3 A summary of local representation theory for $\mathrm{GL}(2)$

All representations from now on will be on a complex vector space. Some references are Gelbart (Automorphic forms on adele groups), Bump (Automorphic forms and representations) and Goldfeld–Hundley (Automorphic representations and L-functions for the general linear group, Vol. I).

3.1 The p-adic case

Fix a prime p. We put $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $Z = \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} \right\}$ the center, A the diagonal subgroup, $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$ the standard unipotent, $B = AN$ the standard Borel and $K = \mathrm{GL}_2(\mathbb{Z}_p)$ the standard maximal compact.

Recall that a representation (π, V) of G is called **admissible** if (i) π is smooth, and (ii) for each compact open subgroup K' of G, the set of K'-fixed vectors,

$$V^{K'} = \left\{ v \in V : \pi(k)v = v \text{ for all } k \in K' \right\},$$

is finite dimensional. The local component of an automorphic representation is admissible, so these are the representations we are interested in classifying.

In fact, any smooth irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ is admissible, so the class of irreducible smooth representations is the same as the class of irreducible admissible representations, but this was not known at the time of the classification of the latter set and is inherently needed in the theory we recall below.

3.1.1 Finite-dimensional representations

The finite-dimensional representations are relatively easy: any irreducible smooth finite-dimensional representation of G is 1-dimensional (Schur's lemma), and these are all of the form $g \mapsto \chi(\det g)$ where χ is a smooth 1-dimensional representations, i.e., characters, of $\mathbb{Q}_p^\times$.

The characters of $\mathbb{Q}_p^\times$ can be described as follows. (See, e.g., Paul Sally's article "An introduction to p-adic fields, harmonic analysis and the representation theory of SL_2.")

We can write any $x \in \mathbb{Q}_p^\times$ uniquely as $p^n u$ where $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_\mathfrak{p}^\times$ is a unit. This gives an isomorphism $\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}_p^\times$. The characters of $\mathbb{Z}$ are just given by $n \mapsto e^{sn}$ for $s \in \mathbb{C}$, which for our purposes we will rewrite in the form $p^{-ns'}$ where $s' = -s/\ln p$. Hence we can write any character χ of $\mathbb{Q}_p^\times$ as

$$\chi(x) = p^{-ns}\omega(u) = |x|^s \omega(u), \quad (x = p^n u,\, u \in \mathbb{Z}_p^\times)$$

for some $s \in \mathbb{C}$ and ω a character of $\mathbb{Z}_p^\times$.

Any character ω of $\mathbb{Z}_p^\times$ is unitary (has image in S^1). By smoothness (in fact continuity), ω has some higher unit group $\mathbb{Z}_p^{(n)} = 1 + p^n\mathbb{Z}_p$ $(n > 0)$ or $\mathbb{Z}_p^{(0)} = \mathbb{Z}_p^\times = \mathrm{GL}_1(\mathbb{Z}_p)$ in its kernel. Note these subgroups $\mathbb{Z}_p^{(n)}$ for $n \geq 0$ are open compact subgroups of $\mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$, i.e., they are analogous to the family of compact open subgroups K_n of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ we defined earlier. The quotient $\mathbb{Z}_p^\times/\mathbb{Z}_p^{(n)}$ is a finite abelian group, specifically $\mathbb{Z}_p^\times/\mathbb{Z}_p^{(n)} \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$, which has order $p^{n-1}(p-1)$ if $n \geq 1$ (and order 1 if $n = 0$). Hence ω may be viewed as a character of some finite abelian group $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

We say χ has **conductor** $c(\chi) = n$ if n is minimal such that $\mathbb{Z}_p^{(n)}$ is contained in the kernel of ω (or equivalently, of χ). If $c(\chi) = 0$, i.e., $\omega = 1$, we say χ is **unramified**; otherwise χ is **ramified**.

