Automorphic Representations
Fall 2011 Notes

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Preface

This notes are the sequel to my Modular Forms notes from Spring 2011.

Recall the basic setup. The upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with the hyperbolic metric has as its orientation-preserving isometry group $\mathrm{PSL}_2(\mathbb{R})$, acting by linear fractional transformations. Let Γ be a congruence subgroup of $\mathrm{PSL}_2(\mathbb{R})$, such as $\mathrm{PSL}_2(\mathbb{Z})$. Then a modular form f of weight k for Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which satisfies the transformation law

$$f\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right] = (cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ z \in \mathfrak{H}.$$

(It must also satisfy a holomorphy condition at the cusps.)

Since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathfrak{H}$ and the stabilizer of the point $i \in \mathfrak{H}$ is

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\},$$

we may identify $\mathfrak{H}$ with the quotient $\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}(2)$ via the map $\mathrm{PSL}_2(\mathbb{R}) \to \mathfrak{H}$ given by

$$g \mapsto g \cdot i.$$

Therefore, we can think of f on $\mathrm{PSL}_2(\mathbb{R})$ which is right-invariant by $\mathrm{SO}(2)$ and has a left-transformation property by Γ.

Consequently, the function $\phi_f : \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ given by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cz+d)^{-k} f \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is left-invariant under Γ, i.e., ϕ_f is a function of $\Gamma \backslash \mathrm{PSL}_2(\mathbb{R})$. We call ϕ_f a *(classical) automorphic form*. Now ϕ_f is not right-invariant by $\mathrm{SO}(2)$, but one can check that it has a right-transformation property by $\mathrm{SO}(2)$. So the passage from modular forms to classical modular forms trades right-$\mathrm{SO}(2)$-invariance with left-Γ-invariance, as well as the (left) Γ transformation law for a (right) $\mathrm{SO}(2)$ transformation law.

For simplicity, say $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Then one can reformulate things adelically because

$$Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})/K \simeq \Gamma \backslash \mathrm{PSL}_2(\mathbb{R}).$$

Here Z denotes the center of $\mathrm{GL}(2)$, the *adèles* $\mathbb{A}$ are a certain subring of $\mathbb{R} \times \prod_p \mathbb{Q}_p$, and K is a certain nice subgroup of $\mathrm{GL}_2(\mathbb{A})$. (One could work with $\mathrm{PSL}_2(\mathbb{A})$ or $\mathrm{PGL}_2(\mathbb{A})$ instead, but it will be most convenient to work with $\mathrm{GL}_2(\mathbb{A})$.)

This means we can lift ϕ_f to a function of $\mathrm{GL}_2(\mathbb{A})$, where it is now called an *adelic automorphic form.* In fact, $\phi_f \in L^2(Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$. Now for any group G and subgroup Γ, G acts on $L^2(\Gamma\backslash G)$ by right translation, i.e.,

$$R(g)\phi(x) = \phi(xg), \quad x, g \in G.$$

Here R is called the *right regular representation* of G on $L^2(\Gamma\backslash G)$.

Consequently, ϕ_f generates a representation π of $\mathrm{GL}_2(\mathbb{A})$ on a subspace V of $L^2(Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$. Namely

$$V = \langle R(g)\phi_f \rangle$$

$$\pi(g)\phi = R(g)\phi \in V, \quad g \in G, \, \phi \in V.$$

In other words, π is the restriction of the right regular representation R to the space V spanned by the translates of ϕ_f.

While all of this may seem excessively complicated, there are two big advantages of this approach. First, it allows one to unify the notions of (elliptic) modular forms, Hilbert modular forms, Siegel modular forms, and various other generalizations, by viewing them as automorphic form or representations on $G(\mathbb{A})$ for an appropriate group G.

Secondly, the representation π has a decomposition $\pi = \otimes_p \pi_p \otimes \pi_\infty$ where each π_p is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and π_∞ is a representation of $\mathrm{GL}_2(\mathbb{R})$. Consequently, this allows us to study modular forms via the representation theory of GL_2.

This subject has two parts: the local theory (e.g., the study of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$) and the global theory (the study of automorphic forms and representations on $\mathrm{GL}_2(\mathbb{A})$). While it is the global theory that has the direct applications to number theory, it is the local theory that carries the arithmetic information. Namely, the Fourier coefficients a_p for a modular form f are encoded in the local components π_p of the corresponding global automorphic representation π.

See my note "A brief over of modular and automorphic forms" for a somewhat broader and more detailed introduction than the above.

One of the main goals of the course is to understand, at least roughly, how the passage from modular forms to automorphic representations looks—in particular, given f, know what the local components π_p are. Consequently, most of the course will be devoted to understanding the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Even so, we will not be able to prove everything, but will have to take some results for granted.

