Calculus II (Fall 2015)
Practice Problems for Exam 1

Note: Section divisions and instructions below are the same as they will be on the exam, so you will have a better idea of what to expect, though I will leave spaces for answers on the actual exam. However, this is not quite a mock exam as there are more practice problems here than there will be on the actual exam. Nevertheless, I recommend you treat it as a mock exam (prepare first and try to finish on your own), with a rough goal of finishing in 2–2.5 hours. Then check your solutions with those posted on D2L, ask questions if needed, and go back and make sure you can do the problems on your own.

Instructions: Write your name and discussion section number (11–16) at the top. Read all instructions.

No notes, text, calculators, etc. are allowed. Please answer the questions in the space provided below. You may use the back of the pages as scratch paper. If you run out of space for your answer on the front, you may continue it on the back provided you note that your answer continues on the back. Each section has additional instructions below. Maximum score: 0 points.

1 True or False

Instructions: Circle T or F for each question. No work is needed. Each problem is worth 0 points. In this section, a and b are real numbers with $a \leq b$, $f(x)$ and $g(x)$ denote a continuous functions on $\mathbb{R}$, and R is the region between $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$.

1. T F If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
2. T F Any continuous function on $[a, b]$ is differentiable.
3. T F Any continuous function on $[a, b]$ is integrable.
4. T F If a function is not continuous at some point in $[a, b]$, then it is not integrable on $[a, b]$.
5. T F $\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)\,dx$
6. T F $\int (f(x)g(x))\,dx = \int f(x)dx \cdot \int g(x)\,dx$
7. T F $\frac{d}{dx} \int_a^b \sin(x^4)\,dx = \sin(x^4)$
8. T F $\int_{-\pi}^{\pi} x \cos x\,dx = 2 \int_0^{\pi} x \cos x\,dx$.
9. T F The area of R (as above) is $\int_a^b (f(x) - g(x))\,dx$.
10. T F If S is the solid obtained by rotating R around the x-axis, the volume of S is $\int_a^b \pi(f(x)^2 - g(x)^2)\,dx$.
11. T F If S is the solid obtained by rotating R around the line $x = a$, the volume of S is $\int_a^b 2\pi(x - a)(f(x) - g(x))\,dx$.

2 Quick questions

Instructions: For each question in this section, you are not required to show any work. However, in the event that your answer is incorrect, you may receive partial credit for work you write down. Each problem is worth 0 points. **Box in your final answer.**

12. Compute $\int_0^3 (x^2+1)\,dx$.

13. Compute $\int_0^{\pi/9} \sin 3x\,dx$.

14. Compute $\int x^2 \cos(x^3)\,dx$.

15. Compute $\int (2\sqrt{x}+1)^5\,dx$.

16. Compute $\int \sec^2 x \tan x\,dx$.

17. Compute $\int \frac{\cos x}{\sin^2 x}\,dx$.

18. Compute the area under $y = x^2 - 5$ from $x = 0$ to $x = 2$.

19. Approximate the area under $y = \sqrt{x}$ between $x = 0$ and $x = 1$ by computing the corresponding Riemann sum with $n = 4$ equal width partitions using the right endpoint rule.

3 Patient Problems

Instructions: For each problem below, show your work. Each problem is worth 0 points. You may use the following identities: $1+2+\cdots+n^2 = \frac{n(n+1)}{2}$ and $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$. **For calculations, box in your final answer**.

20. Using Riemann sums and the definition of integrals, compute the area under the curve $y = x^2$ from $x = 0$ to $x = 2$.

21. Sketch the region bounded by the curves $y = 2x^2 - 7$ and $y = 5 - x^2$, and compute the area.

22. Sketch the region bounded by the curves $y = \sqrt{x}$, $y = 2 - x$ and the x-axis. Compute its area.

23. Let R be the region in the xy-plane under the curve $y = \sqrt{x}$ between $x = 0$ and $x = 1$, and let S be the solid obtained by rotating R about the line $x = -2$. Sketch R and S, and determine the volume of S.

24. Let R be the region in the xy-plane above the x-axis which is bounded by $y = x$ and $y = x^3$. Let S be the solid obtained by rotating R about the y-axis. Sketch R and S, and determine the volume of S.

25. Suppose the Oscar Mayer Wienermobile, initially at rest at time 0, travels due north and accelerates at a rate of $\frac{3t}{2}$ mph/s at time t (measured in seconds). What is the average speed of the Wienermobile from time $t = 0$ to $t = 10$?

Solutions

For the first two parts, explanations are not required, but I'll provide some for your benefit. Sketches I will omit.

Note: Answers can be simplified in different ways, so if what you wrote down is different, it's not necessarily wrong (e.g., # 15). Just check to make sure it equals what I wrote. And if you think you find an error in one of my solutions, please let me know so I can double check it.