This means the only unramified characters of $\mathbb{Q}_p^\times$ are $|\cdot|_p^s$, which is unitary if and only if s is purely imaginary.

Further for a given conductor n, there are only finitely many possibilities for ω; to be precise $p-2$ possibilities if $n = 1$ and $p^{n-2}(p-1)^2$ if $n > 1$. Again, $\chi(x) = |x|_p^s \omega(u)$ is unitary if and only if $\mathrm{Re}(s) = 1$.

3.1.2 Principal series representations

Let ω_1 and ω_2 be two normalized unitary characters of $\mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ and $s_1, s_2 \in \mathbb{C}$. Then one can consider the characters χ_1 and χ_2 of $\mathbb{Q}_p^\times$ given by

$$\chi_i(x) = \omega_i(x)|x|_p^{s_i}.$$

Consequently, $\chi = (\chi_1, \chi_2)$ extends to a character of the Borel B by

$$\chi \left[\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right] = \chi_1(a)\chi_2(b).$$

We define the **normalized parabolic induction** of χ to be

$$V(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \text{ smooth} \mid f \left[\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right] = \chi_1(a)\chi_2(b) \left| \frac{a}{b} \right|^{\frac{1}{2}} f(g) \right\}.$$

Note Goldfeld and Hundley work with the non-normalized induction

$$\mathcal{V}_{nn}(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \text{ smooth} \mid f \left[\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right] = \chi_1(a)\chi_2(b) f(g) \right\}.$$

It is clear one can go between the two via

$$V(\chi_1, \chi_2) = \mathcal{V}_{nn}(\chi_1 |\cdot|^{1/2}, \chi_2 |\cdot|^{-1/2}).$$

The normalization factor of $|a/b|^{1/2}$ makes relations among and conditions on these representations, as we will note below. Therefore, we will work with the normalized induction from now on, which is standard.

We call $V(\chi_1, \chi_2)$ the **principal series representation** of G induced from (χ_1, χ_2). Here the action of G on $V(\chi_1, \chi_2)$ is given by right translation, i.e.,

$$g \cdot f(x) = f(xg), \quad \text{for } g, x \in G,\ f \in V(\chi_1, \chi_2).$$

For example, one has

Lemma 3.1.1. *The contragredient of $V(\chi_1, \chi_2)$, denoted $\check{V}(\chi_1, \chi_2)$ or $\widetilde{V}(\chi_1, \chi_2)$ is equivalent to $V(\chi_1^{-1}, \chi_2^{-1})$.*

Recall the following

Theorem 3.1.2. *The principal series $V(\chi_1, \chi_2)$ is admissible. It is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|_p^{\pm 1}$.*

If $\chi_1\chi_2^{-1} = |\cdot|_p$, then $V(\chi_1, \chi_2)$ contains an irreducible admissible subspace of codimension 1, called a **special representation**.

If $\chi_1\chi_2^{-1} = |\cdot|_p^{-1}$, then $V(\chi_1, \chi_2)$ contains an invariant 1-dimensional subspace whose quotient is irreducible. This quotient is also called a **special representation**.

(If one works with non-normalized induction for the principal series, the above conditions on $\chi_1\chi_2^{-1}$ become $\chi_1\chi_2^{-1} = |\cdot|_p^2$ and $\chi_1\chi_2^{-1} = 1$.)

Definition 3.1.3. *If $V(\chi_1, \chi_2)$ is irreducible, we write $\pi(\chi_1, \chi_2) = V(\chi_1, \chi_2)$. If $V(\chi_1, \chi_2)$, we denote by $\pi(\chi_1, \chi_2)$ the corresponding special representation.*

Note we can write a special representation in the form $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2})$ for an arbitrary character χ of $\mathbb{Q}_p^\times$. When $\chi = 1$, we call this the **Steinberg representation** St. Then one can identify $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2})$ with a *twisted* Steinberg representation $St \otimes \chi$.

