Throughout this set, \( V \) denotes a vector space. All of the exercises are fundamental, though for preparation purposes, I’ve included many more than I will be able to fit on an exam. I’ve starred a few which I think you should pay special attention to.

1. Go over the problems on your quiz and previous homeworks, and make sure you can do them correctly.
   Done.

**True/False**

Circle T or F.

2. T F Any two nonzero vectors in \( \mathbb{R}^3 \) are linearly independent.
   False. They could be scalar multiples of each other.

3. T F A minimal spanning set for \( V \) is a basis for \( V \).
   True. A minimal spanning set means it is LI.

4. T F Any subspace of \( \mathbb{R}^2 \) is either a line through the origin or \( \mathbb{R}^2 \).
   False. These are all the \( 1 \)- and \( 2 \)- dimensional subspaces, but there is also the 0-dimensional subspace, i.e., just the origin.

5. T F The span of two nonzero vectors is either a line through the origin or a plane through the origin.
   True. The span must either be a 1- or 2- dimensional subspace of \( \mathbb{R}^2 \).

6. T F The set of polynomials in \( x \) of degree at most 5 form a vector space.
   True. This example was in the text and in lecture.

7. T F There is a linear transformation from \( \mathbb{R}^2 \to \mathbb{R}^3 \) whose image is the cone \( x^2 + y^2 = z^2 \).
   False. The image is not a subspace of \( \mathbb{R}^3 \).

8. T F There is a linear transformation from \( \mathbb{R}^3 \to \mathbb{R}^2 \) whose image is the line \( y = x + 1 \).
   False. The image is not a subspace of \( \mathbb{R}^3 \).

9. T F There is a linear transformation from \( \mathbb{R}^2 \to \mathbb{R}^3 \) whose image is the plane \( z = x + y \).
   True. For example \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \) works, since the columns form a basis for this plane.

10. T F There is a linear transformation from \( \mathbb{R}^2 \to \mathbb{R}^3 \) whose image is \( \mathbb{R}^3 \).
    False. The rank (dimension of the image = 3 here) must be at most the dimension of the domain (2 here).

**Questions**

11. If \( S = \{v_1, v_2, \ldots, v_k\} \subseteq V \), define \( \text{span}(S) \).
    Answer in words: the set of all linear combinations of \( v_1, \ldots, v_k \).
    Answer in symbols: \( \text{span}(S) = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_1, \ldots, a_k \in \mathbb{R}\} \).

12. With \( S \) as above, define what it means for \( S \) to be a basis of \( V \).
span(S) = V and S is LI.

13. With S as above, define what it means for S to be linearly independent.

The equations

\[ a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \]

has only the trivial solution

\[ a_1 = a_2 = \cdots = a_k = 0. \]

14. Find two different bases for \( \mathbb{R}^2 \) (no proof needed).

For example \( S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3/4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \) and \( S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17026 \\ 1 \end{pmatrix} \right\} \).

15. Consider the basis \( S = \{ t^2 + 1, t + 1, 3t^2 - t \} \) for the space of polynomials of degree at most 2. If \( [v]_S = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \), find \( v \).

\[ v = 3(t^2 + 1) + 2(t + 1) - 1(3t^2 - t) = 3t + 6. \]

16. Consider the basis \( S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\} \) of \( \mathbb{R}^4 \). If \( v = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \), find \( [v]_S \).

We want to write \( v = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \), i.e., solve \( \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \). Solving this, we see \( a = \frac{8}{3} \) and \( b = \frac{2}{3} \), so \( [v]_S = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix} \).

Problems

Show your work.

17. Show that the set of vectors of the form \( \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} \) in \( \mathbb{R}^3 \) forms a subspace of \( \mathbb{R}^3 \). Find a basis for this space (no proof needed). What is its dimension? Describe this space geometrically.

Let \( W = \left\{ \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} \right\} \). Let \( \begin{pmatrix} a \\ b \\ a + b \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ a' + b' \end{pmatrix} \in W \) and \( c \in \mathbb{R} \). Then

\[ \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ a' + b' \end{pmatrix} = \begin{pmatrix} a + a' \\ b + b' \\ (a + a') + (b + b') \end{pmatrix} \in W \]

and

\[ c \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ ca + cb \end{pmatrix} \in W, \]

i.e., \( W \) is closed under addition and scalar multiplication, and therefore a subspace of \( \mathbb{R}^3 \).

Alternatively, observe \( \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \), i.e., \( W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \) and therefore is a subspace of \( \mathbb{R}^3 \). (Recall \( \text{span}(S) \) is always a subspace.)
The above shows \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \) is a basis for \( W \). Its dimension is 2, and geometrically it is the plane in \( \mathbb{R}^3 \) determined by \( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \). A nicer way to describe geometrically is simply as the plane \( z = x + y \) (\( W \) is the set of vectors whose 3rd coordinate is the sum of the first 2.)