In the last part of the course, we will sketch the global theory and discuss applications of automorphic representations.

Let me emphasize that this is a very rich field, where many branches of mathematics come together. On the one hand, this is what makes it so interesting, but on the other, it means it requires a large amount of background material to understand. Fortunately, there are many problems you can work on with knowledge of only a small portion of this field. Therefore, approach this subject with much patience and good humour. Understand a little bit at a time. I hope that at the end, you will have some rough idea of the subject, and be able to start reading some of the literature on your own.

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1 Background Material

First we will review some basic information about p-adic fields. This is a modified version of the corresponding section from my Number Theory II notes. Second we will review some basics of representation theory, with a focus on representations of finite groups.

There are many places to read about both of these topics. For example, Svetlana Katok's book *p-adic Analysis Compared with Real* for the p-adic numbers. For representation theory of finite groups, the classic is Serre's *Linear Representations of Finite Groups*. Other recommendations are the first part of Fulton and Harris's *Representation Theory*, and James and Liebeck's *Representations and Characters of Groups*. Fulton and Harris also includes the representation theory of $\mathrm{GL}_2(\mathbb{F}_p)$, which is an illuminating analogue to what we will study.

1.1 p-adic fields

If R is an integral domain, a map $|\cdot| : R \to \mathbb{R}$ which satisfies

(i) $|x| \geq 0$ with equality if and only if $x = 0$,
(ii) $|xy| = |x||y|$, and
(iii) $|x + y| \leq |x| + |y|$

is called an **absolute value** on R. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent on R if $|\cdot|_2 = |\cdot|_1^c$ for some $c > 0$. If we have an absolute value $|\cdot|$ on R, by (ii), we know $|1 \cdot 1| = |1| = 1$. Similarly, we know $|-1|^2 = |1| = 1$, and therefore $|-x| = |x|$ for all $x \in R$.

Now a absolute value $|\cdot|$ on R makes R into a metric space with distance $d(x, y) = |x-y|$. (The fact $|-x| = |x|$ guarantees $|y-x| = |x-y|$ so the metric is symmetric, and (iii) gives the triangle inequality.) Recall that any metric space is naturally embued with a topology. Namely, a basis of open (resp. closed) neighborhoods around any point $x \in R$ is given by the set of open (resp. closed) balls $B_r(x) = \{y \in R : d(x, y) = |x - y| < r\}$ (resp. $\overline{B}_r(x) = \{y \in R : d(x, y) = |x - y| \leq r\}$) centered at x with radius $r \in \mathbb{R}$.

Ostrowski's Theorem says, that up to equivalence, every absolute value on $\mathbb{Q}$ is of one of the following types:

$|\cdot|_0$, the trivial absolute value, which is 1 on any non-zero element
$|\cdot|_\infty$, the usual absolute value on $\mathbb{R}$
$|\cdot|_p$, the **p-adic absolute value**, defined below, for any prime p.

Here the p-adic absolute value defined on $\mathbb{Q}$ is given by

$$|x| = p^{-n}$$

where $x = p^n \frac{a}{b}$ with $p \nmid a, b$. (Note any $x \in \mathbb{Q}$ can be uniquely written as $x = p^n \frac{a}{b}$ where $p \nmid a, b$ and $\frac{a}{b}$ is reduced.)

In particular, if $x \in \mathbb{Z}$ is relatively prime to p, we have $|x| = 1$. More generally, if $x \in \mathbb{Z}$, $|x| = p^{-n}$ where n is the number of times p divides x.

Note any integer $x \in \mathbb{Z}$ satisfies $|x|_p \leq 1$, and $|x|_p$ will be close to 0 if x is divisible by a high power of p. So two integers $x, y \in \mathbb{Z}$ will be close with respect to the p-adic metric if $p^n | x - y$ for a large n, i.e., if $x \equiv y \bmod p^n$ for large n.

Example 1.1.1. *Suppose $p = 2$. Then*

$$|1|_2 = 1,\ \ |2|_2 = \frac{1}{2},\ \ |3|_2 = 1,\ \ |4|_2 = \frac{1}{4},\ \ |5|_2 = 1,\ \ |6|_2 = \frac{1}{2}, \ldots$$

$$|\frac{3}{4}|_2 = 4,\ \ |\frac{12}{17}|_2 = \frac{1}{4},\ \ |\frac{57}{36}|_2 = 4.$$

With respect to $|\cdot|_2$, the closed ball $\overline{B}_{1/2}(0)$ of radius $\frac{1}{2}$ about 0 is simply all rationals (in reduced form) with even numerator. Similarly $\overline{B}_{1/4}(0)$ of radius $\frac{1}{4}$ about 0 is simply all all rationals (in reduced form) whose numerator is congruent to 0 mod 4.