In general, for any representation (π, V) of G and a character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, one can form the twist $(\pi \otimes \chi, V)$ where the action is given by

$$(\pi \otimes \chi)(g)v = \chi(\det g)\pi(g)v.$$

Hence one has that all special representations are obtained as twists of a single one, the Steinberg. Similarly, one has the relation $\pi(\chi_1, \chi_2) \otimes \chi) \sim \pi(\chi_1\chi, \chi_2\chi)$ for twists of principal series, where $\sim$ denotes equivalence.

Theorem 3.1.4. *The irreducible admissible representations $\pi(\chi_1, \chi_2)$ and $\pi(\mu_1, \mu_2)$ (principal series or special) are equivalent if and only if χ_1 and χ_2 equal, in some order, μ_1 and μ_2.*

(This is another statement which is made much nicer by working with normalized induction for the principal series.)

3.1.3 Supercuspidal representations

So now we know three kinds of irreducible admissible representations of G: the 1-dimensionals, the irreducible principal series, and the special representations. There is one more kind: supercuspidal.

To motivate the definition, let us try to imagine proving that all infinite irreducible admissible representations are principal series or special. Let (π, V) be an infinite irreducible admissible representation of G, and consider the subspace

$$V_N = \langle \pi(n)v - v | n \in N,\, v \in V \rangle.$$

It is not hard to see that V_N is invariant under the diagonal subgroup A (in fact, under the Borel). One can then consider the action of B on the quotient

$$V^N = V/V_N,$$

called the **Jacquet module** of V. One can show the Jacquet module is an admissible representation of A, whose dimension is at most 2.

If (π, V) is an irreducible principal series $\pi(\chi_1, \chi_2)$, then the Jacquet module is essentially (χ_1, χ_2). Conversely, whenever the Jacquet module is 2-dimensional, V is a principal series.

If (π, V) is a special representation $\pi(\chi \cdot |\cdot|_p, \chi)$, then the Jacquet module is 1-dimensional and gives back χ. Conversely, whenever the Jacquet module is 1-dimensional, V is a special representation.

There is a third, sneaky possibility—the Jacquet module is *zero*-dimensional!

Definition 3.1.5. *We say an infinite-dimensional irreducible admissible representation* (π, V) *of* G *is* **supercuspidal** *if the Jacquet module* V^N *is 0-dimensional, i.e., if* $V_N = V$.

Exercise 3.1.6. *Let* (π, V) *be a* 1*-dimensional representation of* G. *One can still define the Jacquet module* V^N *as above. Show* V^N *is* 0*-dimensional.*

The Jacquet module, in some sense, gives us the classification of irreducible admissible representations of G (1-dimensional, principal series, special, supercuspidal)—however it may seem unsatisfactory as the supercuspidal guys are essentially *defined* to be the things that aren't one of the types we already know!

The first question to ask would be, do supercuspidal representations exist? The answer is yes, and constructions are known but the theory is more complicated than for principal series. Roughly the idea is that one can induce an irreducible representation of some compact open subgroup K' of G (here one uses "compact induction.") The simplest case comes from taking irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ and lifting them to K via the projection

$$K = \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$$

induced by the isomorphism $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$. These are known as *depth 0* supercuspidal representations.

However, even without knowing the construction of supercuspidal representations (which was not complete at the time of the classification), supercuspidal representations can be shown to have several nice properties (indeed, the classification is not used to show this). For instance, one can put an inner product $\langle \cdot , \cdot \rangle$ on V. Recall a matrix coefficient of (π, V) is a function $f : G \to \mathbb{C}$ given by

$$f(g) = \langle \pi(g)v, v' \rangle$$

for $v, v' \in V$. For supercuspidal π (but not principal series or special representations), the matrix coefficients f have compact support. Further, they are what Harish-Chandra called supercusp forms, i.e.,

$$\int_N f(g_1 n g_2) dn = 0$$

for all $g_1, g_2 \in G$. These turn out to be particularly useful facts, allowing one to prove many things for supercuspidal representations that are not so easy to prove for principal series or special representations.