18. Do the same as the previous problem for the subset \( \{(x, y, z) : x + y + z = 0\} \) of \( \mathbb{R}^3 \).

Similar to above, I’ll omit the solution, and just observe you can write this as the space \( W = \left\{ \begin{pmatrix} a \\ b \\ -a - b \end{pmatrix} \right\} \).

19. Is \( \{(x, y, z) : 2x - 3y + z = 1\} \) a subspace of \( \mathbb{R}^3 \)?

No. It does not contain the origin.

20.* Let \( A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \).

(a) Find a basis for the image of \( A \).
(b) Find a basis for the kernel of \( A \).
(c) Determine rank \( A \) and nullity \( A \).

It row reduces to the identity, so all columns are linearly independent. Recall for this, and subsequent problems, the image of \( A \) is just the span of the columns of \( A \).

Therefore

(a) Any basis for \( \mathbb{R}^3 \) is a basis for \( A \), e.g., the standard basis, or \( \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \).

(b) By the rank nullity theorem (or the row reduced form) the kernel is trivial, i.e., \( \ker A = \{0\} \), so a basis is the empty set.

(c) \( \text{rank}(A) = 3 \) and \( \text{nullity}(A) = 0 \).

Note: I believe we didn’t actually say the basis or \( \{0\} \) is the empty set in class, so it probably wouldn’t be on the exam like this. I meant to give a matrix of rank 2. So on the exam, I might either ask just for the kernel (not a basis) or make sure the kernel is nontrivial.

21. Do the same as the previous problem for \( A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & -2 & 1 & 1 \\ -2 & -2 & -1 & 1 \\ 3 & 3 & 0 & -1 \end{pmatrix} \).

It row reduces to

\[
\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Therefore

(a) the first three columns of \( A \) form a basis for the image;
(b) the kernel is the set of \( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \) with \( w \) free, \( z = \frac{1}{3} w \), \( y = \frac{1}{2} w \) and \( x = -\frac{1}{6} w \), i.e., \( \ker A = \left\{ \begin{pmatrix} -1/6 \\ 1/2 \\ 1/3 \\ w \end{pmatrix} \right\} \) so a basis is \( \begin{pmatrix} 1/2 \\ 1/3 \\ w \end{pmatrix} \); and
(c) \( \text{rank}(A) = 3 \) and \( \text{nullity}(A) = 1 \).
22. Find a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ whose image is the set of vectors of the form $\begin{pmatrix} a \\ 2b-a \\ b \end{pmatrix}$.

Note $\begin{pmatrix} 2b-a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, i.e., we want the image to be the span of $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. Hence

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$$

does the job.

23. Find a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose image is the plane $z = x + y$.

We need a basis for this plane, so take two vectors in the plane which are not LI, say $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Now we need a $3 \times 3$ matrix who columns spans this plane, e.g.,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$  

(Remark: this plane is the space from #17.)

24.* Let $u, v \in V$. Show $\{u, v\}$ is linearly dependent if and only if $u = cv$ or $v = cu$ for some $c \in \mathbb{R}$.

Proof. ($\Rightarrow$) Suppose $\{u, v\}$ is linearly dependent. This means $au + bv = 0$ for $a, b$ not both 0. If $a \neq 0$ then $u = -\frac{b}{a}v$. Otherwise, $a = 0$ so $bv = 0$ with $b \neq 0$, i.e., $v = 0$ so $v = 0 \cdot u$.

($\Leftarrow$) Now suppose $u = cv$ or $v = cu$. If $u = cv$, then $u + (-c)v = 0$, i.e., $u$ and $v$ are linearly dependent. The case $v = cu$ is similar.

25. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $A$ is a projection (i.e., its image is a line or a point). Show $\det A := ad - bc = 0$.

Proof. If $A$ is a projection, this means its rank is 0 or 1, i.e., the columns of $A$ are linearly dependent. If either column is all zeroes, it is clear $\det A = 0$. Otherwise $\begin{pmatrix} a \\ b \end{pmatrix} = k \begin{pmatrix} c \\ d \end{pmatrix}$ for some $k$. Then $\det(A) = ad - bc = a(kb) - b(ka) = 0$.

26.* Let $A : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation. Suppose the kernel of $A$ is a plane in $\mathbb{R}^4$. What can you say about the image of $A$?

$\ker(A)$ being a plane means $\text{nullity}(A) = 2$, so by the Rank-Nullity Theorem, $\text{rank}(A) = 2$, i.e., the image of $A$ is a plane (through the origin) in $\mathbb{R}^3$. 
