Q_{\psi_\sigma}(x,y)^{(k-2)/2}\big).$$

Lemma 3.3. $\overline{\mathbb{L}(f,\chi)} = w_\infty\cdot\mathbb{L}(f,\chi)$.

Proof. This follows from the discussion in [Pop06, §6.1]. To simplify the notation, define

$$\Theta_\psi := I_f\{\tau_0 \to \gamma_\psi(\tau_0)\}\big(Q_\psi(x,y)^{(k-2)/2}\big),$$

which is independent of the choice of τ_0 and the $\Gamma_0(M)$-conjugacy class of ψ. Let $z \mapsto \bar{z}$ denote complex conjugation. A direct computation shows that $\overline{\Theta_\psi} = \Theta_{\psi^*}$ where recall that $\psi^* = \omega_\infty \psi \omega_\infty^{-1}$. From the discussion in §3.1 we have $\sigma_F \cdot [\psi] = [\psi^*]$, and it follows that $\overline{\Theta_\psi} = \Theta_{\sigma_F \psi}$. Taking sums over a set of representatives of optimal embeddings shows that $\overline{\mathbb{L}(f,\chi)} = \chi(\sigma_F) \cdot \mathbb{L}(f,\chi)$. Let H_χ be the field cut out by χ. The description of a system of representatives Σ_c and Σ_c^+ of $\mathrm{Gal}(H_c/F)$ and $\mathrm{Gal}(H_c^+/F)$ in §3.3 shows that if $\chi(\sigma_F) = 1$ then H_χ is contained in H_c, and therefore H_χ is totally real. On the other hand, if $\chi(\sigma_F) = -1$, then H_χ cannot be contained in H_c, and therefore it is not totally real, so it is the product of two imaginary extensions. By definition of the sign of χ, this means that $\mathbb{L}(f,\chi)$ is a real number when χ is even, and is a purely imaginary complex number when χ is odd, and the result follows. ☐

Using the relation

$$L(\pi_g \times \pi_\chi, 1/2) = \frac{4}{(2\pi)^k}\left(\left(\frac{k-2}{2}\right)!\right)^2 L(f/F, \chi, k/2),$$

it follows from Lemma 3.3 that Theorem 3.1 can be rewritten in the following form:

$$L(f/F,\chi,k/2) = \frac{(2\pi i)^{k-2} \cdot \Omega_f^2 \cdot w_\infty}{\left(\left(\frac{k-2}{2}\right)!\right)^2 \cdot \alpha^2 \cdot (Dc^2)^{(k-1)/2}} \cdot \mathbb{L}(f,\chi)^2. \tag{12}$$

Remark 3.4. By the lemma, the sign w_∞ should also appear in equation (28) of [BD09], as the left hand side of that equation is not positive when χ is odd. However, the main result in [BD09] still follows as this sign will cancel out with a sign arising from Gauss sums as in our argument below.

4. p-adic L-functions

Recall the notation introduced in §2.4: f_∞ is the Hida family passing through the weight two modular form f of level $N = Mp$ associated to the elliptic curve E by modularity; U is a connected neighborhood of 2 in the weight space $\mathcal{X}$; $\mu_*^\pm$ is a measure-valued modular symbol satisfying the property that for all integers $k \in U$, $k \geq 2$, there is $\lambda^\pm(k) \in \mathbb{C}_p^\times$ such that $\rho_k(\mu_*^\pm) = \lambda^\pm(k) I_{f_k}^\pm$ and $\lambda^\pm(2) = 1$.

4.1. p-adic L-function of real quadratic fields.

For any $Q \in \mathcal{F}_{Dc^2}$ and $\kappa \in U$, define

$$Q(x,y)^{(\kappa-2)/2} = \exp_p\left(\frac{\kappa-2}{2}\log_q\left(\langle Q(x,y)\rangle\right)\right)$$

where $\exp_p$ is the p-adic exponential and for $x \in \mathbb{Q}_p$, we let $\langle x \rangle$ denote the principal unit of x, satisfying $x = p^{\mathrm{ord}_p(x)}\zeta\langle x\rangle$ for a $(p-1)$-th root of unity ζ. Recall the Hida family f_∞ introduced in §2.4.

