Calculus II (Fall 2015)
Practice Problems for Exam 2

Note: Section divisions and instructions below are the same as they will be on the exam, so you will have a better idea of what to expect, though I will leave spaces for answers on the actual exam. However, this is not quite a mock exam as there are more practice problems here than there will be on the actual exam. Nevertheless, I recommend you treat it as a mock exam (prepare first and try to finish on your own), with a rough goal of finishing in 2–2.5 hours. Then check your solutions with those posted online, ask questions if needed, and go back and make sure you can do the problems on your own.

Instructions: Write your name and discussion section number (11–16) at the top. Read all instructions.

No notes, text, calculators, etc. are allowed. Please answer the questions in the space provided below. You may use the back of the pages as scratch paper. If you run out of space for your answer on the front, you may continue it on the back provided you note that your answer continues on the back. Each section has additional instructions below. Maximum score: $\ln 1$ points.

1 Tantalizingly True or Foolishly False

Instructions: Circle T or F for each question. No work is needed. Each problem is worth $\lim_{t\to\infty} e^{-t}$ points.

1. T F Let $f : A \to B$ be one-to-one. Then $f^{-1} : B \to A$ exists.

   Solution: False, you need f to be one-to-one and onto. (In practice, one can just restrict the domain and codomain.)

2. T F When defined, $f^{-1}(x) = f(x)^{-1}$.

   Solution: False, the first is the inverse function, the second denotes the reciprocal.

3. T F $\lim_{t\to 1} \ln t = 0$.

   Solution: True: Yes, $\ln x$ is continuous at 0, so in fact the limit is just $\ln 1 = 0$.

4. T F $\ln(5e) = 1 + \ln 5$.

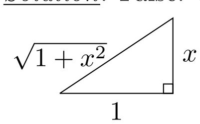
   Solution: True: $\ln(5e) = \ln 5 + \ln e = \ln 5 + 1$.

5. T F The image of $\sin^{-1} x$ is all real numbers.

   Solution: False: While $\sin x$ makes sense for all real x, you cannot think of $\sin^{-1}$ as a function from $[-1, 1]$ to $\mathbb{R}$ because sin is not one-to-one. So we define $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

6. T F $\cos(\tan^{-1} x) = \sqrt{1 + x^2}$.

   Solution: False: draw the triangle

   to see that $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

7. T F $\int \frac{dx}{x} = \ln x + C$

   Solution: False: that's only for $x > 0$. In general, you need $\ln |x| + C$.

8. T F $\int_0^\infty \frac{1}{x^2} dx$ converges.

   Solution: False: $\int_0^\infty \frac{1}{x^2} dx = \int_0^1 \frac{1}{x^2} dx + \int_1^\infty \frac{1}{x^2} dx$, and the second improper integral converges, but the first does not.

2 Enlightening Exercises

Instructions: For each problem below, **show your work**. **For calculations, box in your final answer**. Each problem is worth $\int_{-1}^{1} xe^{-x^2}\,dx$ points.

9. Compute $\lim_{x\to 0^+} x \ln x$.

   Solution: You should get 0. See Example 6 on p. 473.

10. Using the formula for $\frac{d}{dx}e^x$, prove $\frac{d}{dx}a^x = a^x \ln a$ (for $a > 0$).

    Solution: See p. 415.

11. Using integrals, prove a formula for the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (for $a, b > 0$).

    Solution: You should get πab. See Example 2 on p. 503.

12. Compute $\int \frac{1}{\sqrt{x^2-9}}dx$.

    Solution: This is Example 5 on p. 505 with $a = 3$. Use the substitution $x = 3\sec\theta$ so $dx = 3\sec\theta\tan\theta\,d\theta$ and $x^2 - 9 = 9\sec^2\theta - 9 = 9\tan^2\theta$. Then

$$\int \frac{1}{\sqrt{x^2-9}}dx = \int \frac{1}{\sqrt{9\tan^2\theta}} 3\sec\theta\tan\theta\,d\theta = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta$$
$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \boxed{\ln|\tfrac{x}{3} + \tfrac{\sqrt{x^2-9}}{3}| + C.}$$

    For the last step, use the triangle on p. 505. One can also further simplify as in the book.