One should think of supercuspidal representations as the representations that are actually native to GL(2), whereas the principal series and special representations (and 1-dimensionals) all come from representations of GL(1). (Even though supercuspidals can be constructed by induction from subgroups like $K = \mathrm{GL}_2(\mathbb{Z}_p)$, this is still GL(2), just over $\mathbb{Z}_p$ instead of $\mathbb{Q}_p$.)

We remark that this notion still holds when one works with representations of other groups, such as $\mathrm{GL}_n(\mathbb{Q}_p)$: there are "native" representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ which are supercuspidal. Roughly, other representations can be constructed by inducing a representation $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$ of a parabolic $P = MN$, where the Levi subgroup

$$M \simeq \mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \mathrm{GL}_{n_2}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{Q}_p)$$

with $n_1 + n_2 + \cdots + n_k = n$ and ρ_i being a supercuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ (here by a supercuspidal of $\mathrm{GL}_1(\mathbb{Q}_p)$ we just mean a character of $\mathbb{Q}_p^\times$). (To be precise, one should perhaps allow ρ_i to be a *discrete series* representations—which for $\mathrm{GL}_2(\mathbb{Q}_p)$, means supercuspidal or special.)

3.1.4 Classification

Here we summarize the classification. Write $\pi_1 \sim \pi_2$ for π_1 and π_2 being equivalent.

Theorem 3.1.7. *Let π be an irreducible admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Then π is one of the following disjoint types, where χ, χ_1 and χ_2 are arbitrary characters of $\mathbb{Q}_p^\times$.*

(i) irreducible principal series $\pi(\chi_1, \chi_2)$, i.e., $\chi_1\chi_2^{-1} \neq |\cdot|_p^{\pm}$; we have $\pi(\chi_1, \chi_2) \sim \pi(\chi_2, \chi_1)$ and no other equivalences;

(ii) a special representation, which we may write in the form $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2}) = St \otimes \chi$, and $St \otimes \chi \sim St \otimes \chi' \iff \chi = \chi'$;

(iii) a supercuspidal representation;

(iv) 1-dimensional, of the form $\chi \circ \det$.

Recall the **central character** ω_π of π is the character of $Z \simeq \mathbb{Q}_p^\times$ satisfying

$$\pi(zg) = \omega(z)\pi(g), \quad z \in Z,\, g \in G.$$

Because the 1-dimensional representations will not arise as local components of global automorphic representations, we will exclude them in the discussion which follows. One often works with representations of $\mathrm{PGL}_2(\mathbb{Q}_p) = G/Z$. It is easy to see that the irreducible admissible representations of $\mathrm{PGL}_2(\mathbb{Q}_p)$ are same as representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character. We remark that for $\pi = \pi(\chi_1, \chi_2)$ (irreducible principal series or special), $\omega_\pi = \chi_1\chi_2$.

Exercise 3.1.8. *(a) Check that for any representation of G, $\omega_{\pi\otimes\chi} = \chi^2\omega_\pi$.*

(b) Deduce the following corollary.

Corollary 3.1.9. *The irreducible admissible representations of $\mathrm{PGL}_2(\mathbb{Q}_p)$ are of one of the following types*

(i) irreducible principal series $\pi(\chi, \chi^{-1})$ where $\chi \neq |\cdot|_p^{\pm 1/2}$ is an arbitrary character of $\mathbb{Q}_p^\times$;

(ii) a quadratic twist *of Steinberg: $St \otimes \chi$ where $\chi^2 = 1$;*

(iii) a supercuspidal representation of G with trivial central character;

(iv) 1-dimensional, of the form $\chi \circ \det$ where $\chi^2 = 1$.

There is some further classification one can do. For instance, one can consider which representations are unitary.