1. T: this is FTC II
2. F: Think about $|x|$
3. T: true–this follows from of FTC I. So integrals of continuous functions closed intervals always exist but derivatives do not (this is not true on open intervals—e.g., think of $\frac{1}{x}$).
4. F: We can still integrate sometimes, e.g., jump discontinuities.
5. T
6. F
7. F: Read the question carefully—the definite integral gives some constant, so the derivative is 0.
8. F: Since x is odd and $\cos x$ is even, $x \cos x$ is odd, thus by symmetry the integral on the left is 0. (A picture shows the integral on the right is not zero.)
9. F (but almost T): You need to assume $f(x) \geq g(x)$ on $[a, b]$ for this to be true. In general, you need absolute values.
10. F (but almost T): You need something like $f(x), g(x) \geq 0$ on $[a, b]$. To see why, think about $f(x) = x$ and $g(x) = -x$. Then rotating R around the x-axis gives the same solid as rotating the region under $y = x$ around the x-axis (a cone). Here when you rotate the top half R about the x-axis, it passes through the bottom part of R. So one point of view is that the issue is R is bigger than it needs to be to generate S.
11. F (but almost T): Actually, I meant to make this statement true, but I forgot to write down the assumption that $f(x) \geq g(x)$. If $f(x) \geq g(x)$, this is what you get from the method of cylindrical shells: we'll have cylinders of height $f(x) - g(x)$ ($|f(x) - g(x)|$ in general) and radius $x - a$. (Note since we rotate around $x - a$, there is no issue about R being too big as in the previous problem.)
12. You get $\frac{x^3}{3} + x\Big]_0^3 = 9 + 3 = \boxed{12.}$ Neat! Maybe every answer will from now on just be the problem number.
13. Let $u = 3x$ so $du = 3dx$, i.e., $dx = \frac{du}{3}$. Then the integral is

    $$\int_{u(0)}^{u(\pi/9)} \sin u \frac{du}{3} = -\frac{1}{3} \cos u\Big]_0^{\pi/3} = -\frac{1}{3}(\frac{1}{2} - 1) = \boxed{\tfrac{1}{6}.}$$

    Hmm... maybe not every answer will be the problem number.
14. Let $u = x^3$ so $du = 3x^2 dx$. Then the integral is

    $$\int \cos(u) \frac{du}{3} = \frac{1}{3} \sin u + C = \boxed{\tfrac{1}{3} \sin x^3 + C.}$$

15. Let $u = 2\sqrt{x} + 1$ so $du = x^{-1/2}dx$, i.e., $dx = \sqrt{x}du$. Note $\sqrt{x} = \frac{u-1}{2}$. Then our integral is

$$\int u^5 \sqrt{x}\, du = \int u^5 (\frac{u-1}{2})\, du = \frac{1}{2}\int (u^6 - u^5)\, du = \frac{1}{2}(\frac{u^7}{7} - \frac{u^6}{6}) + C.$$

Then we can substitute back (and, if we want, factor out a u^6) to get

$$\boxed{\tfrac{1}{2}(2\sqrt{x}+1)^6(\tfrac{1}{7}(2\sqrt{x}+1) - \tfrac{1}{6}) + C.}$$

16. This problem is neato-torpedo, because we can do it two ways to get two different answers which are both right!

Solution 1: Let $u = \tan x$ so $du = \sec^2 x dx$. Then our integral is

$$\int u\, du = \frac{u^2}{2} + C = \boxed{\tfrac{\tan^2 x}{2} + C.}$$

Solution 2: Let $u = \sec x$ so $du = \sec x \tan x dx$. Then our integral is

$$\int u\, du = \frac{u^2}{2} + C = \boxed{\tfrac{\sec^2 x}{2} + C.}$$

What's going on here? Note that

$$\sec^2 x - \tan^2 x = \frac{1 - \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} = 1,$$

in other words, $\tan^2 x$ and $\sec^2 x$ differ by a constant so the above expressions are really the same after appropriately modifying C. (This issue is one reason forgetting the $+C$ will lead to mental anguish—without the $+C$'s you get two different answers which are not equal.)

17. Rewrite it as

$$\int \frac{\cos x}{\sin x}\frac{1}{\sin x}\, dx = \int \cot x \csc x\, dx = \boxed{-\csc x + C}.$$

18. The area is

$$\int_0^2 (x^2 - 5)\, dx = \frac{x^3}{3} - 5x\Big]_0^2 = \frac{8}{3} - 10 = \boxed{-\tfrac{22}{3}.}$$

The area is negative because most of the region lies below the x-axis.

19. The right end points are $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ and $\frac{4}{4}$. So our Riemann sum is

$$(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1)\frac{1}{4} = \boxed{\tfrac{1}{8}(3 + \sqrt{2} + \sqrt{3})}.$$

20. The n-th Riemann sum, using right endpoints, is

$$R_n = \sum_{i=1}^{n} \underbrace{\left(\frac{2i}{n}\right)^2}_{\text{height}} \cdot \underbrace{\frac{2}{n}}_{\text{width}} = \frac{2}{n^3}(2^2 + 4^2 + \cdots + (2n)^2).$$

(Here the right endpoints are $\frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n}{n}$ so interval widths are all $\frac{2}{n}$.) Factoring out 2^2 and using the formula for the sum of the first n squares I gave you, we get

$$R_n = \frac{8}{n^3}(1^2 + 2^2 + \cdots n^2) = \frac{8n(n+1)(2n+1)}{6n^3} = \frac{4}{3}(2 + \frac{3}{n} + \frac{1}{n^2}) \to \boxed{\tfrac{8}{3}} \quad \text{as } n \to \infty.$$

(Note: it is probably a good idea to double check your answer with what you get using the FTC/power rule.)

