Note: There are a few lectures that won't align nicely with sections in the text, and so I'm labelling them with letters so as not to be confused with section numbers. These are some notes meant to supplement some of these video lectures. They contain some details not in the videos (especially review material/references to previous material), and conversely the videos may contain some details (primarily illustrations and examples) that are not in these notes.

Let A be an $n \times n$ matrix.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the "standard" associated linear transformation, i.e., $A = [T]$, the matrix for T with respect to the standard bases.

Recall/observe:

- We say A is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix B such that $AB = BA = I$ (the identity matrix), in which case B is called the **inverse** of A and is written as $B = A^{-1}$. (Section 10.4)
- If A has an inverse, it is unique. (*Proof:* Suppose B, C are inverses of A. Then $B = B(AC) = (BA)C = C$, i.e., B and C must be the same.)
- To check if B is the inverse of A, it suffices to check that $AB = I$ or that $BA = I$. (Corollary 11.2.5)
- A is invertible if and only if T is *invertible*, i.e., if and only if T an isomorphism. (Corollary 11.2.4)
- If T is invertible with inverse S, then $A^{-1} = [S]$, the standard matrix of S. (This follows from Proposition 11.2.3, i.e., the fact that matrix multiplication corresponds to composition of linear transformations. See also the proof of Corollary 11.2.4.) Geometrically, this means that the inverse matrix of A represents undoing the transformation T (e.g., if A corresponds to rotation counterclockwise by 90 degrees in the plane, then A^{-1} corresponds to rotation clockwise by 90 degrees in the plane.)
- A is invertible $\iff$ $\operatorname{rank}(T) = n$ (i.e., $\operatorname{Im}(T) = \mathbb{R}^n$) $\iff$ $\ker T = \{0\}$ (i.e., the nullity of T is 0). (Use previous bullet point + Exercise 17 in Chapter 8)
- Define the **column rank** to be the dimension of the column space (the span of the columns; see Section 13.1) of A. Then A is invertible $\iff$ the column rank of A is n. (Use previous bullet point + fact that column rank of A is the rank of T.)

Let us now think about what we know for small n.

Example 1. ($n = 1$) It is clear that the 1×1 matrix $A = (a)$ is invertible if and only if $a \neq 0$. In this case $A^{-1} = (a^{-1})$.

Example 2. ($n = 2$) The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if the **determinant** $\det A := ad - bc$ is nonzero, in which case $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. (This was Proposition 10.4.1)

Now we want to explain in the general case how to (i) determine a matrix is invertible, and (ii) compute its inverse in case it is. We will apply the methods of Chapter 13 on solving linear systems.

Lemma 1. *Let $\overline{A}$ be a reduced echelon form of A. Then A and $\overline{A}$ have the same column rank.*

Proof. Consider the homogeneous linear system $AX = \mathbf{0}$, where X is a column vector of length n, and $\mathbf{0}$ is the all zero column vector of length n. Then process of row reduction from Section 13.2 transforms the augmented matrix $[A|\mathbf{0}]$ to $[\overline{A}|\mathbf{0}]$, where $\overline{A}$ is a reduced echelon form of A. (Note that none of the operations in the row reduction process will modify the augmented column of zeros.)

Consequently, the set of solutions to $AX = 0$ is the same as the set of solutions to $\overline{A}X = 0$. However, the set of homogenous solutions X to $AX = 0$ is precisely the kernel of T (Theorem 13.1.3), thus the column rank of A is $n - m$, where $m = \dim \ker T$ is the dimension of the (vector space) of homogenous solutions. By the same reasoning, the column rank of $\overline{A}$ is also $n - m$. □

Proposition 1. *A is invertible if and only if its reduced echelon form $\overline{A}$ is the identity matrix.*

We remark that the proof implies that, for invertible A, there is only one reduced echelon form of A.

Proof. Suppose $\overline{A}$ is some reduced echelon form of A. By the lemma, we know that A is invertible $\iff$ the column rank of $\overline{A}$ is n.

For any matrix in reduced echelon form, each row is either a row of all zeroes or has a leading 1. Thus $\overline{A}$ has column rank n if and only if it has n leading 1's. (More generally, the column rank of a matrix in reduced echelon form is the number of leading 1's.) From the definition of reduced echelon form, this happens if and only if A is the identity matrix. (Think it out.) □

Proposition 2. *Suppose A is invertible, and let $I = I_n$ be the $n \times n$ identity matrix. Then row reduction to reduced echelon form transforms the augmented matrix $[A|I]$ into $[I|A^{-1}]$.*

Proof. Let B be the inverse of A, and write $B = [B_1 B_2 \dots B_n]$ where B_i is the i-th column of B. Let e_i be the i-th standard basis vector, i.e., the i-th column of I. Then

$$AB = I$$

is equivalent to

$$AB_1 = e_1, \quad AB_2 = e_2, \quad \dots, \quad AB_n = e_n.$$

So to solve for B means simultaneously solving these n linear systems where here the B_i's are the unknown vectors.

Row reduction of each augmented matrix $[A|e_i]$ must give us $[I|B_i]$ since the corresponding systems have the same solutions. Instead of doing the row reduction n times, we can simply consider the augmented matrix with n-augmented columns $[A|I] = [A|e_1 e_2 \dots e_n]$ and row reduce once to then obtain $[I|B_1 B_2 \dots B_n] = [I|B] = [I|A^{-1}]$. □

One can adapt the proof of Proposition 2 to give an alternate proof of Proposition 1.

Example 3. Determine if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is invertible, and if so find its inverse.

Solution. Using the first row as a pivot row, we zero out the first column to give $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$. Now using the second row as a pivot row, we zero out the second column to get the reduced echelon form $\overline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Since this is not the identity matrix, A is not invertible. (It has column rank 2.)

Example 4. Determine if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ is invertible, and if so find its inverse.

Solution. In this case A is invertible, and we will just do a single row reduction of the augmented matrix $[A|I]$ to see that both (i) A is invertible and (ii) find its inverse.

We begin with the augmented matrix

$$[A|I] = \begin{pmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}.$$

Use the first row as a pivot row, and subtract the first row from the third row to get

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \end{pmatrix}.$$

We could use the second row as a pivot row now, but since the third row is already in the form we want the second row to be in, I will just swap the 2nd and 3rd rows:

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \end{pmatrix}.$$

Now we use the second row as a pivot row, and subtract the second row from the first row:

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 0 & -1 & | & 1 & 1 & -1 \end{pmatrix}.$$

Scale the 3rd row by -1:

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{pmatrix}.$$

Use the 3rd row as a pivot, and zero out the last column by subtracting the 3rd row from the 1st row to get our reduced echelon form

$$[\overline{A}|B] = \begin{pmatrix} 1 & 0 & 0 & | & 2 & 1 & -1 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{pmatrix}.$$

Since we were able to reduce to the identity matrix on the left, this means that A is invertible and the inverse is

$$B = A^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

It's always good to double check your answer, especially as it's easy to make mistakes, so we check that indeed

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus Propositions 1 and 2, together with the process of row reduction, give us elementary procedures (algorithms) to determine if matrices are invertible and if so, what their inverse is.

In Section 14.3 we will give an alternative approach to matrix inversion: A is invertible if and only if the *determinant* of A is nonzero, and one can compute the inverse using the *cofactor matrix*. However, the methods we presented in this section are in some sense simpler, if a bit tedious.