Definition 4.1. Let $Q \in \mathcal{F}_{Dc^2}$ and let γ_{τ_Q} be the generator of the stabilizer of the root τ_Q of $Q(z,1)$, chosen as in Definition 2.1.

(1) Let $r \in \mathbb{P}^1(\mathbb{Q})$. The *partial square root p-adic L-function* attached to f_∞, a choice of sign $\pm$, and Q is the function of $\kappa \in U$ defined by

$$\mathcal{L}_p^\pm(f_\infty/F, Q, \kappa) = \int_{(\mathbb{Z}_p^2)'} Q(x,y)^{(\kappa-2)/2}\, d\mu_*^\pm\{r \to \gamma_{\tau_Q}(r)\}(x,y).$$

(2) Let χ be a character of G_c^+. The *square root p-adic L-function* attached to f_∞, a choice of sign $\pm$, and χ is the function of $\kappa \in U$ defined by

$$\mathcal{L}_p^{\pm}(f_\infty/F, \chi, \kappa) = \sum_{\sigma \in G_c^+} \chi^{-1}(\sigma)\mathcal{L}_p^{\pm}(f_\infty/F, Q^\sigma, \kappa).$$

(3) The *p-adic L-function* attached to f_∞, the sign $\pm$, and χ is

$$L_p^{\pm}(f_\infty/F, \chi, \kappa) = \left(\mathcal{L}_p^{\pm}(f_\infty/F, \chi, \kappa)\right)^2.$$

Let $\chi : G_c^+ \to \{\pm 1\}$ be a quadratic ring class character. Let ϵ be the sign of χ and set $w_\infty = \epsilon$. Denote $\mu_* = \mu_*^{w_\infty}$, $\Omega_{f_k} = \Omega_{f_k}^{w_\infty}$, $\lambda(k) = \lambda^{w_\infty}(k)$ and $L_p(f_\infty/F, \chi, k) = L_p^{w_\infty}(f_\infty/F, \chi, k)$. Recall the newform $f_k^\sharp$ whose p-stabilization is the weight k specialization of the Hida family f_∞ introduced in §2.4. Define the algebraic part of the central value of the L-function of the newform $f_k^\sharp$ twisted by χ to be

$$L^{\mathrm{alg}}(f_k^\sharp/F, \chi, k/2) = \frac{\left((\frac{k-2}{2})!\right)^2 \sqrt{D}c}{(2\pi i)^{k-2} \cdot \Omega^2_{f_k^\sharp}} \cdot L(f_k^\sharp/K, \chi, k/2).$$

Theorem 4.2. *For all integers $k \in U$, $k \geq 2$, we have*

$$L_p(f_\infty/F, \chi, k) = \lambda(k)^2 \cdot \alpha^2 \cdot (1 - a_p(k)^{-2}p^{k-2})^2 \cdot (Dc^2)^{(k-2)/2} \cdot L^{\mathrm{alg}}(f_k^\sharp/F, \chi, k/2)$$

where the rational number α is defined in (6).

Proof. By definition,

$$\begin{aligned}\mathcal{L}_p(f_\infty/F, Q, k) &= \int_{(\mathbb{Z}_p^2)'} Q(x,y)^{(k-2)/2}\, d\mu_*\{r \to \gamma_{\tau_Q}(r)\}(x,y) \\ &= \lambda(k)(1 - a_p(k)^{-2}p^{k-2}) I_{f_k^\sharp}\{r \to \gamma_{\tau_Q}(r)\}(Q^{(k-2)/2}),\end{aligned}$$

where the last equality follows from [BD09, Proposition 2.4] and therefore we get, in the notation of Section 3.4,

$$L_p(f_\infty/F, \chi, k) = \lambda(k)^2(1 - a_p(k)^{-2}p^{k-2})^2 \cdot \mathbb{L}(f_k^\sharp, \chi)^2.$$

