LOCAL ROOT NUMBERS, BESSEL MODELS, AND A CONJECTURE OF GUO AND JACQUET

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Dedicated to the memory of Steve Rallis

ABSTRACT. Let E/F be a quadratic extension of number fields and D a quaternion algebra over F containing E. Let π_D be a cuspidal automorphic representation of $\mathrm{GL}(n, D)$ and π its Jacquet–Langlands transfer to $\mathrm{GL}(2n)$. Guo and Jacquet conjectured that if π_D is distinguished by $\mathrm{GL}(n, E)$, then π is symplectic and $L(1/2, \pi_E) \neq 0$, where π_E is the base change of π to E. When n is odd, Guo and Jacquet also conjectured a converse. The converse does not always hold when n is even, but we conjecture it holds if and only if certain local root number conditions are satisfied, which is if and only if the corresponding generic representation of the split special orthogonal group $\mathrm{SO}(2n + 1)$ has a special E-Bessel model. We use the theta correspondence to relate E-Bessel periods on $\mathrm{SO}(5)$ with $\mathrm{GL}(2, E)$ periods on $\mathrm{GL}(2, D)$, and deduce part of our conjecture when $n = 2$.

1. Introduction

Let F be a number field and $\mathbb{A}$ its adele ring. Let G and H be algebraic groups defined over F with common center Z, and suppose H is a closed subgroup of G. In this paper, a (cuspidal) automorphic representation means an irreducible unitary (cuspidal) automorphic representation. We say a cuspidal representation π of $G(\mathbb{A})$ with trivial central character is H-distinguished if the period integral

$$\mathcal{P}_H(\phi) := \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\, dh$$

defines a nonzero linear form on π.

Let E/F be a quadratic extension of number fields and $X(E\!:\!F)$ denote the set of isomorphism classes of quaternion algebras over F which split over E. For $D \in X(E\!:\!F)$, let $\mathrm{JL} = \mathrm{JL}_D$ denote the Jacquet–Langlands correspondence of representations from an inner form $\mathrm{GL}(n, D)$ to $\mathrm{GL}(2n)$ defined by Badulescu [2] and Badulescu–Renard [3], and LJ_D denote its inverse. For a cuspidal representation π of $\mathrm{GL}(2n, \mathbb{A})$, π_E denotes the base change of π to $\mathrm{GL}(2n, \mathbb{A}_E)$, and $X(E\!:\!F\!:\!\pi)$ denotes the set of $D \in X(E\!:\!F)$ for which $\pi_D = \mathrm{LJ}_D(\pi)$ exists as a (necessarily cuspidal) representation of $\mathrm{GL}(n, D)(\mathbb{A})$. Note since the matrix algebra M_2 lies in $X(E\!:\!F)$, $X(E\!:\!F\!:\!\pi)$ also contains M_2, in which case $\mathrm{LJ}_{M_2}(\pi) = \pi$. Recall that a cuspidal representation π of $\mathrm{GL}(2n)$ is called symplectic if $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$, which is equivalent to being a lift from a generic cuspidal representation of the split orthogonal group $\mathrm{SO}(2n + 1)$ by the descent of Ginzburg–Rallis–Soudry (see [18]) or Arthur's trace formula [1].

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For each $D \in X(E\colon F)$, fix an embedding $E \hookrightarrow D$, which gives an embedding $\mathrm{GL}(n, E) \hookrightarrow \mathrm{GL}(n, D)$.

Conjecture 1 (Guo–Jacquet [19])**.** *(1) Fix $D \in X(E\colon F)$. Let π_D be a cuspidal representation of $\mathrm{GL}(n, D)(\mathbb{A})$ with trivial central character which has a cuspidal transfer $\pi = \mathrm{JL}(\pi_D)$ to $\mathrm{GL}(2n, \mathbb{A})$. If π_D is $\mathrm{GL}(n, E)$-distinguished, then π is symplectic and $L(1/2, \pi_E) \neq 0$.*

(2) Suppose n is odd. Let π be a cuspidal representation of $\mathrm{GL}(2n, \mathbb{A})$ with trivial central character. If π is symplectic and $L(1/2, \pi_E) \neq 0$, then there exists a $D \in X(E\colon F\colon \pi)$ such that the representation $\pi_D = \mathrm{LJ}_D(\pi)$ of $\mathrm{GL}(n, D)(\mathbb{A})$ is $\mathrm{GL}(n, E)$-distinguished.

We call part (2) the converse direction of the Guo–Jacquet conjecture, and our goal here is to study the converse direction for n even, though our Conjecture 3 below also partially refines the Guo–Jacquet converse when n is odd.

A few remarks on this conjecture are in order. First, the case $n = 1$ was already established by Waldspurger [34]. Waldspurger further proved that, when $n = 1$, there is a unique such D in part (2), and by work of Tunnell [32] and Saito [31], this D can be determined uniquely in terms of local root numbers. For $n > 1$ odd, it is also reasonable to expect that the D in part (2) is unique and is determined by root numbers. On the other hand, for n even when the converse direction of the Guo–Jacquet conjecture holds, we have the following non-uniqueness conjecture.

Conjecture 2 (Feigon–Martin–Whitehouse [4])**.** *Suppose n is even. Let π be a symplectic cuspidal representation of $\mathrm{GL}(2n, \mathbb{A})$. If there exists $D \in X(E\colon F\colon \pi)$ such that $\mathrm{LJ}_D(\pi)$ is $\mathrm{GL}(n, E)$-distinguished, then $\mathrm{LJ}_D(\pi)$ is $\mathrm{GL}(n, E)$-distinguished for all $D \in X(E\colon F\colon \pi)$.*

Conjectures 1(1) and 2 are proved in [4] using a simple relative trace formula, under the assumptions that π is supercuspidal at some place split in E, and E/F is split at all even and archimedean places.

There are a couple of reasons the D in the converse direction might be unique for n odd, but not for n even. The first reason, geometric, is due to the fact that when one considers the relevant relative trace formula comparison for this problem, the "regular elliptic" double $\mathrm{GL}(n, E)$-double cosets for $\mathrm{GL}(n, D)$, as D varies in $X(E\colon F)$, correspond to distinct $\mathrm{GL}(n) \backslash \mathrm{GL}(2n) / \mathrm{GL}(n)$-double cosets precisely when n is odd (see [19] and [4]). Specifically, for D_1, D_2 distinct elements of $X(E\colon F)$, the regular elliptic $\mathrm{GL}(n, E)$-double cosets for $\mathrm{GL}(n, D_1)$ match with those for $\mathrm{GL}(n, D_2)$ when n is even. This allows for the relative trace formula comparison between $\mathrm{GL}(n, E)$-periods on $\mathrm{GL}(n, D_1)$ and and $\mathrm{GL}(n, E)$-periods on $\mathrm{GL}(n, D_2)$ to get the results on Conjecture 2 in [4].

The second reason is spectral. The uniqueness when $n = 1$ (and the analogous uniqueness in the orthogonal Gross–Prasad conjectures) follows from local dichotomy: if π_v and $\pi_{D,v}$ are corresponding local representations of $\mathrm{GL}(2, F_v)$ and $\mathrm{GL}(1, D_v)$, then exactly one of π_v and $\pi_{D,v}$ is $\mathrm{GL}(1, E_v)$-distinguished. Prasad (cf. [29]) observed that local dichotomy is related to the character identity $\chi_{\pi_v} = -\chi_{\pi_{D,v}}$. The analogous character identity for representations π_v and $\pi_{D,v}$ of $\mathrm{GL}(2n, F_v)$ and $\mathrm{GL}(n, D_v)$ is, however, $\chi_{\pi,v} = (-1)^n \chi_{\pi_{D,v}}$. This suggests one may have local dichotomy in our situation if and only if n is odd. In fact, at least when $\pi_{D,v}$ is a discrete series representation, this is a special case of a conjecture of Prasad and Takloo–Bighash [30, Conjecture 1].