Definition 3.1.10. *Let (π, V) be an admissible representation of G. Then π is* **unitary** *(or* **unitarizable***) if there exists a positive-definite invariant Hermitian form on V, i.e., there is a positive-definite Hermitian form $(\,,\,)$ on V such that*

$$(\pi(g)v, \pi(g)w) = (v, w) \quad \textit{for all } g \in G,\, v, w \in V.$$

The above definition also makes sense for representations of $\mathbb{Q}_p^\times$.

Lemma 3.1.11. *A character χ of $\mathbb{Q}_p^\times$ is unitary if and only it is of the form $\omega|\cdot|_p^{ir}$ where $r \in \mathbb{R}$ and ω is a finite order character.*

Proof. First observe a 1-dimensional representation χ is unitary if and only if its image lies in S^1: for $z, w \in \mathbb{C}$, $x \in \mathbb{Q}_p^\times$ and $(\ ,\ )$ a Hermitian form on $V = \mathbb{C}$, we have

$$(\chi(x)z, \chi(x)w) = \chi(x)(z, \chi(x)w) = \chi(x)\overline{\chi(x)}(z, w) = |\chi(x)|^2(z, w).$$

Now by the classification of characters of $\mathbb{Q}_p^\times$ given in Section 3.1.1, we can write $\chi = \omega|\cdot|_p^s$ for some ω of finite order and $s \in \mathbb{C}$. Since ω is finite order, it has image in S^1. Therefore χ is unitary if and only if $|\cdot|_p^s$ has image in S^1, which is equivalent to s being purely imaginary. □

Theorem 3.1.12. *Let π be an irreducible admissible representation of G. Then π is unitary if and only if π is one of the following types:*

(i-a) (continuous series) an irreducible principal series $\pi(\chi_1, \chi_2)$ where χ_1, χ_2 are both unitary;

(i-b) (complementary series) an irreducible principal series $\pi(\chi, \overline{\chi}^{-1})$ where $\chi = |\cdot|_p^\sigma$, $0 < \sigma < 1$;

(ii) a special representation with unitary central character; or

(iii) a supercuspidal representation with unitary central character.

Conjecturally, only types (i-a), (ii) and (iii) should occur as local components of automorphic representations. We will say more about this when we move to the global theory.

3.1.5 Conductors

To each type of representation (i)–(iii) in the above theorem is associated some data which is used in connection with the study of modular and automorphic forms.

First we discuss ramification. For $n \geq 0$, let

$$K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in p^n\mathbb{Z}_p \right\}.$$

In particular $K(0) = K$. This is a local (p-adic) analogue of the congruence subgroup $\Gamma_0(N)$, where n is the largest power of p such that $p^n|N$.

Definition 3.1.13. *Let (π, V) be an infinite-dimensional irreducible admissible representation of G. Let $n \geq 0$ be minimal such that $V^{K(n)} \neq \{0\}$. We say the* **conductor** *of π is $c(\pi) = n$. If $c(\pi) = 0$, we say π is* **unramified***; otherwise it is* **ramified**.

The conductor is always finite, and an important fact is that $V^{K(c(\pi))}$ is 1-dimensional. A vector in $V^{K(c(\pi))}$ is called a **new vector** or **new form**, and is analogous to the notion of new forms in the sense of modular forms.

Theorem 3.1.14. *(i) For an irreducible principal series $\pi = \pi(\chi_1, \chi_2)$, $c(\pi) = c(\chi_1) + c(\chi_2)$.*

(ii) For a special representation $St \otimes \chi$, $c(\pi) = 1$ if χ is unramified; otherwise $c(\pi) = 2c(\chi)$.

(iii) If π is supercuspidal, then $c(\pi) \geq 2$.