Using (12) gives the result. □

4.2. **Mazur–Kitagawa p-adic L-functions.** Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \{\pm 1\}$ be a primitive quadratic Dirichlet character of conductor m. Suppose that $\chi(-1) = (-1)^{(k-2)/2}w_\infty$ and put $\Omega_{f_k} = \Omega_{f_k}^{w_\infty}$, $\lambda(k) = \lambda^{w_\infty}(k)$ and $\mu_* = \mu_*^{w_\infty}$. For $k \in U$ a positive integer define

$$L^{\mathrm{alg}}(f_k^\sharp, \chi, k/2) = \frac{\tau(\chi)((k-2)/2)!}{(-2\pi i)^{(k-2)/2}\Omega_{f_k^\sharp}} L(f_k^\sharp, \chi, k/2)$$

as the *algebraic part* of the central special value of $L(f_k^\sharp, \chi, s)$, where $\tau(\chi) = \sum_{a=1}^m \chi(a)e^{2\pi i a/m}$ denotes the Gauss sum of the character χ. The Mazur–Kitagawa p-adic L-function is defined as

$$L_p(f_\infty, \chi, k, s) = \sum_{a=1}^m \chi(pa) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \left(x - \frac{pa}{m}y\right)^{s-1} y^{k-s-1}\, d\mu_*\{\infty \to pa/m\}$$

and satisfies the following *interpolation formula*: for all integers $k \in U$ with $k \geq 2$ we have

$$L_p(f_\infty, \chi, k, k/2) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{(k-2)/2})^2 L^{\mathrm{alg}}(f_k^\sharp, \chi, k/2). \tag{13}$$

4.3. **A factorization formula for genus characters.** Let $\chi : G_c^+ \to \{\pm 1\}$ be a primitive character, and let $\chi_{D_1 d} : \mathbb{Q}(\sqrt{D_1 d}) \to \{\pm 1\}$ and $\chi_{D_2 d} : \mathbb{Q}(\sqrt{D_2 d}) \to \{\pm 1\}$ be the associated quadratic Dirichlet characters.

Theorem 4.3. *The following equality*

$$L_p(f_\infty/F, \chi, \kappa) = \alpha^2 \cdot (Dc^2)^{(\kappa-2)/2} \cdot L_p(f_\infty, \chi_{D_1 d}, \kappa, \kappa/2) \cdot L_p(f_\infty, \chi_{D_2 d}, \kappa, \kappa/2)$$

holds for all $\kappa \in U$, where the rational number α is defined in (6).

Proof. Let $\chi_{D_i d}$ denote the quadratic characters associated with the extension $\mathbb{Q}(\sqrt{D_i d})$. Since p is inert in F, we have $\chi_D(p) = -1$, and therefore from (11) we get

$$\chi_{D_1 d}(p) = -\chi_{D_2 d}(p).$$

It follows that the Euler factor $(1 - a_p(k)^{-2} p^{k-2})^2$ appearing in Theorem 4.2 is equal to the product of the two Euler factors

$$(1 - \chi_{D_1 d}(p) a_p(k)^{-1} p^{(k-2)/2})^2 \quad \text{and} \quad (1 - \chi_{D_2 d}(p) a_p(k)^{-1} p^{(k-2)/2})^2$$

appearing in (13). By comparison of Euler factors, we see that for all even integers $k \geq 4$ in U we have

$$L(f_k^\sharp/F, \chi, s) = L(f_k^\sharp, \chi_{D_1 d}, s) \cdot L(f_k^\sharp, \chi_{D_2 d}, s). \tag{14}$$

Therefore, from Theorem 4.2 and the factorization formula (14) it follows that for all even integers $k \geq 4$ in U the following factorization formula holds:
(15)

$$L_p(f_\infty/F, \chi, k) = \left(\frac{\alpha^2 \cdot \sqrt{D} c \cdot (Dc^2)^{(k-2)/2} \cdot w_\infty}{\tau(\chi_{D_1 d}) \cdot \tau(\chi_{D_2 d})} \right) \cdot L_p(f_\infty, \chi_{D_1 d}, k, k/2) \cdot L_p(f_\infty, \chi_{D_2 d}, k, k/2).$$

Since $D_i d$ are fundamental discriminants, $\tau(\chi_{D_i d}) = \sqrt{D_i d}$ (interpreting $\sqrt{x}$ as $i\sqrt{|x|}$ for $x < 0$), so $\frac{\sqrt{D} c}{\tau(\chi_{D_1 d}) \cdot \tau(\chi_{D_2 d})} = w_\infty$, and the formula in the statement holds for all even integers $k \geq 4$ in U. Since $\mathbb{Z} \cap U$ is a dense subset of U, and the two sides of equation (15) are continuous functions in U, they coincide on U. $\square$