Exercise 1.1.2. *Prove $|\cdot|_p$ is an absolute value on $\mathbb{Q}$.*

Recall, for a space R with an absolute value $|\cdot|$, one can define Cauchy sequences (x_n) in R—namely, for any $\epsilon > 0$, $|x_m - x_n| < \epsilon$ for all m, n large. One forms the completion of R with respect to $|\cdot|$ by taking equivalence classes of Cauchy sequences. Everyone knows that the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ is $\mathbb{R}$.

Definition 1.1.3. *The field $\mathbb{Q}_p$ of* ***p-adic numbers*** *is defined to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.*

The usual way to write down an explicit p-adic number is the following. Consider a sequence in $\mathbb{Q}$ given by

$$\begin{aligned}
x_0 &= a_{-d}p^{-d} + a_{1-d}p^{1-d} + \cdots + a_0 \\
x_1 &= a_{-d}p^{-d} + a_{1-d}p^{1-d} + \cdots + a_0 + a_1 p \\
x_2 &= a_{-d}p^{-d} + a_{1-d}p^{1-d} + \cdots + a_0 + a_1 p + a_2 p^2 \\
&\vdots
\end{aligned}$$

where $d \in \mathbb{Z}$ is fixed and each $0 \le a_i < p$. Then $|x_{n+1} - x_n|_p = |a_{n+1}p^{n+1}|_p = \frac{1}{p^{n+1}}$ (unless $x_{n+1} = x_n$, in which case it is of course 0). Hence $x = (x_n)$ is a Cauchy sequence, and we can write it more succinctly as a formal Laurent series

$$x = a_{-d}p^{-d} + a_{1-d}p^{1-d} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots . \tag{1.1}$$

Assuming $a_{-d} \neq 0$, we see that $|x|_p = p^d$.

That any p-adic number can be written in the above form, follows from this simple exercise.

Exercise 1.1.4. *Suppose (x_n) is a Cauchy sequence in $(\mathbb{Q}, |\cdot|_p)$. Show that $|x_n - x|_p \to 0$ for some $x \in \mathbb{Q}_p$ of the form* (1.1)*.*

Hence, the $\mathbb{Q}_p$'s are an arithmetic analogue of $\mathbb{R}$, just being completions of the absolute values on $\mathbb{Q}$ ($\mathbb{Q}$ is already complete with respect to the trivial absolute value—$\mathbb{Q}$ is totally disconnected with respect to $|\cdot|_0$). This approach to constructing $\mathbb{Q}_p$ gives both an absolute value and a topology on $\mathbb{Q}_p$, which are the most important things to understand about $\mathbb{Q}_p$.

Precisely, write any $x \in \mathbb{Q}_p$ as

$$x = a_m p^m + a_{m+1}p^{m+1} + \cdots ,\ \ a_m \neq 0$$

for some $m \in \mathbb{Z}$. Then we define the **p-adic (exponential) valuation**(or **ordinal**) of x to be

$$\mathrm{ord}_p(x) = m.$$

Then

$$|x|_p = p^{-m} = p^{-\mathrm{ord}_p(x)}.$$

Definition 1.1.5. *The* **ring of integers** *of $\mathbb{Q}_p$, or the* **p-adic integers**, *are*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \mathrm{ord}_p(x) \geq 0\} = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Proposition 1.1.6. *$\mathbb{Z}_p$ is a closed (topologically) subring of $\mathbb{Q}_p$.*

Proof. That $\mathbb{Z}_p$ is closed is immediate from the definition since $x \mapsto |x|_p$ is continuous. Observing that

$$\mathbb{Z}_p = \left\{ \sum_{n \geq 0} a_n p^n \right\},$$

it is easy to see $\mathbb{Z}_p$ is a ring. □

Corollary 1.1.7. *The group of units $\mathbb{Z}_p^\times$ of $\mathbb{Z}_p$ is*

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : \mathrm{ord}_p(x) = 0\} = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

Proof. Suppose $x \in \mathbb{Z}_p$ is invertible, i.e., $x^{-1} \in \mathbb{Z}_p$. Then

$$|x|_p |x^{-1}|_p = |1|_p = 1.$$

However $x, x^{-1} \in \mathbb{Z}_p$ implies $|x|_p$, $|x^{-1}|_p \leq 1$. Thus $|x|_p = |x^{-1}|_p = 1$. Hence $\mathbb{Z}_p^\times \subseteq \{x \in \mathbb{Q}_p : |x|_p = 1\}$.