Now we will formulate a conjecture which will tell us when the converse to the Guo–Jacquet conjecture should hold for n even. This conjecture is already suggested by the work of (Gan)–Gross–Prasad [17], [11], Mao–Rallis [25] and Valverde [33]. First we make some necessary definitions.

The E-Bessel subgroup R_E of the split special orthogonal group $\mathrm{SO}(2n+1)$ is a (unique up to conjugacy) spherical subgroup of the form $\mathrm{SO}(2)\,S$ where S is a certain unipotent subgroup of $\mathrm{SO}(2n+1)$ and here $\mathrm{SO}(2,F) \simeq E^\times/F^\times$. Fix a certain character ψ_E of S and extend it to a character of R_E so that the $\mathrm{SO}(2)$-action is trivial. We refer to [6, pp. 92–93] for a precise definition of (R_E, ψ_E), where it is denoted by $(R_\lambda, \chi_\lambda)$ with $E = F\left(\sqrt{\lambda}\right)$. We say a representation σ of $\mathrm{SO}(2n+1,\mathbb{A})$ has a (special) E-Bessel model (or is (R_E,ψ_E)-distinguished) if the period

$$\mathcal{P}_{R_E,\psi_E}(\phi) := \int_{R_E(F)\backslash R_E(\mathbb{A})} \phi(r)\psi_E(r)\,dr$$

is not identically zero on σ.

In [16], Gross and Prasad associated certain local symplectic root number characters to a pair of representations $\sigma_v \otimes \tau_v$ of $\mathrm{SO}(2n+1,F_v) \times \mathrm{SO}(W,F_v)$ where W is an even-dimensional orthogonal space. These characters are characters of the component group of the Langlands parameter associated to $\sigma_v \otimes \tau_v$ (a finite elementary abelian 2-group). When $\mathrm{SO}(W) \simeq \mathrm{SO}(2)$ and τ_v is trivial, we denote this character restricted to the the component group of the Langlands parameter associated to σ_v by $\chi_{\sigma,v}$. See Section 5 for the precise defintion.

Conjecture 3. *Let π be a cuspidal representation of $\mathrm{GL}(2n,\mathbb{A})$ corresponding to a generic cuspidal representation σ of $\mathrm{SO}(2n+1,\mathbb{A})$. The following are equivalent:*

(1) *π is $\mathrm{GL}(n,E)$-distinguished;*
(2) *σ has a special E-Bessel model; and*
(3) *$L(1/2,\pi_E) \neq 0$ and the local root number characters $\chi_{\sigma,v}$ are trivial for all v.*

In particular, if π is $\mathrm{GL}(n,E)$-distinguished, we have $\epsilon(1/2,\pi_{E,v}) = 1$ for all v.

We note this agrees with the above-mentioned results of Waldspurger, Tunnell and Saito when $n = 1$. This, combined with Conjecture 2, provides a local root number criterion for when the converse to the Guo–Jacquet conjecture holds for n even. It also suggests that when n is odd, if local dichotomy is true, that the D in Conjecture 1(2) may be determined by local root number conditions, as in the $n = 1$ case and, more generally, the (Gan)–Gross–Prasad conjectures.

The equivalence of (2) and (3) is a particular case of the (Gan)–Gross–Prasad conjectures for $\mathrm{SO}(2n+1)\times\mathrm{SO}(2)$ using the fact that $L(s,\pi_E) = L(s,\sigma\times 1_E)$, where 1_E denotes the trivial representation of $\mathrm{SO}(2)$. Specifically, the Vogan L-packet Π_σ of σ consists of the automorphic representations of pure inner forms of $\mathrm{SO}(2n+1)$ with identical Langlands parameter as σ, and Π_σ is conjecturally parameterized by the irreducible characters of the component group of σ. The local Gross–Prasad conjecture [17] tells us there is at most one $\sigma' \in \Pi_\sigma$ such that σ' has a special E-Bessel model (in the sense that is locally does everywhere), and that $\sigma' = \sigma$ if and only if $\chi_{\sigma,v}$ is trivial for all v. Then the global Gan–Gross–Prasad conjecture [11] says that σ satisfying these local conditions has a global E-Bessel model if and only if $L(1/2,\sigma\times 1_E) \neq 0$.

We note that the condition on $\epsilon(1/2, \pi_{E,v}) = 1$ at the end of the conjecture is a consequence of the local root number condition in (3) together with the local Langlands correspondence. In fact, for $n = 1$, these local epsilon factor conditions for π_E are precisely the local root number conditions in (3). For general n, the condition that $\epsilon(1/2, \pi_{E,v}) = 1$ for all v means that the member σ' of the Vogan L-packet of σ which locally everywhere has an E-Bessel model is actually a representation of $\mathrm{SO}(2n+1)$, rather than just a pure inner form of $\mathrm{SO}(2n+1)$, but for $n > 1$ the conditions $\epsilon(1/2, \pi_{E_v}) = 1$ do not suffice to guarantee σ' is the generic representation σ of $\mathrm{SO}(2n+1)$.

Since the local Gross–Prasad conjecutures are now (essentially) known for orthogonal pairs (Waldspurger [35], Moeglin–Waldspurger [27]), the equivalence of (2) and (3) would follow from the $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$ case of global Gan–Gross–Prasad (modulo some standard conjectures taken for granted in [35] and [27].)

Our first main result is

Theorem 4. *When $n = 2$, conditions (1) and (2) of Conjecture 3 are equivalent.*

We prove this by using the theta correspondence for $\mathrm{GSp}(4)$ and $\mathrm{GSO}(3,3)$ to express the E-Bessel period on a generic representation σ of $\mathrm{SO}(5)$ in terms of the $\mathrm{GL}(2,E)$-period for the corresponding π on $\mathrm{GL}(4)$. We in fact do this for twisted periods, i.e., periods twisted by characters of $\mathrm{GL}(1,E)$. See (12) and (14) for the definition of these periods. This period relation was initially announced over 17 years ago by the first author, and has also recently appeared independently in the work of Prasad and Takloo–Bighash [30, Section 13].

Now we derive a couple of consequences, which implicitly assume certain not-yet-proven (but likely soon to be) results briefly discussed below. The first is

Corollary 5. *When $n = 2$, part (1) of Conjecture 3 implies that $\chi_{\sigma,v}$ is trivial for all v, as asserted in part (3).*

This is a consequence of the above theorem together with the local Gross–Prasad conjecture on epsilon factors for $\mathrm{SO}(5) \times \mathrm{SO}(2)$. As remarked above, local Gross–Prasad conjectures are known in greater generality by [35] and [27] modulo certain standard conjectures related to tempered L-packets and local twisted trace formulas. For the specific case of $\mathrm{SO}(5) \times \mathrm{SO}(2)$, the local Gross–Prasad conjectures are known unconditionally apart from even residual characteristic by Prasad–Takloo-Bighash [30].

Furthermore, the present authors [7] proved the global Gan–Gross–Prasad conjecture for $\mathrm{SO}(5) \times \mathrm{SO}(2)$ (which essentially coincides with a conjecture of the first author and Shalika [9]) where one takes the trivial representation on $\mathrm{SO}(2)$, under some local hypotheses analogous to the above result of [4]. The proof uses a simple version of the first relative trace formula proposed in [9] together with the corresponding fundamental lemma proved in [8]. The results of [7] however depend on the abovementioned not-yet-proven assumptions made in [35], [27], as well as the stabilization of the trace formula assumed in [1]. (No such assumptions are needed for the analogous results in [4] on Conjectures 1(1) and 2.) Admitting the same assumptions gives the following consequence of Theorem 4 and [7].