21. The points of intersection will happen when $2x^2 - 7 = 5 - x^2$, i.e., when $3x^2 = 12$, i.e., when $x = \pm 2$. Let $f(x) = 5 - x^2$ and $g(x) = 2x^2 - 7$. Then $f(x) \geq g(x)$ on $[-2, 2]$, so the area is

$$\int_{-2}^{2} ((5-x^2)-(2x^2-7))\,dx = \int_{-2}^{2} (12-3x^2)\,dx = 2\int_0^2 (12-3x^2)\,dx = 2(12x-x^3)]_0^2 = 2(24-8) = \boxed{32.}$$

(I used symmetry in the middle, but of course you don't have to.)

22. Can I go home now? Haven't we done enough? No? Okay.

Note $\sqrt{x} = 2 - x$ when $x = 1$, so the region is the region under $y = \sqrt{x}$ from 0 to 1 together with the region under the curve $y = 2 - x$ from 1 to 2. Thus the area is

$$\int_0^1 \sqrt{x}\,dx + \int_1^2 (2-x)\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 + (2x - \frac{x^2}{2})\Big]_1^2 = \frac{2}{3} + (4-2) - (2 - \frac{1}{2}) = \boxed{\tfrac{7}{6}.}$$

Alternatively, you could integrate dy—if you do this, you don't need to break the region up into two integrals, since R is the region to the left of $x = 2 - y$ and to the right of $x = y^2$ for $0 \leq y \leq 1$. Thus you can also write the area as

$$\int_0^1 ((2-y)-y^2)\,dy = 2y - \frac{y^2}{2} - \frac{y^3}{3}\Big]_0^1 = (2 - \frac{1}{2} - \frac{1}{3}) = \boxed{\tfrac{7}{6}.}$$

23. We can do this by cross-sections (discs/washers) or cylindrical shells. Again, you can do this either way, but you should be able to solve it both ways, just in case the problem you get is much harder one way than the other.

Solution 1 (cross-sections): If we take horizontal cross-sections (with respect to y), we will get annuli (washers) with outer radius 3 and inner radius $x + 2 = y^2 + 2$. Thus the volume is

$$\int_0^1 \pi(3^2 - (y^2+2)^2)\,dy = \pi\int_0^1 (5 - 4y^2 - y^4)\,dy = (5y - \frac{4y^3}{3} - \frac{y^5}{5})\Big]_0^1 = \boxed{\tfrac{52\pi}{15}.}.$$

Solution 2 (cylindrical shells): If we approximate by cylinders, the cylinder about $x = -2$ through x has radius $x + 2$ and height $\sqrt{x}$. So our volume is

$$\int_0^1 2\pi(x+2)\sqrt{x}\,dx = 2\pi\int_0^1 (x^{3/2} + 2x^{1/2})\,dx = 2\pi(\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2})\Big]_0^1 = \boxed{\tfrac{52\pi}{15}.}$$

24. Again, you can do this problem either with cross-sections or cylindrical shells. Here R is the region below $y = x$ and above $y = x^3$ between $x = 0$ and $x = 1$. I think cylindrical shells is slightly easier here so I will do that. We use cylinders of radius x and the height $x - x^3$, so by our formula in class we see the volume is

$$\int_0^1 2\pi x(x-x^3)\,dx = 2\pi\int_0^2 (x^2 - x^4)\,dx = 2\pi(\frac{x^3}{3} - \frac{x^5}{5})\Big]_0^1 = \boxed{\tfrac{4\pi}{15}.}$$

25. Write $a(t)$ for acceleration and $v(t)$ for velocity. We know

$$v(t) = \int a(t)\,dt = \int \frac{3t}{2}\,dt = \frac{3t^2}{4} + C$$

for some C. Since the Wienermobile starts off at rest, $v(0) = 0$, which means $C = 0$. The average velocity is

$$\frac{1}{10}\int_0^{10} v(t)\,dt = \frac{1}{10}\int_0^{10} \frac{3t^2}{4} = \frac{1}{10}\cdot\frac{t^3}{4}\Big]_0^{10} = \frac{1000}{40} = \boxed{25 \text{ mph.}}$$

(Remember the average value of a function $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. But in this case, you should already know that average velocity is (distance traveled)/(time spent), so you could probably reason this out for yourself by realizing distance traveled is $\int_0^{10} v(t)\,dt$.)

Note: since units are given in the problem, you are expected to include units (in this case, mph) in your final answer.

If you're curious, note the speed of the Wienermobile after 10 seconds is $v(10) = \frac{3}{4} \cdot 10^2 = 75$ mph, so the average velocity over this period is considerably less than half the final velocity on this interval. To make sense of this, draw a graph of $v(t)$.