13. Compute $\int \frac{x}{\sqrt{x^2-9}}dx$.

    Solution: One could do this with the same trig substitution $x = 3\sec\theta$ as in the last problem, which will boil things down to an integral of $\sec^2\theta$, but the easier thing to do is either substitution $u = x^2 - 9$ or $u = \sqrt{x^2-9}$. If we take $u = x^2 - 9$, we have $du = 2x\,dx$. Then

$$\int \frac{x}{\sqrt{x^2-9}}dx = \int u^{-1/2}\frac{du}{2} = u^{1/2} + C = \boxed{\sqrt{x^2-9} + C.}$$

    Alternatively, if we take $u = \sqrt{x^2-9}$ we get $du = \frac{x}{\sqrt{x^2-9}}dx$ so

$$\int \frac{x}{\sqrt{x^2-9}}dx = \int du = u + C = \boxed{\sqrt{x^2-9} + C.}$$

14. Compute $\int \frac{x^3+1}{x^2+1}dx$.

    Solution: Here we should first divide, and we get

$$\int \frac{x^3+1}{x^2+1}dx = \int \left(x - \frac{x-1}{x^2+1}\right)dx = \frac{x^2}{2} - \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx = \frac{x^2}{2} - \int \frac{x}{x^2+1}dx + \tan^{-1} x.$$

    Using the substitution $u = x^2 + 1$, this equals

$$\frac{x^2}{2} - \int \frac{1}{u}\frac{du}{2} - \tan^{-1} x = \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + C = \boxed{\tfrac{x^2}{2} - \ln\sqrt{x^2+1} + \tan^{-1} x + C.}$$

15. Evaluate $\int \frac{1}{x^3+x}dx$.

<u>*Solution*</u>: We factor the bottom and use partial fractions to write

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Comparing numerators shows $(A+B)x^2 + Cx + A = 1$, so $A+B = C = 0$ and $A = 1$. Thus $B = -A = -1$, and

$$\int \frac{1}{x^3+x}dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \boxed{\ln|x| - \ln\sqrt{x^2+1} + C,}$$

where we integrated $\frac{x}{x^2+1}$ with the substitution $u = x^2+1$ as in the previous exercise.

16. Compute $\int \cos^3 x\,dx$.

<u>*Solution*</u>: You should get $\sin x - \frac{1}{3}\sin^3 x + C$ (or something equivalent). See Example 1 on p. 495.

17. Evaluate $\int x^2 \sin^2 x\,dx$.

<u>*Solution*</u>: First we use the half-angle identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$\int x^2\sin^2 x\,dx = \frac{1}{2}\int (x^2 - x^2\cos 2x)dx = \frac{x^3}{6} - \frac{1}{2}\int x^2\cos 2x\,dx.$$

Now we do the latter integral using integration by parts with $u = x^2$, $du = 2x\,dx$, $dv = \cos 2x\,dx$, $v = \frac{\sin 2x}{2}$ to get

$$\int x^2\cos 2x\,dx = \frac{1}{2}x^2\sin 2x - \int x\sin 2x\,dx.$$

A second integration by parts with $u = x$, $du = dx$, $dv = \sin 2x\,dx$, $v = -\frac{\cos 2x}{2}$ gives

$$\int x^2\cos 2x\,dx = \frac{1}{2}x^2\sin 2x - (-\frac{1}{2}x\cos 2x + \int \frac{\cos 2x}{2}\,dx) = \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{\sin 2x}{4} + C.$$

Putting it all together gives

$$\boxed{\tfrac{x^3}{6} - \tfrac{1}{4}x^2\sin 2x - \tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C.}$$

18. Evaluate $\int \ln x\,dx$.

<u>*Solution*</u>: Use integration by parts to get $x\ln x - x + C$. See Example 2 on p. 489. $\int \tan^{-1} x dx$ is another example where you might not think to use integration by parts, but it works (Example 5 on p. 491).

19. Evaluate $\int \frac{dx}{e^x+1}$.

<u>*Solution*</u>: Try $u = e^x + 1$. Then $du = e^x\,dx$, so $dx = \frac{du}{e^x} = \frac{du}{u-1}$. Then

$$\int \frac{dx}{e^x+1} = \int \frac{1}{u}\frac{du}{u-1} = \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

where we used partial fractions to write $\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$ and solve to get $A = -1, B = 1$. Then our integral evaluates to

$$\ln|u-1| - \ln|u| + C = \ln e^x - \ln|e^x+1| + C = \boxed{x - \ln|e^x+1| + C.}$$

Another approach is to let $u = e^x$. Then we get

$$\int \frac{dx}{e^x + 1} = \int \frac{1}{u+1} \frac{du}{e^x} = \int \frac{1}{u+1} \frac{du}{u}$$

and again do partial fractions to get the same answer.

(Yet another way to do the problem is write $\frac{1}{e^x+1} = \frac{e^x+1-e^x}{e^x+1} = 1 - \frac{e^x}{e^x+1}$. Then one can do the problem with the substitution $u = e^x + 1$ but avoid the partial fractions.)

20. Evaluate $\int e^x \sin x \, dx$.

<u>*Solution*</u>: This is one of those where you use integration by parts twice and end up with the same integral, and solve for it. You should get $\frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C$. See Example 4 on p. 490.