Corollary 6. *Suppose $n = 2$ and E is split at all archimedean places. Let σ be a generic, everywhere locally tempered cuspidal representation of $\mathrm{SO}(5, \mathbb{A})$ which is supercuspidal at some place split in E and corresponds to a cuspidal π on $\mathrm{GL}(4, \mathbb{A})$.*

Then Conjecture 3 holds for π and σ, and Conjecture 1(1) holds for $\pi_D = \mathrm{LJ}_D(\pi)$ for any $D \in X(E\!:\!F\!:\!\pi)$.

As in Gan–Takeda [13], we can in fact consider more generally a theta correspondence for GSp(4) and $\mathrm{GSO}(V_D)$ where V_D is a 6-dimensional quadratic space associated to D. Note that $\mathrm{GSO}(V_D)$ is closely related to $\mathrm{GL}(2, D)$. Consequently, at least when D is split at each infinite place, we show that, for a cuspidal π_D of $\mathrm{GL}(2, D)$ with theta lift $\Theta(\pi_D)$ on GSp(4), $\Theta(\pi_D)$ having a (twisted) E-Bessel period is equivalent to π_D having a (twisted) $\mathrm{GL}(2, E)$-period. (See also [30, Section 13].) When D is ramified at some infinite place, then one needs to replace $\Theta(\pi_D)$ with the theta lift $\Theta(\pi_D)^+$ on $\mathrm{GSp}(4)^+$, which is a finite index subgroup of GSp(4), as $\Theta(\pi_D)$ is not necessarily irreducible in this case. In any case, this gives a criterion for the converse to Jacquet–Guo when $n = 2$.

Theorem 7. *Suppose π is a cuspidal symplectic representation of $\mathrm{GL}(4, \mathbb{A})$. Then there exists $D \in X(E\!:\!F\!:\!\pi)$ such that $\pi_D = \mathrm{LJ}_D(\pi)$ is $\mathrm{GL}(2, E)$-distinguished if and only if the theta lift $\Theta(\pi_D)^+$ to $\mathrm{GSp}(4, \mathbb{A})^+$ for some $D \in X(E\!:\!F\!:\!\pi)$ has a special E-Bessel model.*

In particular, admitting the assumptions in [35] and [27] for the local Gross–Prasad conjectures, the converse to Jacquet–Guo when $n = 2$ can only hold when $\epsilon(1/2, \pi_{E,v}) = 1$ for all v, which is the final assertion in Conjecture 3.

Now assume D is split at each infinite place, in which case $\mathrm{GSp}(4)^+ = \mathrm{GSp}(4)$. The passage from Theorem 7 to Conjecture 2 should be related to the transfer of Shalika models between $\mathrm{GL}(2, D)$ and GL(4). Namely, in light of the local Gross–Prasad conjectures and the fact that $\Theta(\pi)$ is generic if nonzero, one wants to show that, if some π_D is $\mathrm{GL}(2, E)$-distinguished, then $\Theta(\pi_D)$ is generic for all $D \in X(E\!:\!F\!:\!\pi)$, which is equivalent to π_D having a Shalika model [13, Cor 3.2]. If this were the case, then all $\Theta(\pi_D)$ would be identical by strong multiplicity one for generic representations of GSp(4) [24]. At least, this gives the following partial result towards Conjecture 2.

Theorem 8. *Let $D \in X(E\!:\!F)$ be split at each infinite place. Suppose π_D is a cuspidal representation of $\mathrm{GL}(2, D)(\mathbb{A})$ such that π_D has a Shalika model and $\pi = \mathrm{JL}(\pi_D)$ is also cuspidal. Then π_D is $\mathrm{GL}(2, E)$-distinguished if and only if π is.*

Hence, Conjecture 2 for $n = 2$ would follow from Theorem 8 if the following were true: if π_D is $\mathrm{GL}(2, E)$-distinguished for some $D \in X(E\!:\!F\!:\!\pi)$ split at each archimedean place, then π_D has a Shalika model for all $D \in X(E\!:\!F\!:\!\pi)$. Jacquet and the second author [21] conjectured that, for cuspidal representations π_D of $\mathrm{GL}(2, D)$ and $\pi = \mathrm{JL}(\pi_D)$ of GL(4), a Shalika model for π_D implies a Shalika model for π. In [13], Gan–Takeda proved this, and also showed the converse holds if and only if π_D avoids certain local obstructions (being a principal series induced from two representations of $D_v^\times$ with trivial central character at places where D ramifies). Consequently, Conjecture 2 says that if π_D is $\mathrm{GL}(2, E)$-distinguished, it should also avoid the local obstructions in [13]. We do not address this here.

Now we discuss evidence for Conjecture 3 when $n > 2$.

First we remark that for arbitrary n, a global argument of Prasad [29] in the study of trilinear forms for GL(2) suggests the following principle: when the nonvanishing of periods is related to the nonvanishing of L-values, an epsilon factor criterion is required for the nonvanishing of a given period. In our particular case, this idea, together with the Guo–Jacquet conjecture, is suggestive of the final statement in Conjecture 3 about $\epsilon(1/2, \pi_{E,v})$. In other words, in light of the (Gan)–Gross–Prasad conjectures, π being $\mathrm{GL}(n,E)$-distinguished should imply some functorial transfer σ' of π to the split $\mathrm{SO}(2n+1)$ has an E-Bessel model. The more refined notion that this σ' should be the generic representation of $\mathrm{SO}(2n+1)$ is suggested by previous work on relative trace formula identities, which we now describe.

Let σ be a cuspidal representation of $\mathrm{SO}(2n+1,\mathbb{A})$ which corresponds to a cuspidal representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{A})$ via the theta correspondence. Here we take

$$\mathrm{Sp}(2n) = \left\{ g \in \mathrm{GL}(2n) \mid g \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} \right\}, \quad w = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let N' be the standard maximal upper unipotent subgroup of $\mathrm{Sp}(2n)$ and T the standard maximal split torus. We may take for a set of representatives of the $T(F)$-conjugacy classes of nondegenerate characters of $N'(\mathbb{A})$ the characters

$$\theta'_\tau(n) = \psi(n_{1,2} + \cdots + n_{n-1,n} + \tau n_{n,n+1}), \quad n = (n_{ij})$$

where $\tau \in F^\times/(F^\times)^2$.

Mao and Rallis [25] proved a relative trace identity of the following form

$$\mathrm{RTF}_{\mathrm{SO}(2n+1)}(R_E, \psi_E; U, \psi) = \mathrm{RTF}_{\widetilde{\mathrm{Sp}}(2n)}(N', \theta'^{-1}_\tau; N', \theta'), \tag{1}$$

where $E = F(\sqrt{\tau})$, U now denotes the standard maximal unipotent in $\mathrm{SO}(2n+1)$, ψ is a nondegenerate character of U, and $\theta' = \theta'_1$. Here the notation in (1) means that for suitable matching functions f on $\mathrm{SO}(2n+1,\mathbb{A})$ and f' on $\widetilde{\mathrm{Sp}}(2n,\mathbb{A})$ with associated kernels K_f and $K_{f'}$ one has

$$\begin{aligned}\int_{R_E(F)\backslash R_E(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} & K_f(r,u)\psi_E(r)\psi(u)\, dr\, du \\ &= \int_{N'(F)\backslash N'(\mathbb{A})} \int_{N'(F)\backslash N'(\mathbb{A})} K_{f'}(n_1,n_2)\theta'^{-1}_\tau(n_1)\theta'(n_2)\, dn_1\, dn_2.\end{aligned}$$