Similarly, if $|x|_p = 1$, we see $|x^{-1}|_p = 1$ so $x \in \mathbb{Z}_p^\times$. □

Exercise 1.1.8. *Let $p = 5$. Determine $\mathrm{ord}_p(x)$ and $|x|_p$ for $x = 4, 5, 10, \frac{217}{150}, \frac{60}{79}$. Describe the (open) ball of radius $\frac{1}{10}$ centered around 0 in $\mathbb{Q}_p$.*

Exercise 1.1.9. *Let $x \in \mathbb{Q}$ be nonzero. Show*

$$|x|_\infty \cdot \prod_p |x|_p = 1.$$

This result will be important for us later.

Despite the fact that $\mathbb{R}$ and $\mathbb{Q}_p$ are analogous in the sense that they are both completions of nontrivial absolute values on $\mathbb{Q}$, there are a couple of fundamental ways in which the p-adic absolute value and induced topology are different from the usual absolute value and topology on $\mathbb{R}$.

Definition 1.1.10. *Let $|\cdot|$ be an absolute value on a field F. If $|x+y| \leq \max\{|x|, |y|\}$, we say $|\cdot|$ is* **nonarchimedean**. *Otherwise $|\cdot|$ is* **archimedean**.

The nonarchimedean triangle inequality, $|x+y| \leq \max\{|x|, |y|\}$, is called the **strong triangle inequality**.

Proposition 1.1.11. *$|\cdot|_\infty$ is archimedean but $|\cdot|_p$ is nonarchimedean for each p.*

Proof. Everyone knows $|\cdot|_\infty$ or $\mathbb{Q}$ or $\mathbb{R}$ is archimedean—this is what we are use to and the proof is just $|1+1|_\infty = 2 > 1 = \max\{|1|_\infty, |1|_\infty\}$.

Now let's show $|\cdot|_p$ is nonarchimedean on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ ($\mathbb{Q}_p$ is the completion of $\mathbb{Q}$), this will imply $|\cdot|_p$ is nonarchimedean on $\mathbb{Q}_p$ also. Let $x, y \in \mathbb{Q}$. Write $x = p^m \frac{a}{b}$, $y = p^n \frac{c}{d}$, where a, b, c, d are relatively prime to p, and $m, n \in \mathbb{Z}$. Without loss of generality, assume $m \leq n$. Then we can write

$$x + y = p^m \left(\frac{a}{b} + p^{n-m} \frac{c}{d} \right) = p^m \frac{ad + p^{n-m}bc}{bd}.$$

Since $n \geq m$, the numerator on the right is an integer. The denominator are relatively prime to p since b, d are, though the numerator is possibly divisible by p (though only if $n = m$ and $p|(ad+bc)$). This means that we can write $x + y = p^{m+k}\frac{e}{f}$ where $e, f \in \mathbb{Z}$ are prime to p and $k \geq 0$. Thus

$$|x+y|_p = p^{-m-k} \leq p^{-m} = \max\left\{p^{-m}, p^{-n}\right\} = \max\{|x_p|, |y_p|\}$$

□

Notice that our proof shows that we actually have equality $|x+y|_p = \max\{|x|_p, |y|_p\}$ (since $k = 0$ above) except possibly in the case $|x|_p = |y|_p$.

Exercise 1.1.12. *Find two integers $x, y \in \mathbb{Z}$ such that*
(i) $|x|_3 = |y|_3 = \frac{1}{3}$ *but* $|x+y|_3 = \frac{1}{9}$.
(ii) $|x|_3 = |y|_3 = |x+y|_3 = \frac{1}{3}$.

Proposition 1.1.13. *Every ball $B_r(x)$ in $\mathbb{Q}_p$ is both open and closed. Thus the singleton sets in $\mathbb{Q}_p$ are closed.*

Using the fact that the balls are closed, one can show that $\mathbb{Q}_p$ is *totally disconnected*, i.e., its connected components are the singleton sets. However the singleton sets are not open, as that would imply $\mathbb{Q}_p$ has the discrete topology, i.e., every set would be both open and closed.

Proof. Each ball is open by definition. The following two exercises show $B_r(x)$ is also closed.

Then for any $x \in \mathbb{Q}_p$, the intersection of the closed sets $\bigcap_{r>0} B_r(x) = \{x\}$, which must be closed. □

Exercise 1.1.14. *Show* $B_r(x) = x + B_r(0) = \{x + y : y \in B_r(0)\}$.