5. The main result

Let the notation be as in the introduction: $E/\mathbb{Q}$ is an elliptic curve of conductor $N = Mp$ with $p \nmid M$, $p \neq 2$, and $F/\mathbb{Q}$ a real quadratic field of discriminant $D = D_F$ such that all primes dividing M are split in F and p is inert in F. Finally, $c \in \mathbb{Z}$ is a positive integer prime to ND and $\chi : G_c^+ \to \{\pm\}$ is a primitive quadratic character of the strict ring class field of conductor c of F. Let w_∞ be the sign of χ, and as above put $\mathcal{L}_p(f_\infty/F, Q, \kappa) = \mathcal{L}_p^{w_\infty}(f_\infty/F, Q, \kappa)$, $\mathcal{L}_p(f_\infty/F, \chi, \kappa) = \mathcal{L}_p^{w_\infty}(f_\infty/F, \chi, \kappa)$ and $L_p(f_\infty/F, \chi, \kappa) = L_p^{w_\infty}(f_\infty/F, \chi, \kappa)$.

We begin by observing that $\mathcal{L}_p(f_\infty/F, Q, 2) = 0$, since its value is $\mu_f\{r \to \gamma_{\tau_Q}(r)\}(\mathbb{P}^1(\mathbb{Q}_p))$, and the total measure of μ_f is zero. For the next result, let w_M be the sign of the Atkin–Lehner involution acting on f. Also, let $\log_E : E(\mathbb{C}_p) \to \mathbb{C}$ denote the logarithmic map on $E(\mathbb{C}_p)$ induced from the Tate uniformization and the choice of the branch $\log_q$ of the logarithm fixed above.

Theorem 5.1. *For all quadratic characters $\chi : G_c^+ \to \{\pm 1\}$ we have*

$$\frac{d}{d\kappa} \mathcal{L}_p(f_\infty/F, \chi, \kappa)_{\kappa=2} = \frac{1}{2} (1 - \chi_{D_1 d}(-M) w_M) \log_E(P_\chi),$$

where P_χ is defined as in (1).

Proof. We have

$$\begin{aligned}\frac{d}{d\kappa}\mathcal{L}_Q(f_\infty/F,\chi,\kappa)_{\kappa=2} &= \frac{1}{2}\int_{(\mathbb{Z}_p^2)'}(\log_q(x-\tau_Q y)+\log_q(x-\bar{\tau}_Q y))\,d\mu_*\{r\to\gamma_{\tau_Q}(r)\} \\ &= \frac{1}{2}(\log_E(P_{\tau_Q})+\log_E(\tau_p P_{\tau_Q})).\end{aligned}$$

By (4), $\tau_p(J_{\tau_Q}) = -w_M J_{\tau_Q}^{\sigma\tau_Q}$ and by [BD09, Proposition 1.8] (whose extension to the present situation presents no difficulties) we know that $\chi(\sigma) = \chi_{D_1 d}(-M)$, so the result follows summing over all Q. □

Theorem 5.2. *Let χ be a primitive quadratic character of G_c^+ with associated Dirichlet characters $\chi_{D_1 d}$ and $\chi_{D_2 d}$. Suppose that $\chi_{D_1 d}(-M) = -w_M$. Then:*

1. *There is a point $\mathbf{P}_\chi$ in $E(H_\chi)^\chi$ and $n\in\mathbb{Q}^\times$ such that $\log_E(P_\chi) = n\cdot\log_E(\mathbf{P}_\chi)$.*
2. *The point $\mathbf{P}_\chi$ is of infinite order if and only if $L'(E/F,\chi,1)\neq 0$.*

Proof. By Theorem 5.1 we have

$$\frac{1}{2}\frac{d^2}{d\kappa^2}L_p(f_\infty/F,\chi,\kappa)_{\kappa=2} = \log_E^2(P_\chi).$$