On the other hand, in his thesis, Valverde [33] proved a relative trace identity of the following form

$$\mathrm{RTF}_{\mathrm{GL}(2n)}(\mathrm{GL}(n,E), 1; N, \theta) = \mathrm{RTF}_{\widetilde{\mathrm{Sp}}(2n)}(N', \theta'^{-1}_\tau; N', \theta'), \tag{2}$$

where N is the standard maximal unipotent of $\mathrm{GL}(2n)$ with a nondegenerate character θ and $E = F(\sqrt{\tau})$. Combining these relative trace identities gives a third one of the form

$$\mathrm{RTF}_{\mathrm{SO}(2n+1)}(R_E, \psi_E; U, \psi) = \mathrm{RTF}_{\mathrm{GL}(2n)}(\mathrm{GL}(n,E), 1; N, \theta). \tag{3}$$

One expects that a refinement of this identity will give a period relation of the form

$$|\mathcal{P}_{R_E,\psi_E}(\varphi)\mathcal{W}(\varphi)|^2 = |\mathcal{P}_{\mathrm{GL}(n,E)}(\phi)\mathcal{W}(\phi)|^2, \tag{4}$$

where φ and ϕ are respectively certain automorphic forms lying in cuspidal representations σ of $\mathrm{SO}(2n+1)$ and π of $\mathrm{GL}(2n)$ which correspond, and $\mathcal{W}$ denotes the appropriate Whittaker period on both groups. In light of the Whittaker period appearing on the left hand side, (4) means that a nonzero $\mathrm{GL}(n,E)$ period on (the necessarily symplectic) π should be equivalent to the a nonzero special E-Bessel period on the associated generic representation σ of $\mathrm{SO}(2n+1)$. This is precisely the equivalence of (1) with (2) in Conjecture 3.

We remark that the Bessel period on the left side of (4) should be related to $L(1/2, \sigma \times 1_E)$ by the Gross–Prasad conjectures and the $\mathrm{GL}(n, E)$-period on the right side of (4) should be related to $L(1/2, \pi_E)$ by the Guo–Jacquet conjecture. See [33] for an expected L-value formula from combining (3) with the Guo–Jacquet conjecture.

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2. Preliminaries

2.1. **Accidental isomorphism.** Here we follow Gan–Takeda [13]. Let F be a number field and D be a quaternion algebra over F. We consider the quadratic space

$$(V_D, q_D) = (D, \mathbb{N}_D) \oplus \mathbb{H}$$

where $\mathbb{H}$ is the hyperbolic plane and $\mathbb{N}_D$ denotes the reduced norm on D. We realize (V_D, q_D) concretely as

$$V_D = \left\{ (a, b; x) := \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \mid a, b \in F,\, x \in D \right\}$$

with

$$q_D\,(a, b; x) = -\det \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} = -ab + \mathbb{N}_D\,(x)\,.$$

Let

$$\mathrm{GO}\,(V_D) = \{ h \in \mathrm{GL}\,(V_D) \mid q_D\,(h\,(X)) = \lambda_D\,(h)\; q_D\,(X),\, \forall X \in V_D \}$$

where λ_D is the similitude. We have $\left\{ (\det h)\,(\lambda_D\,(h))^{-3} \right\}^2 = 1$ for $h \in \mathrm{GO}\,(V_D)$ and the connected component $\mathrm{GSO}\,(V_D)$ of $\mathrm{GO}\,(V_D)$ is given by

$$\mathrm{GSO}\,(V_D) = \left\{ h \in \mathrm{GSO}\,(V_D) \mid (\det h)\,(\lambda_D\,(h))^{-3} = 1 \right\}.$$

We have a homomorphism $\varphi : \mathrm{GL}\,(2, D) \times \mathrm{GL}(1) \to \mathrm{GSO}\,(V_D)$ such that

$$\varphi\,(g, z) \cdot X = z\, g\, X\, {}^t\bar{g} \quad \text{for } g \in \mathrm{GL}\,(2, D),\ z \in \mathrm{GL}(1) \text{ and } X \in V_D.$$

Here $\lambda_D\,(\varphi\,(g, z)) = N\,(g) \cdot z^2$ where N denotes the reduced norm on the central simple algebra $M_2\,(D)$. Indeed we have

$$(\mathrm{GL}\,(2, D) \times \mathrm{GL}(1))\,/\mathrm{Ker}\,\varphi \overset{\varphi}{\simeq} \mathrm{GSO}\,(V_D) \tag{5}$$

where $\mathrm{Ker}\,\varphi = \left\{ \left(z, z^{-2}\right) \mid z \in \mathrm{GL}(1) \right\}$.

2.2. $\mathrm{GL}(2,E)\subset\mathrm{GL}(2,D)$**.** For E a quadratic extension of F, let $X(E:F)$ denote the isomorphism classes of quaternion algebras over F which split over E. As a set of representatives of $X(E:F)$, we may take

$$D_\epsilon=\left\{\begin{pmatrix}\alpha & \beta\epsilon\\ \bar\beta & \bar\alpha\end{pmatrix}\mid \alpha,\beta\in E\right\},\quad \epsilon\in F^\times/\mathrm{N}_{E/F}\left(E^\times\right).$$

Here $\bar\alpha$ denotes the conjugate over F for $\alpha\in E$ and we identify $\alpha\in E$ with $\begin{pmatrix}\alpha & 0\\ 0 & \bar\alpha\end{pmatrix}$ in D_ϵ.

Suppose that $D=D_\epsilon$. Let us define a subspace $V_D^{(0)}$ of V_D by

$$V_D^{(0)}=\left\{v_0(\beta):=\begin{pmatrix}0 & \left(\begin{smallmatrix}0 & \epsilon\beta\\ \bar\beta & 0\end{smallmatrix}\right)\\ \left(\begin{smallmatrix}0 & -\epsilon\beta\\ -\bar\beta & 0\end{smallmatrix}\right) & 0\end{pmatrix}\mid \beta\in E\right\}.$$

Then we have $V_D=V_D^{(0)}\oplus V_D^{(1)}$ where

$$V_D^{(1)}=\left(V_D^{(0)}\right)^\perp=\left\{X\in M_2(E)\mid X={}^t\bar X\right\},$$

i.e. the set of 2 by 2 Hermitian matrices.

Lemma 9. (1) *For $(g,z)\in\mathrm{GL}(2,D)\times\mathrm{GL}(1)$, we have $\varphi(g,z)\cdot V_D^{(0)}=V_D^{(0)}$ if and only if $g\in\mathrm{GL}(2,E)$.*

(2) *For $(g,z)\in\mathrm{GL}(2,E)\times\mathrm{GL}(1)$, we have*

$$\varphi(g,z)\cdot v_0(\beta)=v_0(\beta z\det g)\quad \textit{for } v_0(\beta)\in V_D^{(0)};$$
$$\varphi(g,z)\cdot X=z\,g\,X\,{}^t\bar g\quad \textit{for } X\in V_D^{(1)}.$$

Here $\bar g$ is the conjugate over F of $g\in\mathrm{GL}(2,E)$.