Corollary 3.1.15. *Let π be an infinite-dimensional irreducible admissible representation. Then π is unramified if and only if π is an* unramified principal series*, i.e., an irreducible principal series $\pi(\chi_1, \chi_2)$ with both χ_1 and χ_2 unramified.*

One reason to understand this is the following: if $f \in S_k(N)$ is a new form, then it gives rise to an irreducible admissible infinite-dimensional local representation π_p for each p. To apply representation theory to modular forms, one wants to understand the representations π_p in the sense of our classification above. For each p, the conductor $c(\pi_p) = n_p$ where n_p is the largest power of p such that $p^{n_p}|N$.

In particular, if $N = 1$, then π_p is an unramified principal series for all p. If $N = p_1 \cdots p_k$ where all the p_j's are distinct, then π_p is either Steinberg or ramified principal series (with conductor 1) for any $p = p_j$, and π_p is an unramified principal series for all other p.

3.1.6 L- and ϵ- factors

To the infinite-dimensional irreducible admissible representations π of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ one can associate certain functions called local L- and ϵ- factors. When patched together these will give global L- and ϵ- factors attached to automorphic representations.

One way to construct the L-factors is as follows. Suppose π has a Kirillov model $\mathcal{K}$. Then for $\phi \in \mathcal{K}$, one can define the **zeta integral**

$$Z(s, \phi) = \int_{\mathbb{Q}_p^\times} \phi(y)|y|_p^{s-1/2} d^\times y.$$

Then the L-factor $L(s, \pi)$ should be defined so it is the "gcd" of the local zeta functions $Z(s, \phi)$ as ϕ ranges over $\mathcal{K}$. More precisely, for each ϕ, there is a polynomial h_ϕ such that $Z(s, \phi) = h_\phi(p^{-s})L(s, \pi)$. In fact, for some ϕ, $h_\phi = 1$. Put another way, for a well chosen ϕ, we have $L(s, \pi) = Z(s, \pi)$.

This is carried out for $\mathrm{GL}_1(\mathbb{Q}_p)$, i.e., for characters of $\mathbb{Q}_p^\times$, in Tate's thesis.

We remark these zeta integrals are analogous to the construction of the completed L-function $\Lambda(s, f)$ of a modular form f via the *Mellin transform*:

$$\Lambda(s, f) = \int_0^\infty f(iy)|y|^s d^\times y.$$

The fact that one needs choose an appropriate $\phi \in \mathcal{K}$ to get the L-function from the zeta integral is analogous to the fact that in the above Mellin transform definition of $\Lambda(s, f)$, one needs to choose f to be, say, a normalized Hecke eigen cusp form to define a nice L-function with an Euler product.

Definition 3.1.16. *For certain irreducible admissible representations π of G, we define the* **local L-factor** $L(s, \pi)$ *as follows.*

(i) For an irreducible principal series $\pi = \pi(\chi_1, \chi_2)$, we set

$$L(s, \pi) = \frac{1}{(1 - \alpha_1 p^{-s})(1 - \alpha_2 p^{-s})}$$

where $\alpha_i = \chi_i(p)$ if χ_i unramified and $\alpha_i = 0$ if χ_i is ramified.

(ii) For a special representation $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2}) = St \otimes \chi$, we set

$$L(s, \pi) = \frac{1}{1 - \alpha p^{-s}}$$

where $\alpha = \chi(p)|p|_p^{1/2} = p^{-1/2}\chi(p)$ if $\chi|\cdot|_p^{1/2}$ is unramified, and $\alpha = 0$ else.

(iii) For π supercuspdial, we set

$$L(s, \pi) = 1.$$

For simplicity, we only define ϵ-factors for $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Definition 3.1.17. *Let ψ be the standard additive character of $\mathbb{Q}_p$, and π be an irreducible admissible representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The* **local ϵ-factor** *$\epsilon(s,\pi,\psi)$ attached to π is*

$$\epsilon(s,\pi,\psi) = \epsilon p^{c(\pi)(1/2-s)}$$

where $\epsilon = \pm 1$. Specifically

(i) if $\pi = \pi(\chi,\chi^{-1})$ is an irreducible principal series, then $\epsilon = \chi(-1)$.