On the other hand, by the factorization of Theorem 4.3 we have

$$L_p(f_\infty/F,\chi,\kappa) = \alpha^2\cdot(Dc^2)^{(k-2)/2}\cdot L_p(f_\infty,\chi_{D_1 d},\kappa,\kappa/2)\cdot L_p(f_\infty,\chi_{D_2 d},\kappa,\kappa/2),$$

where the integer α is defined in (6). Let $\mathrm{sign}(E,\chi_{D_i d}) = -w_N\chi_{D_i d}(-N)$, where w_N is the sign of the Atkin–Lehner involution at N. This is the sign of the functional equation of the complex L-series $L(E,\chi_{D_i d},s)$. Since

$$\chi_{D_1 d}(-N)\cdot\chi_{D_2 d}(-N) = \chi_D(-N) = -1,$$

we may order the characters $\chi_{D_1 d}$ and $\chi_{D_2 d}$ in such a way that $\mathrm{sign}(E,\chi_{D_1 d}) = -1$ and $\mathrm{sign}(E,\chi_{D_2 d}) = +1$. So $\chi_{D_1 d}(-N) = w_N$ and since $\chi_{D_1 d}(-M) = -w_M$ it follows that $\chi_{D_1 d}(p) = -w_p = a_p$. So the Mazur–Kitagawa p-adic L-function $L_p(f,\chi_{D_1 d},\kappa,s)$ has an exceptional zero at $(\kappa,s) = (2,1)$, and its order of vanishing is at least 2. We may apply [BD07, Theorem 5.4], [Mok11, Sec. 6] and [Mok, Theorem 3.1], which show that there is a global point $\mathbf{P}_{\chi_{D_1 d}}\in E(\mathbb{Q}(\sqrt{D_1 c}))$ and a rational number $\ell_1\in\mathbb{Q}^\times$ such that

$$\frac{d^2}{d\kappa^2}L_p(f_\infty,\chi_{D_1 d},\kappa,\kappa/2)_{\kappa=2} = \ell_1\log_E^2(\mathbf{P}_{\chi_{D_1 d}}),$$

and this point is of infinite order if and only if $L'(E,\chi_{D_1 d},1)\neq 0$. Moreover, $\ell_1\equiv L^{\mathrm{alg}}(f,\psi,1)$ mod $(\mathbb{Q}^\times)^2$ for any primitive Dirichlet character ψ such that $L(f,\psi,1)\neq 0$, $\psi(p) = -\chi_{D_1 d}(p)$, and $\psi(\ell) = \chi_{D_1 d}(\ell)$ for all $\ell\mid M$. Now

$$\ell_2 = \frac{1}{2}L_p(f_\infty,\chi_{D_2 d},2,1) = L^{\mathrm{alg}}(E,\chi_{D_2 d},1)$$

is a rational number which is non-zero if and only if $L(E,\chi_{D_2 d},1)\neq 0$. In this case, $\ell_1\ell_2$ is a square: choose $t\in\mathbb{Q}^\times$ such that $t^2 = \ell_1\ell_2$ if $\ell_2\neq 0$ and $t = 1$ otherwise, and let $\mathbf{P}_\chi = \mathbf{P}_{\chi_{D_1 d}}$ in the first case and 0 otherwise. Now the first part of the theorem follows setting $n = \alpha\cdot t$. Finally, for the second part note that $L(E,\chi_{D_2 d},1)\neq 0$ if and only if $L'(E/F,\chi,1)\neq 0$ thanks to the factorization (14). □

References

[BD07] Massimo Bertolini and Henri Darmon, *Hida families and rational points on elliptic curves*, Invent. Math. **168** (2007), no. 2, 371–431. MR 2289868

[BD09] ———, *The rationality of Stark–Heegner points over genus fields of real quadratic fields*, Ann. of Math. (2) **170** (2009), no. 1, 343–370. MR 2521118

[BSV] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, *Reciprocity laws for diagonal classes and rational points on elliptic curves,*, preprint 2017.

[Coh78] Harvey Cohn, *A classical invitation to algebraic numbers and class fields*, Springer-Verlag, New York-Heidelberg, 1978, With two appendices by Olga Taussky: "Artin's 1932 Göttingen lectures on class field theory" and "Connections between algebraic number theory and integral matrices", Universitext. MR 506156

[Dar01] Henri Darmon, *Integration on $\mathscr{H}_p \times \mathscr{H}$ and arithmetic applications*, Ann. of Math. (2) **154** (2001), no. 3, 589–639. MR 1884617

[Das05] Samit Dasgupta, *Stark–Heegner points on modular Jacobians*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 3, 427–469. MR 2166341

[DR] Henri Darmon and Victor Rotger, *Stark-Heegner points and generalized Kato classes*, preprint 2017.