Proof. Let us write $g\in\mathrm{GL}(2,D)$ as $g=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ with $a_{ij}=\begin{pmatrix}\alpha_{ij} & \epsilon\beta_{ij}\\ \bar\beta_{ij} & \bar\alpha_{ij}\end{pmatrix}\in D_\epsilon$. By a direct computation, we have

$$\varphi(g,z)\cdot v_0(\beta)=\begin{pmatrix}a & X\\ \bar X & b\end{pmatrix},\quad X=\begin{pmatrix}\gamma & \epsilon\delta\\ \bar\delta & \bar\gamma\end{pmatrix}$$

where

$$\begin{aligned}
a&=z\epsilon\,\mathrm{tr}_{E/F}\left(\left(\bar\beta_{11}\alpha_{12}-\alpha_{11}\bar\beta_{12}\right)\beta\right);\\
b&=z\epsilon\,\mathrm{tr}_{E/F}\left(\left(\bar\beta_{21}\alpha_{22}-\alpha_{21}\bar\beta_{22}\right)\beta\right);\\
\gamma&=z\epsilon\left(\alpha_{12}\bar\beta_{21}-\alpha_{11}\bar\beta_{22}\right)\beta+z\epsilon\left(\beta_{11}\bar\alpha_{22}-\beta_{12}\bar\alpha_{21}\right)\bar\beta;\\
\delta&=z\left(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}\right)\beta-z\epsilon\left(\beta_{11}\beta_{22}-\beta_{12}\beta_{21}\right)\bar\beta.
\end{aligned}$$

Here we note that for $s,t\in E$, we have $s\beta+t\bar\beta=0$ for all $\beta\in E$ if and only if $s=t=0$. Thus $\varphi(g,z)\cdot V_D^{(0)}=V_D^{(0)}$ implies

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\bar\beta_{12} & \bar\beta_{22}\\ -\bar\beta_{11} & -\bar\beta_{21}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad \det\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\neq 0.$$

Hence $\beta_{11}=\beta_{12}=\beta_{21}=\beta_{22}=0$, i.e. $g\in\mathrm{GL}(2,E)$. The rest is clear. $\square$

2.3. **Theta correspondence for similitudes.** Let W be the space of four dimensional row vectors over F with the symplectic form

$$\langle w_1, w_2 \rangle = w_1 \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} {}^t w_2 \quad \text{for } w_1, w_2 \in W.$$

Then the symplectic similitude group $\mathrm{GSp}(W)$ is defined by

$$\mathrm{GSp}(W) = \{ g \in \mathrm{GL}(4) \mid \langle w_1 g, w_2 g \rangle = \nu(g) \langle w_1, w_2 \rangle \ \ (\forall w_1, w_2 \in W) \}.$$

Let $\mathbb{W}_D = V_D \otimes_F W$. Then $\mathbb{W}_D$ is equipped with a symplectic form:

$$\ll \ , \ \gg = (\ , \ ) \otimes \langle \ , \ \rangle$$

where

$$(v_1, v_2) = q_D(v_1 + v_2) - q_D(v_1) - q_D(v_2) \quad \text{for } v_1, v_2 \in V_D.$$

The group $\mathrm{GSp}(\mathbb{W})$ acts on $\mathbb{W}$ from the right and we have a homomorphism:

$$i : \mathrm{GSp}(W) \times \mathrm{GSO}(V_D) \to \mathrm{GSp}(\mathbb{W})$$

defined by

$$(v \otimes w) \cdot i(g, h) = h^{-1} v \otimes wg.$$

Concerning similitudes we have

$$\nu_{\mathbb{W}}(i(g, h)) = \nu(g) \cdot \lambda_D(h)^{-1}.$$

Now let

$$R_D = i^{-1}(\mathrm{Sp}(\mathbb{W})) = \{ (g, h) \in \mathrm{GSp}(W) \times \mathrm{GSO}(V_D) \mid \nu(g) = \lambda_D(h) \}.$$

Clearly $\mathrm{Sp}(W) \times \mathrm{SO}(V_D)$ is a subgroup of R_D.

Let us fix a non-trivial character ψ of $\mathbb{A}/F$ and consider the Weil representation ω_D of $\mathrm{Sp}(W, \mathbb{A})$ on the Schwartz-Bruhat space $\mathcal{S} = \mathcal{S}((V_D \oplus V_D)(\mathbb{A}))$. We recall that for $\phi \in \mathcal{S}$ we have:

$$(6) \quad \omega_D(m(g)) \phi(v_1, v_2) = |\det g|^3 \cdot \phi((v_1, v_2) g), \quad m(g) = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix};$$

$$(7) \quad \omega_D(u(b)) \phi(v_1, v_2) = \psi\left[\frac{1}{2} \mathrm{tr}(\mathrm{Gr}(v_1, v_2) b) \right] \cdot \phi(v_1, v_2), \quad u(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix}.$$

Here $\mathrm{Gr}(v_1, v_2)$ denotes the Gram matrix, i.e. $\mathrm{Gr}(v_1, v_2) = \begin{pmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_1, v_2) & (v_2, v_2) \end{pmatrix}$.

For $h \in \mathrm{GSO}(V_D, \mathbb{A})$ and $\phi \in \mathcal{S}$, let

$$L(h) \phi(v_1, v_2) = |\lambda_D(h)|^{-3} \cdot \phi\left(h^{-1} v_1, h^{-1} v_2\right).$$

By Harris–Kudla [20, Lemma 5.1.2], for $g \in \mathrm{Sp}(W, \mathbb{A})$ and $h \in \mathrm{GSO}(V_D, \mathbb{A})$, we have

$$(8) \qquad L\left(h^{-1}\right) \omega_D(g) \, L(h) = \omega_D \left[\begin{pmatrix} 1_2 & 0 \\ 0 & \lambda_D(h) 1_2 \end{pmatrix}^{-1} g \begin{pmatrix} 1_2 & 0 \\ 0 & \lambda_D(h) 1_2 \end{pmatrix} \right].$$

Since the group R_D is isomorphic to $\mathrm{Sp}(W) \rtimes \mathrm{GSO}(V_D)$ via

$$R_D \ni (g, h) \mapsto \left(g \begin{pmatrix} 1_2 & 0 \\ 0 & \nu(g)^{-1} 1_2 \end{pmatrix}, h \right) \in \mathrm{Sp}(W) \rtimes \mathrm{GSO}(V_D)$$

where $\mathrm{GSO}(V_D)$ acts on $\mathrm{Sp}(W)$ by (8), we may define a representation of $R_D(\mathbb{A})$ on $\mathcal{S}$, which we denote as ω_D by abuse of notation, as the following by (8):

$$\omega_D(g,h)\,\phi(v_1,v_2) = \omega_D\left(g\begin{pmatrix}1_2 & 0\\ 0 & \nu(g)^{-1}\,1_2\end{pmatrix}\right)L(h)\,\phi(v_1,v_2)\,. \tag{9}$$

For $\phi \in \mathcal{S}$, a theta function $\theta_\phi : R_D(\mathbb{A}) \to \mathbb{C}$ is defined by

$$\theta_\phi(r) = \sum_{x \in (V_D \oplus V_D)(F)} \omega_D(r)\,\phi(x)\,.$$

Suppose that π_D is a cuspidal representation of $\mathrm{GL}_2(D)(\mathbb{A})$ with central character μ^2. By (5), we may regard $\pi_D \boxtimes \mu$ as a cuspidal representation of $\mathrm{GSO}(V_D)(\mathbb{A})$. Here we note that the central character of $\pi_D \boxtimes \mu$ is μ.

We define the subgroup $G(\mathbb{A})^+ = \mathrm{GSp}(4,\mathbb{A})^+$ of $\mathrm{GSp}(W)(\mathbb{A})$ by

$$G(\mathbb{A})^+ = \{g \in \mathrm{GSp}(W)(\mathbb{A}) \mid \exists h \in \mathrm{GSO}(V_D)(\mathbb{A}) \text{ such that } \nu(g) = \lambda_D(h)\}\,.$$

Then for $f \in \pi_D \boxtimes \mu$ and $\phi \in \mathcal{S}$, we define $\theta(\phi, f) : G(\mathbb{A})^+ \to \mathbb{C}$ by

$$\theta(\phi,f)(g) = \int_{\mathrm{SO}(V_D)(F)\backslash \mathrm{SO}(V_D)(\mathbb{A})} \theta_\phi(g, hh_g)\; f(hh_g)\; dh. \tag{10}$$

Here h_g is an element of $\mathrm{GSO}(V_D, \mathbb{A})$ satisfying $\nu(g) = \lambda_D(h_g)$ and the right hand side of (10) does not depend on the choice of h_g. Then we have

$$\theta(\phi,f)(\gamma g) = \theta(\phi,f)(g) \quad \text{for } \gamma \in G(F)^+ := \mathrm{GSp}(W)(F) \cap G(\mathbb{A})^+$$

and

$$\theta(\phi,f)(zg) = \mu(z)\;\theta(\phi,f)(g) \quad \text{for } z \in Z(\mathbb{A}) \tag{11}$$

where Z is the center of $\mathrm{GSp}(W)$.