Exercise 1.1.15. *Show that $B_r(0)$ is closed for any $r \in \mathbb{R}$.*

Your proof of the second exercise should make use of the fact that $|\cdot|_p$ is a *discrete* absolute value, i.e., the valuation $\mathrm{ord}_p : \mathbb{Q}_p \to \mathbb{R}$ actually has image $\mathbb{Z}$, which is a discrete subset of $\mathbb{R}$. In other words, the image of $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$, namely $p^{\mathbb{Z}}$, is discrete in $\mathbb{R}$ except for the limit point at 0. On the other hand, the image of the ordinary absolute value $|\cdot|_\infty$ on $\mathbb{R}$ is a *continuous* subset of $\mathbb{R}$, namely $\mathbb{R}_{\geq 0}$.

Another strange, but nice thing, about analysis on $\mathbb{Q}_p$ is that a series $\sum x_n$ converges if and only if $x_n \to 0$ in $\mathbb{Q}_p$.

While these are some very fundamental differences between $\mathbb{R}$ and $\mathbb{Q}_p$, you shouldn't feel that $\mathbb{Q}_p$ is too unnatural—just different from what you're familiar with. To see that $\mathbb{Q}_p$ isn't too strange, observe the following:

Proposition 1.1.16. *$\mathbb{Q}_p$ and $\mathbb{R}$ are both Hausdorff and locally compact, but not compact.*

Proof. The results for $\mathbb{R}$ should be familiar, so we will just show them for $\mathbb{Q}_p$.

Recall a space is Hausdorff if any two points can be separated by open sets. $\mathbb{Q}_p$ is Hausdorff since it is a metric space: namely if $x \neq y \in \mathbb{Q}_p$, let $d = d(x, y) = |x + y|_p$. Then for $r \leq \frac{d}{2}$, $B_r(x)$ and $B_r(y)$ are open neighborhoods of x and y which are disjoint.

Recall a Hausdorff space is locally compact if every point has a compact neighborhood. Around any $x \in \mathbb{Q}_p$, we can take the closed ball $\overline{B}_r(x)$ of radius r. This is a closed and (totally) bounded set in a complete metric space, and therefore compact. (In fact one could also take the open ball $\overline{B}_r(x)$, since we know it is closed from the previous exercise.)

Perhaps more instructively, one can show $\overline{B}_r(x)$ is sequentially compact in $\mathbb{Q}_p$, which is equivalent to compactness being a metric space. We may take a specific r if we want, say $r = 1$. Further since $\overline{B}_1(x) = x + \overline{B}_1(0)$ by the exercise above, it suffices to show $\overline{B}_1(0) = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \mathbb{Z}_p$ is sequentially compact. If

$$
\begin{aligned}
x_1 &= a_{10} + a_{11}p + a_{12}p^2 + \cdots \\
x_2 &= a_{20} + a_{21}p + a_{22}p^2 + \cdots \\
x_3 &= a_{30} + a_{31}p + a_{32}p^2 + \cdots \\
&\vdots
\end{aligned}
$$

is a Cauchy sequence, then for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n|_p < \epsilon$ for all $m, n > N$. Take $\epsilon = p^{-r}$ for $r > 0$. Then $|x_m - x_n|_p < \epsilon = p^{-r}$ means $x_m \equiv x_n \bmod p^{r+1}$, i.e., the coefficients of $1, p, p^2, \ldots, p^r$ must be the same for all x_m, x_n with $m, n > N$. Let $a_0, a_1, \ldots, a_r$ denote these coefficients. We can do this for larger and larger r (note that $a_0, \ldots, a_{r-1}$ will never change) to get a sequence (a_n), and then it is clear that the above sequence converges to

$$x = a_0 + a_1p + a_2p^2 + \cdots \in \mathbb{Z}_p.$$

This provides a second proof of local compactness.

To see $\mathbb{Q}_p$ is not compact, observe the sequence $x_1 = p^{-1}$, $x_2 = p^{-2}$, $x_3 = p^{-3}, \ldots$ has no convergent subsequence. Geometrically, $|x_n| = p^n$, so this is a sequence of points getting further and further from 0, and the distance to 0 goes to infinity. □

We remark that $\mathbb{Q}$, with either usual subspace topology coming from $\mathbb{R}$ or the one coming from $\mathbb{Q}_p$, is a space which is not locally compact. The reason is any open neighborhood about a point is not complete—the limit points are contained in the completion of $\mathbb{Q}$ (w.r.t. to whichever absolute value we are considering), but not in $\mathbb{Q}$. (The trivial absolute value $|\cdot|_0$ induces the discrete topology on $\mathbb{Q}$, meaning single points are open sets, so it is trivially locally compact.)

The general definition of a **local field** is a locally compact field with a non-discrete topology, hence we see that $\mathbb{Q}_p$ and $\mathbb{R}$ are local fields, whereas $\mathbb{Q}$ (with the usual topology) is not. Any local field of characteristic 0 will be a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$. These will be the completions of number fields (a finite extension of $\mathbb{Q}$), which are in contrast called **global fields**.