[FMP17] Daniel File, Kimball Martin, and Ameya Pitale, *Test vectors and central L-values for* GL(2), Algebra Number Theory **11** (2017), no. 2, 253–318. MR 3641876

[GKZ87] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. **278** (1987), no. 1-4, 497–562. MR 909238

[GM13] Xavier Guitart and Marc Masdeu, *Computation of ATR Darmon points on nongeometrically modular elliptic curves*, Exp. Math. **22** (2013), no. 1, 85–98. MR 3038785

[GM14] ———, *Overconvergent cohomology and quaternionic Darmon points*, J. Lond. Math. Soc. (2) **90** (2014), no. 2, 495–524. MR 3263962

[GM15a] ———, *Elementary matrix decomposition and the computation of Darmon points with higher conductor*, Math. Comp. **84** (2015), no. 292, 875–893. MR 3290967

[GM15b] ———, *A p-adic construction of ATR points on $\mathbb{Q}$-curves*, Publ. Mat. **59** (2015), no. 2, 511–545. MR 3374616

[GMcS15] Xavier Guitart, Marc Masdeu, and Mehmet Haluk ¸Sengün, *Darmon points on elliptic curves over number fields of arbitrary signature*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 2, 484–518. MR 3384519

[Gre09] Matthew Greenberg, *Stark–Heegner points and the cohomology of quaternionic Shimura varieties*, Duke Math. J. **147** (2009), no. 3, 541–575. MR 2510743

[GS16] Matthew Greenberg and Marco Seveso, *p-adic families of cohomological modular forms for indefinite quaternion algebras and the Jacquet-Langlands correspondence*, Canad. J. Math. **68** (2016), no. 5, 961–998. MR 3536925

[GSS16] Matthew Greenberg, Marco Adamo Seveso, and Shahab Shahabi, *Modular p-adic L-functions attached to real quadratic fields and arithmetic applications*, J. Reine Angew. Math. **721** (2016), 167–231. MR 3574881

[LRV12] Matteo Longo, Victor Rotger, and Stefano Vigni, *On rigid analytic uniformizations of Jacobians of Shimura curves*, Amer. J. Math. **134** (2012), no. 5, 1197–1246. MR 2975234

[LRV13] ———, *Special values of L-functions and the arithmetic of Darmon points*, J. Reine Angew. Math. **684** (2013), 199–244. MR 3181561

[LV14] Matteo Longo and Stefano Vigni, *The rationality of quaternionic Darmon points over genus fields of real quadratic fields*, Int. Math. Res. Not. IMRN (2014), no. 13, 3632–3691. MR 3229764

[LV16] ———, *Quaternionic Darmon points on abelian varieties*, Riv. Math. Univ. Parma (N.S.) **7** (2016), no. 1, 39–70. MR 3675402

[Mok] Chung Pang Mok, *On a theorem of Bertolini–Darmon about rationality of Stark–Heegner points over genus fields of real quadratic fields*, preprint.

[Mok11] ———, *Heegner points and p-adic L-functions for elliptic curves over certain totally real fields*, Comment. Math. Helv. **86** (2011), no. 4, 867–945. MR 2851872

[MW09] Kimball Martin and David Whitehouse, *Central L-values and toric periods for* GL(2), Int. Math. Res. Not. IMRN (2009), no. 1, Art. ID rnn127, 141–191. MR 2471298

[Pop06] Alexandru A. Popa, *Central values of Rankin L-series over real quadratic fields*, Compos. Math. **142** (2006), no. 4, 811–866. MR 2249532

[RS12] Victor Rotger and Marco Adamo Seveso, *$\mathscr{L}$-invariants and Darmon cycles attached to modular forms*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1955–1999. MR 2984593

[Sev12] Marco Adamo Seveso, *p-adic L-functions and the rationality of Darmon cycles*, Canad. J. Math. **64** (2012), no. 5, 1122–1181. MR 2979580

[Tri06] Mak Trifković, *Stark–Heegner points on elliptic curves defined over imaginary quadratic fields*, Duke Math. J. **135** (2006), no. 3, 415–453. MR 2272972

Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy
E-mail address: mlongo@math.unipd.it
E-mail address: sthuyan@gmail.com

Department of Mathematics, University of Oklahoma, Norman, OK 73072 USA
E-mail address: kmartin@math.ou.edu