Let $\Theta(\pi_D \boxtimes \mu)^+$ be the set of $\theta(\phi, f)$, where ϕ and f run over $\mathcal{S}$ and π_D respectively. This is irreducible as a $G^+(\mathbb{A})$-module by Gan [10, Proposition 2.5] together with the necessary local Howe conjecture proved in this case by Gan-Takeda [14].

We have $G(\mathbb{A})^+ = \mathrm{GSp}(W)(\mathbb{A})$ if and only if D splits at all archimedean places of F as discussed in Gan-Takeda [13, Remark 2.3]. When $G(\mathbb{A})^+ \subsetneq \mathrm{GSp}(W)(\mathbb{A})$, we extend $\theta(\phi, f)$ to a function of $\mathrm{GSp}(W)(\mathbb{A})$ by insisting that it is left $\mathrm{GSp}(W)(F)$ invariant and zero outside $\mathrm{GSp}(W)(F)\;G(\mathbb{A})^+$. Let $\Theta(\pi_D \boxtimes \mu)$ be the representation of $\mathrm{GSp}(W)(\mathbb{A})$ whose space is spanned by $\mathrm{GSp}(W)(\mathbb{A})$ translates of $\theta(\phi, f)$, where ϕ and f run over $\mathcal{S}$ and π_D respectively. We note that when $G(\mathbb{A})^+ \subsetneq \mathrm{GSp}(W)(\mathbb{A})$, the representation $\Theta(\pi_D \boxtimes \mu)$ is of finite multiplicity but not necessarily irreducible.

3. Pull Back of the Bessel Period

We take $\eta \in E^\times$ such that $E = F(\eta)$ and $\eta^2 = d \in F$. Let us define a torus T of GL_2 by

$$T = \left\{g \in \mathrm{GL}_2 \mid {}^t r\, S_d\, r = \det r \cdot S_d\right\} \quad \text{where } S_d = \begin{pmatrix}1 & 0\\ 0 & -d\end{pmatrix}.$$

Then we have

$$T = \left\{\begin{pmatrix}a & bd\\ b & a\end{pmatrix} \mid a^2 - b^2 d \in \mathrm{GL}_1\right\} \simeq E^\times$$

where we identify $r = \begin{pmatrix} a & bd \\ b & a \end{pmatrix}$ with $a + b\eta \in E^\times$. We regard T also as a subgroup of $\mathrm{GSp}(W)$ via

$$T \ni r \mapsto \begin{pmatrix} r & 0 \\ 0 & \det r \cdot {}^t r^{-1} \end{pmatrix} \in \mathrm{GSp}(W).$$

Let U be the unipotent radical of the Siegel parabolic subgroup of $\mathrm{GSp}(W)$, i.e.

$$U = \left\{ u(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} \mid b = {}^t b \right\}.$$

Let Ω be a character of $T(\mathbb{A})/T(F)$ such that $\Omega\mid_{Z(\mathbb{A})} \cdot \mu = 1$ and χ be a non-trivial character of $\mathbb{A}/F$. For an automorphic form $\varPhi$ on $\mathrm{GSp}(W)$ with central character μ, we define a Bessel period of type (E, Ω) by

$$\mathcal{B}_{E,\Omega,\chi}(\varPhi) = \int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \Omega(r)\, \chi_E(u)\, \varPhi(ur)\, du\, dr \tag{12}$$

where χ_E is a character of $U(\mathbb{A})$ defined by

$$\chi_E(u(b)) = \chi\left[\mathrm{tr}(S_d\, b)\right].$$

For $\lambda \in F^\times$, let χ' be a character of $\mathbb{A}/F$ given by $\chi'(x) = \chi(\lambda x)$. Then we have

$$\mathcal{B}_{E,\Omega,\chi'}(\varPhi) = \mathcal{B}_{E,\Omega,\chi}\left(R\begin{pmatrix} 1_2 & 0 \\ 0 & \lambda \cdot 1_2 \end{pmatrix} \varPhi\right)$$

where R denotes the right regular representation. Thus the dependence on χ is not essential.

The first named author announced the following proposition at the Special Session on Theta Correspondences and Automorphic Forms in AMS Spring Central Sectional Meeting #909, Iowa City, March, 1996. After writing this paper, we realized this result has recently appeared independently in [30, Theorem 11].

Proposition 10. *Let $D = D_\epsilon \in X(E\!:\!F)$. Let π_D be a cuspidal representation of $\mathrm{GL}(2, D)$ whose central character is μ^2, so that we may consider the representation $\pi_D \boxtimes \mu$ of $\mathrm{GSO}(V_D)$. We identify $f \in \pi_D$ with an element of $\pi_D \boxtimes \mu$ defined by*

$$f(\varphi(g, z)) = f(g)\, \mu(z) \quad \text{for } (g, z) \in \mathrm{GL}(2, D)(\mathbb{A}) \times \mathrm{GL}(1, \mathbb{A}).$$

Let Ω be a character of $T(\mathbb{A})/T(F)$ such that $\Omega\mid_{Z(\mathbb{A})} \cdot \mu = 1$. Let χ be a character of $\mathbb{A}/F$ defined by $\chi(x) = \psi(\epsilon x)$ where ψ is the character used for theta correspondence. Then we have

$$\mathcal{B}_{E,\Omega,\chi}(\theta(\phi, f)) = \int_{G_1(\mathbb{A})\backslash G_0(\mathbb{A})} L(\varphi(h_0))\, \phi(x_0)\, \mathcal{P}_{D,\Omega}\left[(\pi_D \boxtimes \mu)(h_0)\, f\right] dh_0. \tag{13}$$

Here

$$G_0 = \left\{(g, z) \in \mathrm{GL}(2, D) \times \mathrm{GL}(1) \mid N(g) \cdot z^2 = 1\right\},$$
$$G_1 = \left\{(g, z) \in \mathrm{GL}(2, E) \times \mathrm{GL}(1) \mid z \cdot \det g = 1\right\},$$

$x_0 = \left(v_1^{(0)}, v_2^{(0)}\right)$ *with*

$$v_1^{(0)} = \begin{pmatrix} 0 & \left(\begin{smallmatrix} 0 & \epsilon \\ 1 & 0 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & -\epsilon \\ -1 & 0 \end{smallmatrix}\right) & 0 \end{pmatrix} \quad \textit{and} \quad v_2^{(0)} = \begin{pmatrix} 0 & \left(\begin{smallmatrix} 0 & \epsilon\eta \\ -\eta & 0 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & -\epsilon\eta \\ \eta & 0 \end{smallmatrix}\right) & 0 \end{pmatrix},$$

and

$$\mathcal{P}_{D,\Omega}(f') = \int_{\mathbb{A}^\times \mathrm{GL}(2,E)\backslash \mathrm{GL}(2,\mathbb{A}_E)} f'(h)\, \Omega(\det h)\, dh \quad \text{for } f' \in \pi_D. \tag{14}$$

Proof. For simplicity, we shall write $\mathcal{B}$ for $\mathcal{B}_{E,\Omega,\chi}$. Let us define $I : G(\mathbb{A})^{+} \to \mathbb{C}$ by

$$I(g) = \int_{U(F)\backslash U(\mathbb{A})} \chi_E(u)\ \theta(\phi, f)(ug)\ du$$

so that we may write

$$\mathcal{B}(\theta(\phi, f)) = \int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \Omega(r)\ I(r)\ dr.$$