To try to minimize the background required, we will mostly work with $\mathbb{Q}_p$, though the theory we develop extends without difficulty to any finite extension.

1.2 Representation theory

Definition 1.2.1. *Let G be a group and V be a vector space over a field F. A* **representation** *π of G on V is a homomorphism*

$$\pi : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ denotes the group of automorphisms of V. The representation is denoted by (π, V), π, or sometimes just V. Here V is called the **representation space** *of V.*

*If $\dim V = n < \infty$, we say π is an n-***dimensional representation** *over F. Otherwise, we say π is* **infinite dimensional**. *If π is a 1-dimesional representation, we also refer to π as a* **(linear) character**.

Example 1.2.2. *Let G and V be arbitrary. The map $\pi : G \to \mathrm{GL}(V)$ given by $\pi(g) = \mathrm{id}$ is always a representation, called the* **trivial representation** *of G on V.*

If we simply say the trivial representation *of G, without other context, it is taken for granted we mean the 1-dimensional trivial representation $\pi : G \to \mathrm{GL}_1(F)$.*

Example 1.2.3. *Let $G = C_n$, the cyclic group of order n, and take $F = \mathbb{C}$. Write $G = \langle x | x^n = 1\rangle$. Then the map*

$$\pi_1 : G \to \mathrm{GL}(\mathbb{C}) = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times \quad \textit{given by} \quad \pi_1(x^j) = e^{2\pi i j/n}$$

is a representation (in fact, linear character) of G. Note that since π_1 is a homomorphism, it is determined by its values on a set of generators, so one can define π_1 simply by $\pi_1(x) = e^{2\pi i/n}$.

The homomorphism

$$\pi_2 : G \to \mathrm{GL}(\mathbb{C}^2) = \mathrm{GL}_2(\mathbb{C}) \quad \textit{defined by} \quad \pi_2(x) = \begin{pmatrix} e^{2\pi i k_1/n} & \\ & e^{2\pi k_2/n} \end{pmatrix}$$

is a 2-dimensional representation of G for any fixed $k_1, k_2 \in \mathbb{Z}$. One can analogously define an n-dimensional representation of C_n.

While π_2 in the example above gives an action of C_n on a 2-dimensional space $V = \mathbb{C}^2$, the action is entirely determined by what it does to the two orthogonal subspaces $\mathbb{C}e_1$ and $\mathbb{C}e_2$, where $\{e_1, e_2\}$ is the standard basis for V. Thus we should be able to decompose π_2 into two 1-dimensional representations.

Definition 1.2.4. *Let (π, V) be a representation of a group G. We say W is a* **(G-)invariant subspace** *of V if $\{\pi(g)w : g \in G, w \in W\} \subseteq W$. A* **subrepresentation** *of a representation (π, V) of a group G is an invariant subspace W of V together with the homomorphism*

$$\pi_W : G \to \mathrm{GL}(W) \quad \textit{given by} \quad \pi_W(g)w = \pi(g)w.$$

For a subrepresentation W, the **quotient** *of (π, V) by (π_W, W) is the representation*

$$\pi^W : G \to \mathrm{GL}(V/W) \quad \textit{given by} \quad \pi^W(g)(v + W) = \pi(g)v + W.$$

If $V = W_1 \oplus \cdots \oplus W_k$ with each W_i invariant, we say π is a **direct sum** *of $\pi_{W_1}, \ldots \pi_{W_k}$ and write*

$$\pi = \pi_{W_1} \oplus \cdots \oplus \pi_{W_k}.$$

Since the maps π_W and π^W are naturally defined given W, we typically just refer to the subrepresentation and quotient representation as W and V/W.

Exercise 1.2.5. *(i) Check the quotient is a well-defined representation.*

(ii) For π_2 from the previous example, determine the subrepresentations and their quotients. Is π_2 a direct sum?

For any representation (π, V), the subspaces $\{0\}$ and V are always invariant.

Definition 1.2.6. *We say (π, V) is* **irreducible** *if there are no invariant subspaces other than $\{0\}$ and V.*

Example 1.2.7. *Any 1-dimensional representation is irreducible. In particular, the trivial one is.*

Proposition 1.2.8. *Let G be a finite group and (π, V) be a representation of G. If π is irreducible, then π is finite dimensional.*

This is of course not true for general groups, though if G is *compact*, e.g., $\mathrm{Gal} = \mathrm{SO}(2)$, then the irreducible *continuous* representations are finite dimesional.

