

Version 1.1 (5/4/16)

Changes in v1.1: Fixed typo in 1(b) and rephrased 1(c).

## Instructions

In this project we consider some different population models and basic numerical methods in their analysis. In this context, let us say that a population model (with given initial conditions) is *stable* if, under this model, all species coexist indefinitely. If one or more species goes extinct, call the model *unstable*.

This project should be a typed report and represent your own work, which will include Mathematica code and output as well as your own analysis and interpretations. When possible, write up your analysis in a way that would make sense to a non-scientific audience.

You will be asked to use (i) the Euler method, (ii) the Runge–Kutta method and (iii) the power series method in this project. Part of the goal of the project is to better understand these methods, so you are required to code these methods in Mathematica yourself. (For the power series method, you should first do some initial analysis by hand.)

As a rough guideline, I expect the write-up to be about 6–12 pages, not including your Mathematica code and output/graphs. Your exposition is important and will constitute roughly half of your project grade—it should show that you understand the models in question as well as the numerical methods involved, as well as that you can communicate ideas clearly. To a large extent, you may organize your report how you like (e.g., include code as it comes up or as an appendix), but please address 1, 2 and 3 in order.

To try to make sure you are on the right track, during the final two lab meetings I will come around and ask you to show me what you have, so please try to get a good chunk of the project done before those meetings. If you like, you may submit a draft or partial draft to me (by email is fine) the last week of class to get feedback before you turn in your final report. You are also welcome to come talk to me in my office, either during the last week of class or Mon May 9.

If you have questions on what is expected, please ask me.

The project is due (printed and stapled) at our final meeting **Tue May 10, 1:30pm, PHSC 356** (not that you and I can never meet afterwards, but you know what I mean).

## The project

1. One issue with the basic Lotka–Volterra equations we considered for predator-prey systems is that in the absence of predators, the prey grow without bound. We can modify the system just like we increased our sophistication of a model of a single population from the Malthusian model to the logistic model (cf. Sec. 1.3 of text). Consider the following predator-prey system

$$\begin{aligned}y' &= y(2.1 - .00006y - 0.15z) \\z' &= z(-1.2 + .0006y),\end{aligned}$$

where  $y(t)$  and  $z(t)$  represent rabbit and tiger populations at time  $t$ .

- (a) Give an explanation for why the formulas above make a somewhat reasonable model for the system. (I don't expect any justification of the actual values of

the coefficients, as they are just made up (see references below), but give an interpretation for what each term might represent, at least accounting for the sign of each coefficient.) Describe some limitations of the model.

- (b) Plot a phase portrait of the system along with the  $y$ - and  $z$ -nullclines, i.e., the phase portrait together with curve determined by  $y' = 0$  as well as the curve determined by  $z' = 0$ . Explain how to interpret the phase portrait and the meaning of the nullclines in terms of populations. Find the equilibrium solutions (i.e., with  $y' = z' = 0$ ) and describe the possible long-term behaviours of the system. Under what initial conditions the system will be stable versus unstable?
- (c) Suppose we have initial conditions  $y(0) = 3000$ ,  $z(0) = 20$ . Use each of (i) Euler's method, (ii) the Runge–Kutta method to numerically solve this system well enough for you to confidently describe the long-term behaviour of this system (and in particular for you to conclude if it is stable or unstable). Then try (iii) the power series method and compare the results with (i) and (ii). Does it seem like the solutions can be expressed in terms of basic functions you know (polynomials, exponentials, trig functions, etc)? Explain the ideas of each of these methods in your own words, how you implemented them (i.e., explain your code in your own words), and how you determined what level of approximation to use. Furthermore, compare the rate at which methods (i) and (ii) converge to the actual solution. Discuss pros and cons of these different methods, and what method seems preferable.
2. Another type of interaction population system one can consider besides predator-prey is competition models. That is, one species is not feasting on the other, but rather both are competing for some resources. Consider the following competition model

$$\begin{aligned}x' &= x(3 - .00009x - .00003y) \\y' &= y(2.1 - .00003x - .00006y),\end{aligned}$$

where  $x(t)$  and  $y(t)$  represent field mice and rabbit populations.

- (a) Give an explanation for why the formulas above make a somewhat reasonable model for the system. Describe some limitations of the model.
- (b) Plot a phase portrait of the system along with the  $x$ - and  $y$ -nullclines. Explain how to interpret the phase portrait and the meaning of the nullclines in terms of populations. Find the equilibrium solutions and describe the possible long-term behaviours of the system. Under what initial conditions the system will be stable versus unstable?
- (c) Suppose we have initial conditions  $x(0) = y(0) = 3000$ . Approximate solutions to this system well enough for you to confidently describe the long-term behaviour of this system (and in particular for you to conclude if it is stable or unstable). (You may use Euler's method, Runge–Kutta or a variant, power series, or a combination—your choice.) Explain what method you chose and why, how you implemented it and how you determined what level of approximation was sufficient to understand the long-term behaviour.

3. We can similarly consider multiple interacting species. Let's put together our predator-prey and competition models above into the system

$$\begin{aligned}x' &= x(3 - .00009x - .00003y - 0.15z) \\y' &= y(2.1 - .00003x - .00006y - 0.15z) \\z' &= z(-1.2 + .0006x + .0006y),\end{aligned}$$

where  $x(t)$ ,  $y(t)$  and  $z(t)$  represent field mice, rabbit and tiger populations.

- Give an explanation for why the formulas above make a somewhat reasonable model for the system. Describe some limitations of the model.
- Suppose we have initial conditions  $x(0) = y(0) = 3000$  and  $z(0) = 20$ . Approximate solutions to this system well enough for you to confidently describe the long-term behaviour of this system (and in particular for you to conclude if it is stable or unstable). (You may use Euler's method, Runge-Kutta or a variant, power series, or a combination—your choice.) Explain what method you chose and why, how you implemented it and how you determined what level of approximation was sufficient to understand the long-term behaviour.
- Compare the behaviour of this system to the previous two examples (with corresponding initial populations). Describe how the interaction with 3 species is different than the interaction with just 2 species. Can you make a case that a system with more or fewer species is likely to be stable? Do you think this depends much upon the initial conditions?
- (Bonus, i.e., optional) Since this system has more than 2 variables, we can't talk about a (2-d) phase portrait. However, there are a couple of ways one can try to visualize this system. One can plot 3-d trajectories in 3-d phase space, or plot various 2-d phase portraits corresponding to the  $xy$ -,  $xz$ -, and  $yz$ -planes. Can you use one of these methods to describe possible long-term behaviours of the system (not fixing the above initial conditions)? Does this help you address the questions in (c)?

## References

This project is based on the numerical example (not apparently based on a model of specific species) in the paper "Interspecific competition, predation and species diversity: a comment" by Cramer and May (*J. theor. Biol.*, 1972), which is briefly discussed in Chapter 8 of the book *Mathematical Modelling with Case Studies* by B. Barnes and G.R. Fulford (2nd ed., CRC Press, 2009). If you're interested, that book has a number of interesting examples of models, e.g. a model for the so-called "lemming mass suicides" based on actual data.