We have

$$I(g) = \int_{\mathrm{SO}(V_D)(F)\backslash \mathrm{SO}(V_D)(\mathbb{A})} \left(\int_{U(F)\backslash U(\mathbb{A})} \chi_E(u) \sum_{x \in (V_D \oplus V_D)(F)} \omega_D(ug, hh_g)\phi(x)\ f(hh_g)\ du \right) dh.$$

By (7), the inner integral becomes

$$\sum_{x \in X_0} \omega_D(g, hh_g)\phi(x)\ f(hh_g)$$

where

$$X_0 = \left\{ x = (v_1, v_2) \in (V_D \oplus V_D)(F) \mid \mathrm{Gr}(v_1, v_2) = -\begin{pmatrix} 2\epsilon & 0 \\ 0 & -2\epsilon d \end{pmatrix} \right\}.$$

Since $x_o \in X_0$ and the group $\mathrm{SO}(V_D)(F)$ acts transitively on X_0 by Witt's theorem, we have

$$\begin{aligned} &\sum_{x \in X_0} \omega_D(g, hh_g)\phi(x)\ f(hh_g) \\ &\quad = \sum_{\gamma \in \mathrm{SO}\left(V_D^{(1)}\right)(F)\backslash \mathrm{SO}(V_D)(F)} \omega_D(g, hh_g)\phi\left(\gamma^{-1}x_0\right)\ f(hh_g) \\ &\quad = \sum_{\gamma \in \mathrm{SO}\left(V_D^{(1)}\right)(F)\backslash \mathrm{SO}(V_D)(F)} \omega_D(g, \gamma hh_g)\phi(x_0)\ f(hh_g), \end{aligned}$$

where we identify $\mathrm{SO}\left(V_D^{(1)}\right)$ with the group of elements $g \in \mathrm{SO}(V_D)$ such that $g\mid_{V_D^{(0)}} = 1$. By telescoping the integral, we have

$$I(g) = \int_{\mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})\backslash \mathrm{SO}(V_D)(\mathbb{A})} \omega_D(g, hh_g)\phi(x_0) \left(\int_{\mathrm{SO}\left(V_D^{(1)}\right)(F)\backslash \mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})} f(h_1 hh_g)\ dh_1 \right) dh.$$

For $r = \begin{pmatrix} a & bd \\ b & a \end{pmatrix} \in T$, we identify r with $a + b\eta \in E^{\times}$ and let

$$h_r = \varphi\left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \mathrm{GSO}(V_D).$$

Then we have

$$\mathcal{B}\left(\theta\left(\phi,f\right)\right)=\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\Omega\left(r\right)\int_{\mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})\backslash \mathrm{SO}(V_D)(\mathbb{A})}\omega_D\left(r,hh_r\right)\phi\left(x_0\right)$$
$$\int_{\mathrm{SO}\left(V_D^{(1)}\right)(F)\backslash \mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})}f\left(h_1hh_r\right)\,dh_1\,dh\,dr.$$

Since

$$\left(v_1^{(0)},v_2^{(0)}\right)\begin{pmatrix}a & bd\\ b & a\end{pmatrix}=\left(av_1^{(0)}+bv_2^{(0)},bdv_1^{(0)}+av_2^{(0)}\right)=h_r\left(v_1^{(0)},v_2^{(0)}\right),$$

we have

$$\begin{aligned}\omega_D\left(r,hh_r\right)\phi\left(x_0\right)&=\omega_D\begin{pmatrix}r & 0\\ 0 & {}^tr^{-1}\end{pmatrix}L\left(hh_r\right)\phi\left(x_0\right)\\&=\left|\det r\right|^3L\left(hh_r\right)\phi\left(x_0r\right)\\&=L\left(h_r^{-1}hh_r\right)\phi\left(x_0\right).\end{aligned}$$

Thus by the change of variable $h\mapsto h_rhh_r^{-1}$, we have

$$\mathcal{B}\left(\theta\left(\phi,f_D\right)\right)=\int_{\mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})\backslash \mathrm{SO}(V_D)(\mathbb{A})}L\left(h\right)\phi\left(x_0\right)\,\mathcal{P}'\left(\left(\pi_D\boxtimes\mu\right)\left(h\right)f\right)\,dh,$$

where

$$\mathcal{P}'\left(f'\right)=\int_{\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times}\Omega\left(r\right)\int_{\mathrm{SO}\left(V_D^{(1)}\right)(F)\backslash \mathrm{SO}\left(V_D^{(1)}\right)(\mathbb{A})}f'\left(h_1h_r\right)\,dh_1\,dr$$

for $f'\in\pi_D\boxtimes\mu$. By Lemma 9, we have $G_1=\varphi^{-1}\left(\mathrm{SO}\left(V_D^{(1)}\right)\right)$ and we may identify G_1 with $\{g\in\mathrm{GL}\left(2,E\right)\mid\det g\in\mathrm{GL}\left(1,F\right)\}$. Hence

$$\mathcal{P}'\left(f'\right)=\int_{\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times}\Omega\left(r\right)\int_{\mathbb{A}^\times G_1(F)\backslash G_1(\mathbb{A})}f'\left(g_1\begin{pmatrix}r & 0\\ 0 & 1\end{pmatrix}\right)\mu\left(\det g_1\right)^{-1}\,dg_1\,dr.$$

Further we have

$$G_1\cap\left\{\begin{pmatrix}r & 0\\ 0 & 1\end{pmatrix}\mid r\in E^\times\right\}=\left\{\begin{pmatrix}r & 0\\ 0 & 1\end{pmatrix}\mid r\in F^\times\right\}$$

and hence

$$\mathcal{P}'\left(f'\right)=\int_{\mathbb{A}^\times\mathrm{GL}(2,E)\backslash\mathrm{GL}(2,\mathbb{A}_E)}f'\left(h\right)\,\Omega\left(\det h\right)\,dh=\mathcal{P}_{D,\Omega}\left(f'\right).$$

The rest is clear since $G_0=\varphi^{-1}\left(\mathrm{SO}\left(V_D\right)\right)$. □

Corollary 11. *The cuspidal representation π_D has period $\mathcal{P}_{D,\Omega}$ if and only if $\Theta\left(\pi_D\boxtimes\mu\right)^+$ has Bessel period $\mathcal{B}_{E,\Omega,\chi}$.*

Proof. This follows from the pull-back formula (13) by a standard argument such as the one in Gan and Savin [12, pp. 2718–2719]. □

4. Proofs of theorems

To conclude our results as stated in the introduction, we apply a result of Gan–Takeda [13]. We note that while they assume D is split at each archimedean place because this assumption is present in Badulescu [2], their work is now valid for arbitrary D by the results of Badulescu–Renard [3] on the archimedean Jacquet–Langlands correspondence. We will also use strong multiplicity one for generic representations of GSp(4). This was proved by Jiang–Soudry [24], where they had to assume the base field was totally real to use the second-term Siegel–Weil identity. By recent work of Gan–Takeda–Qiu [15], this is now valid over arbitrary number fields.

Proof of Theorem 4. First note the hypothesis in Conjecture 3 that π corresponds to a generic cuspidal representation σ implies $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$ (Ginzburg–Rallis–Soudry descent [18]), which is equivalent to π having a Shalika model by a result of Jacquet–Shalika [22], cf. [23, Theorem 2.2]. On the other hand, by [13, Corollary 3.2], this is equivalent to $\Theta(\pi \boxtimes 1)$ being generic. By strong multiplicity one for generic representations of GSp(4) [24], the irreducible representation $\Theta(\pi \boxtimes 1)$ is isomorphic to σ, viewed as a representation of GSp(4) with trivial central character. Now apply Corollary 11 with $\mu = 1$ for D split. $\square$

Proof of Theorem 7. Immediate from Corollary 11. $\square$

Proof of Theorem 8. By Corollary 11 with $\mu = 1$, π_D is $\mathrm{GL}(2, E)$-distinguished if and only if $\Theta(\pi_D \boxtimes 1)$ has a special E-Bessel period, and similarly for π and $\Theta(\pi \boxtimes 1)$. By assumption π_D has a Shalika model, which implies π does [13]. Hence $\Theta(\pi_D \boxtimes 1)$ and $\Theta(\pi \boxtimes 1)$ are both generic. Moreover, by strong multiplicity one for generic representations of GSp(4) [24], they must be equal. $\square$

5. Local root numbers and Gross–Prasad conjectures

In this section, we describe the local root number and epsilon factor conditions in Conjecture 3. Further details may be found in [16], [17] and [11].