Proof. Let $v_0 \in V$ be nonzero. Then the linear span W of $\langle \pi(g)v_0 : g \in G\rangle$ is a invariant subspace. To see this take any $w \in W$. Then we can write $w = \sum c_i \pi(g_i)v_0$, where this sum is finite. For any $g \in G$, we see

$$\pi(g)w = \sum c_i \pi(gg_i)v_0 \in W.$$

Since $\pi(1)v_0 = v_0$, we see $W \neq \{0\}$. Hence by irreducibility, $W = V$, but it is clear W is finite dimensional. □

The basic problem in representation theory is to classify all irreducible representations of G. In working with groups with more structure, such as topological groups or algebraic (matrix) groups, one typically restricts this question to a certain class of representations, such as *continuous* or *smooth* representations, that are more natural for the groups at hand. We will wait to discuss this until the next chapter.

All representations of finite groups over $\mathbb{C}$ decompose into a direct sum of irreducible representations, so in this case, knowing the irreducible representations of a group tells us all representations of a group. While this is not true in general, this classification is still crucial to understanding the representation theory of the group.

There are meaningless ways to get new representations out of old ones, such as replacing F or V by something which is isomorphic. A slightly less trivial, but still essentially meaningless, way to get a new representation out of an old one is by conjugation. For example, if π_2 is as in the previous example, we can replace π_2 by $\pi_2^\gamma(g) = \gamma^{-1}\pi_2(g)\gamma$ for any $\gamma \in \mathrm{GL}_2(\mathbb{C})$. If $\gamma = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then π_2^γ simply interchanges k_1 and k_2. More generally, conjugation can be viewed as a special case of composing the representation with either an automorphism of V or an automorphism of G.

Thus this classification of irreducible representations should only be up to a certain notion of equivalence.

Definition 1.2.9. *Let (π_1, V_1) and (π_2, V_2) be representations of G. A linear transformation $A : V_1 \to V_2$ is an* **intertwining map** *if*

$$\pi_2(g)A = A\pi_1(g), \quad g \in G.$$

The set of all such intertwining maps is denoted $\mathrm{Hom}_G(\pi_1, \pi_2) = \mathrm{Hom}_G(V_1, V_2)$. We also write $\mathrm{End}_G(\pi) = \mathrm{Hom}_G(\pi, \pi)$ and $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$. We omit the subscripts G when understood.

We say π_1 and π_2 are **equivalent** *if $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains an invertible transformation $A : V_1 \to V_2$. In this case we write $\pi_1 \cong \pi_2$.*

Note we always have $0 \in \mathrm{Hom}(\pi_1, \pi_2)$.

Example 1.2.10. *Let $G = C_n = \langle x \rangle$,*

$$\pi_1 : G \to \mathrm{GL}_1(\mathbb{C}) \quad \textit{given by} \quad \pi_1(x) = e^{2\pi i/n}$$

$$\pi_2 : G \to \mathrm{GL}_1(\mathbb{C}) \quad \textit{given by} \quad \pi_2(x) = e^{-2\pi i/n}$$

$$\pi_3 : G \to \mathrm{GL}_2(\mathbb{C}) \quad \textit{given by} \quad \pi_3(x) = \begin{pmatrix} e^{2\pi i/n} & \\ & e^{-2\pi i/n} \end{pmatrix}$$

$$\pi_4 : G \to \mathrm{GL}_2(\mathbb{C}) \quad \textit{given by} \quad \pi_4(x) = \begin{pmatrix} e^{-2\pi i/n} & \\ & e^{2\pi i/n} \end{pmatrix}.$$

Let us assume $n > 2$ so $e^{2\pi i/n} \neq e^{-2\pi i/n}$.
Then

$$\mathrm{End}(\pi_1) = \mathrm{Hom}(\pi_1, \pi_1) = \{A \in M_{1\times 1}(\mathbb{C}) : \pi_1(g)A = A\pi_1(g)\} = \mathbb{C}$$

Note if $A \in \mathrm{Hom}(\pi_1, \pi_2)$ this means $A \in \mathbb{C}$ such that

$$\pi_2(x)A = A\pi_1(x) \textit{ for } v \in \mathbb{C} \implies e^{-2\pi i/n}A = Ae^{2\pi i/n} \implies A = 0,$$

i.e., $\mathrm{Hom}(\pi_1, \pi_2) = \{0\}$.