(ii-a) if $\pi = St$, then $\epsilon = -1$.

(ii-b) if $\pi = St \otimes \chi$, χ nontrivial quadratic, then $\epsilon = \chi(-1)$.

3.2 The real case

Now let $G = \mathrm{GL}_2(\mathbb{R})$. Of course, there are no compact open subgroups of G, but a maximal compact subgroup is the orthogonal group $K = \mathrm{O}(2)$. As in the p-adic case, we let $B = AN$ where A is the diagonal subgroup of G and $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \subset G$.

The representation theory for $\mathrm{GL}_2(\mathbb{R})$ (which of course was historically studied before that for $\mathrm{GL}_2(\mathbb{Q}_p)$) largely parallels the representation theory for $\mathrm{GL}_2(\mathbb{Q}_p)$, but the details are quite different due to the very different topologies on these groups. In fact, many problems turn out to be much easier for real groups, whereas others turn out to be much easier for p-adic groups. Nevertheless, Harish-Chandra—who developed much general theory over the reals and p-adics in the 1950's and 1960's—described a philosophy which he called the "Lefschetz principle:" whatever is true for real groups is also be true for p-adic groups, and one should be able to treat them equally. I.e., even though the details are quite different, one should be able to put the theories for $G(\mathbb{Q}_p)$ and $G(\mathbb{R})$ inside a single framework.

In any case, it is not one of our goals to discuss the representation theory for $\mathrm{GL}_2(\mathbb{R})$ in detail. We simply give a summary of facts.

Let (π, V) be a (smooth) representation of G on a Hilbert space V. First we should define admissibility.

In the p-adic case, we defined admissible as the condition that $V^{K'}$ is finite dimensional for any compact open subgroup. Here, we don't have compact open subgroups to work with. Another way to state the p-adic condition is that the restriction $\pi_{K'}$ of π to K' only contains the trivial representation finitely many times. In particular, this means the restriction of π to $\mathrm{GL}_2(\mathbb{Z}_p)$ contains any finite order character $\chi \circ \det$ of $\mathrm{GL}_2(\mathbb{Z}_p)$ at most finitely many times (to see this, restrict further to a compact open subgroup on which $\chi \circ \det$ is trivial).

While one can define admissibility for $G = \mathrm{GL}_2(\mathbb{R})$ in terms of K, it is perhaps simpler to think of it in terms of the compact subgroup

$$K_0 = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < \pi \right\},$$

which has index 2 in $K = \mathrm{O}(2)$. Since $\mathrm{SO}(2)$ is compact and abelian, all its irreducible representations are characters.

Exercise 3.2.1. *Show any continuous character of* $\mathrm{SO}(2)$ *is of the form*

$$\chi_k\left[\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\right] = e^{2\pi ik\theta}$$

for some $k \in \mathbb{Z}$*.*

Further since $\mathrm{SO}(2)$ is compact, any representation of $\mathrm{SO}(2)$ is semisimple, i.e., decomposes as a direct sum of irreducible representations.

Definition 3.2.2. *We say* π *is* **admissible** *if, for any* $k \in \mathbb{Z}$*, the restriction* π_{K_0} *of* π *to* K_0 *contains* χ_k *with finite multiplicity.*

We first state what the classification looks like. Then we will briefly and informally discuss each type of representation.

Theorem 3.2.3. *Let* π *be an irreducible admissible unitary representation of* $\mathrm{PGL}_2(\mathbb{R})$*. Then* k *is one of the following types*

(i) irreducible principal series;
(ii) a weight k *discrete series* D_k *for* $k \geq 2$*;*
(iii) a limit of discrete series D_1^**; or*
(iv) finite-dimensional.