Let k be a local field of characteristic not 2, and let K be either a quadratic field extension or split $(k \oplus k)$. Denote by κ the quadratic character of $k^\times$ associated to K (the trivial character if $K = k \oplus k$). Let V be a split orthogonal space of dimension $2n+1$ over k, and W a 2-dimensional orthogonal space so that $\mathrm{SO}(W)$ is isomorphic to the group of norm 1 elements of K. Fix an irreducible admissible generic representation σ of $\mathrm{SO}(V)$. This is conjecturally associated to a Langlands parameter $\varphi_1 : WD(k) \to \mathrm{Sp}(M_1)$, where $WD(k)$ is the Weil–Deligne group of k and M_1 is a complex symplectic space of dimension $2n$. (The Langlands parametrization should follow from ongoing work of Arthur and the Paris school; see Arthur's book [1] and recent works of Moeglin and Waldspurger, e.g., [26], [36], [28].) The trivial character 1_K of $\mathrm{SO}(W)$ has Langlands parameter $\varphi_2 : WD(k) \to \mathrm{O}(M_2)$, where M_2 is a 2-dimensional complex orthogonal space, and $\varphi_2 \simeq 1 \oplus \kappa$, where 1 denotes the trivial representation.

Let C_{φ_1} (resp. C_{φ_2}) denote the centralizer of the image of φ_1 (resp. φ_2) in $\mathrm{Sp}(M_1)$ (resp. $\mathrm{SO}(M_2)$). The component group of ϕ_i is $A_{\varphi_i} = C_{\varphi_i}/C^0_{\varphi_i}$, which is a finite elementary abelian 2-group. In particular, A_{φ_2} is $\{\pm 1\}$ if $\eta \neq 1$ (i.e., K/k is a field extension), or the trivial group if $\eta = 1$. Let $\varphi = \varphi_1 \otimes \varphi_2 : WD(k) \to \mathrm{Sp}(M_1) \times \mathrm{O}(M_2)$. The component group of φ is $A_\varphi = A_{\varphi_1} \times A_{\varphi_2}$.

For a symplectic representation M of the Weil group $W(k)$, it is well known how to associate a root number $\epsilon_0(M,\psi)$ with respect to an additive character ψ. If M is a symplectic representation of $WD(k)$, consider $\epsilon_0(M)$ by restricting M to $W(k)$. If k is archimedean, set $\epsilon(M)=\epsilon_0(M)$. If k is nonarchimedean with residue order q and M has nilpotent endomorphism N, we set

$$\epsilon(M)=\epsilon_0(M)\det(-\mathrm{Fr}\cdot q^{-1/2}|M^{I,N}),$$

where I the inertia subgroup of $W(k)$, Fr is the geometric Frobenius, and $M^{I,N}=M^I/(\ker N\cap M^I)$. The root number $\epsilon(M)=\pm1$ and is independent of the choice of ψ.

Gross and Prasad [16, Section 10] define a symplectic root number character of A_φ as follows. We can realize any $a=(a_1,a_2)\in A_\varphi$ as an element of order 2 in $C_\varphi=C_{\varphi_1}\times C_{\varphi_2}\subset \mathrm{Sp}(M_1)\times\mathrm{SO}(M_2)$. For an operator x acting on a vector space M, let M^x denote the (-1)-eigenspace of this action. Then, for a as above, set

$$\chi(a)=\epsilon((M_1\otimes M_2)^a)\det(M_2)(-1)^{\dim M_1^{a_1}/2}\det(M_2^{a_2})(-1)^{\dim M_1/2}.$$

This gives a well-defined character of A_φ.

The pure inner forms of $\mathrm{SO}(V)$ are the set of $\mathrm{SO}(V')$ where V' is an orthogonal space of dimension $2n+1$ and trivial discriminant. When K is split, $\mathrm{SO}(W)$ has no nontrivial pure inner forms; otherwise $\mathrm{SO}(W)$ has one nontrivial pure inner form. The Vogan L-packet Π_φ consists of irreducible admissible representations $\sigma'\otimes\tau'$ of a pure inner form $\mathrm{SO}(V')\times\mathrm{SO}(W')$ which have Langlands parameter φ, and it is (conjecturally) parameterized by the irreducible characters of A_φ in such a way that $\sigma\otimes 1_K$ corresponds to the trivial representation. The refined local Gross–Prasad conjecture [17] in our case says there is exactly one element $\sigma'\otimes\tau'\in\Pi_\varphi$ such that σ' has the τ'-Bessel model, and that it is precisely the element of Π_φ corresponding to χ under the Vogan parametrization. Fix this σ'.

In our case, to determine if σ has a local $\mathrm{SO}(W)$-Bessel model, i.e., $\sigma=\sigma'$, it suffices to look at the restriction

$$\chi_\sigma(a_1)=\epsilon(M_1^{a_1}\otimes M_2)\kappa(-1)^{\dim M_1^{a_1}/2}$$

of χ to the first component A_{φ_1}. To see this, we remark that the Vogan parametrization is such that the distinguished representation σ' lives on $\mathrm{SO}(V)$ if and only if $\chi(-1,1)=\chi_\sigma(-1)=1$. Since $\chi(-1,-1)=1$, we see $\chi(1,-1)=1$ (which implies τ' must be the representation 1_K on $\mathrm{SO}(W)$) and that $\chi_\sigma(a_1)=1$ for all $a_1\in A_{\varphi_1}$ implies $\chi(a)=1$ for all $a\in A_\varphi$. This completes our description of the local root number condition in Conjecture 3(3).

The condition on epsilon factors at the end of Conjecture 3 comes from looking at the specific condition

$$\chi_\sigma(-1)=\epsilon(M_1\otimes M_2)\kappa(-1)^n=1, \tag{15}$$

i.e., that σ' is a representation of $\mathrm{SO}(V)$. Via local Langlands conjectures, (15) says that when condition (2) of Conjecture 3 holds, we should have

$$\epsilon(1/2,\pi_v\times 1_{E,v})=\kappa_v(-1)^n, \tag{16}$$

for all places v of F. Here κ_v is the quadratic character attached to E_v/F_v.

Recall that our assertion at the end of Conjecture 3 is the alternate statement that $\epsilon(1/2,\pi_{E,v})=1$ for all v. We claim that this should be the same as (16). There is nothing to show at split places, so assume K/k is a field extension. Denote the restriction of M_1 to $WD(K)$ by $M_{1,K}$. Denote by $1_{W(k)}$ and $1_{W(K)}$ the trivial representations of $W(k)$ and $W(K)$. Then inductivity for epsilon factors of virtual representations of virtual degree 0 tells us

$$\epsilon(M_{1,K})\epsilon(1_{W(K)})^{-2n} = \epsilon(M_1 \otimes M_2)\epsilon(1_{W(k)})^{-2n}\epsilon(\kappa)^{-2n}.$$

In other words,

$$\epsilon(M_{1,K}) = \epsilon(M_1 \otimes M_2)\epsilon(\kappa)^{-2n} = \epsilon(M_1 \otimes M_2)\kappa(-1)^n,$$

which combined with (15) and local Langlands conjectures, implies the claim.

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