One defines principal series as in the p-adic case: begin with two characters χ_1 and χ_2 of $\mathbb{R}$ to give a character of A; extend this to a character of B and induce to G. This is usually irreducible. When it is not, it gives one of the discrete series representations D_k or the limit D_1^*. Thus the representations of types (ii) and (iii) are analogous to the special representations in the p-adic case. (There are no supercuspidal representations in the real case.) The reason we separate out D_1^* is that it is not quite as nice as the D_k's.

To be a little more precise, there are discrete series of the same weight with different central characters, so one should write something like $D_{k,\omega}$, where ω is the central character. If we fix a central character ω, then the parity of k must be compatible with the central character and there is a unique discrete series $D_{k,\omega}$ of weight k. Specifically, if ω is even ($\omega(-1) = 1$) then k must be even, and if ω is odd ($\omega(-1) = -1$), then k must be odd.

For simplicity, we describe the discrete series when $\omega = 1$ (so k must be even). Let $\mathcal{H}$ denote the upper half plane. Let V_k be the space of holomorphic square-integrable functions on $\mathcal{H}$ (w.r.t. the hyperbolic measure on $\mathcal{H}$), i.e.,

$$V_k = \left\{ f : \mathcal{H} \to \mathbb{C} \text{ holomorphic} \;\middle|\; \int_{\mathcal{H}} |f(x+iy)|^2 y^k \frac{dx\,dy}{y^2} < \infty \right\}.$$

The discrete series of weight k is the representation (D_k, V_k) given by

$$(D_k(g)f)(z) = \frac{(\det g)^{k/2}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right) \quad \text{where} \quad g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+.$$

This statement technically needs $g \in \mathrm{GL}_2(\mathbb{R})^+$ (matrices of positive determinant) to make sense as $\mathcal{H}$ is not preserved by the full action of $\mathrm{GL}_2(\mathbb{R})$, but one can extend the above definition to all $g \in \mathrm{GL}_2(\mathbb{R})$ without much difficulty. Observe that when we restrict the discrete series to the subgroup $\mathrm{GL}_2(\mathbb{Z})$, the invariant one dimensional subspaces are the modular forms of weight k. One can show the restriction $D_k|_K$ of the discrete series D_k to $K = \mathrm{SO}(2)$ decomposes as

$$D_k|_K \simeq \bigoplus_{\substack{|j| \geq k \\ j \equiv k \bmod 2}} \chi_j.$$

Finally, we remark that in the real case, irreducible finite-dimensional does not mean 1-dimensional—the standard representation $\rho : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{C})$ is the simplest example. One also has the symmetric powers of the standard representation

$$\mathrm{Sym}^n(\rho) : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_{n+1}(\mathbb{C}).$$

However all irreducible finite-dimensional representations are of the form $\mathrm{Sym}^n(\rho) \otimes (\chi \circ \det)$, where χ is a character of $\mathbb{R}^\times$. (Here $\mathrm{Sym}^1(\rho) = \rho$ and $\mathrm{Sym}^0(\rho)$ is trivial.)

One last thing that we should mention is that one often works with something slightly more general than honest representations of $\mathrm{GL}_2(\mathbb{R})$. Namely, one often works with what are called $(\mathfrak{g}, K)$-*modules*, which are compatible pairs of representations of the Lie algebra $\mathfrak{g}$ of G and of the maximal compact subgroup K of G. Given a unitary representation of $\mathrm{GL}_2(\mathbb{R})$, one naturally gets a pair of representations on $\mathfrak{g}$ and K, which form a $(\mathfrak{g}, K)$-module, though not all $(\mathfrak{g}, K)$-modules are obtained in this way. However the classification for $(\mathfrak{g}, K)$-modules looks the same as the classification for representations of G. Whether one uses $(\mathfrak{g}, K)$-modules or actual representations of G depends upon the model one chooses for automorphic forms. We'll say a little bit about this in the next section.