Now let's determine $\mathrm{Hom}(\pi_1, \pi_3)$. Suppose $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : \mathbb{C} \to \mathbb{C}^2$. Then $A \in \mathrm{Hom}(\pi_1, \pi_3)$ means, for all $j \in \mathbb{Z}$,

$$\pi_3(x^j)A = A\pi_1(x^j) \iff \begin{pmatrix} e^{2\pi ij/n} & \\ & e^{-2\pi ij/n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{2\pi ij/n} \iff a_2 = 0.$$

Hence we see that $\mathrm{Hom}(\pi_1, \pi_3) = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{C} \right\} \simeq \mathbb{C}$.

Finally, we determine $\mathrm{Hom}(\pi_3, \pi_4)$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$. Then $A \in \mathrm{Hom}(\pi_3, \pi_4)$ means, for all $j \in \mathbb{Z}$,

$$\pi_4(x^j)A = A\pi_3(x^j) \iff \begin{pmatrix} e^{-2\pi ij/n} & \\ & e^{2\pi ij/n} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{2\pi ij/n} & \\ & e^{-2\pi ij/n} \end{pmatrix}.$$

It is easy to see this means

$$\mathrm{Hom}(\pi_3, \pi_4) = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbb{C} \right\} \simeq \mathbb{C} \oplus \mathbb{C}.$$

Since this contains invertible elements in $M_2(\mathbb{C})$, we see π_3 and π_4 are equivalent.

Exercise 1.2.11. *Keep the notation of the previous example.*

(i) Determine $\mathrm{End}(\pi_3)$ *and* $\mathrm{Hom}(\pi_3, \pi_2)$*.*

(ii) Consider the representation

$$\pi_5 : G \to \mathrm{GL}_2(\mathbb{C}) \quad \text{given by} \quad \pi_4(x) = \begin{pmatrix} e^{2\pi i/n} & \\ & e^{2\pi i/n} \end{pmatrix}.$$

Determine $\mathrm{End}(\pi_5)$ *and* $\mathrm{Hom}(\pi_5, \pi_3)$*.*

As you might guess from the above example, the space $\mathrm{Hom}(\pi_1, \pi_2)$ tell us when π_1 is (up to equivalence) a subrepresentation of π_2, or more generally, how many subrepresentations π_1 and π_2 have in common. For now, let us just state what happens for finite groups.

Proposition 1.2.12. *Let* G *be a finite group and* (π_1, V_1)*,* (π_2, V_2) *be finite-dimensional representations over* $\mathbb{C}$*.*

(i) **(Schur's lemma)** *Suppose* π_1 *and* π_2 *are irreducible. Then* $\mathrm{Hom}(\pi_1, \pi_2) = \{0\}$ *unless* $\pi_1 \cong \pi_2$*, in which case* $\mathrm{Hom}(\pi_2, \pi_2) \simeq \mathbb{C}$*.*

(ii) Suppose π_1 *is irreducible. Then* $\mathrm{Hom}(\pi_1, \pi_2) \simeq \mathbb{C}^n$ *where* n *is the number of times* π_1 *appears (up to equivalence) in the direct sum decomposition (which is unique up to equivalence) of* π_2*.*

(iii) Suppose $\pi_1 \cong \rho_1 \oplus \cdots \oplus \rho_l$ *and* $\pi_2 \cong \tau_1 \oplus \cdots \oplus \tau_k$ *with each* ρ_i, τ_j *irreducible. Then* $\mathrm{Hom}(\pi_1, \pi_2) \simeq \mathbb{C}^n$ *where* n *is the number of pairs* (i, j) *such that* $\rho_i \cong \tau_j$*.*

(iv) π_1 *is irreducible if and only if* $\mathrm{Hom}(\pi_1, \pi_1) \simeq \mathbb{C}$*.*

We will show Schur's lemma, in greater generality, later in this course. Parts (ii) and (iii) are simple generalizations of (i), and (iv) a consequence of (iii).

Before we move on to the next topic, here are a couple other examples of finite-dimensional representations.

Example 1.2.13. *Let* $G = \mathrm{GL}_n(F)$ *or* $\mathrm{SL}_n(F)$*, where* F *is a field. The map* $\pi : G \to \mathrm{GL}_n(F)$ *given by* $\pi(g) = g$ *is an irreducible* n*-dimensional representation of* G*, called the* **standard representation** *of* G*.*

The map $\check{\pi} : G \to \mathrm{GL}_n(F)$ *given by* $\pi(g) = {}^t g^{-1}$ *is also an* n*-dimensional representation of* G*, called the* **contragredient representation** *associated to* π*. (One can make this definition for any* G, π*).*

A nontrivial 1*-dimensional representation of* G *is given by the map* $g \mapsto \det(g)$*.*

Example 1.2.14. *Let* π *be the standard representation of* $G = \mathrm{GL}_2(F)$*, and let* e_1, e_2 *be the standard basis for* $V = F^2$*. Let* $\mathrm{Sym}^2(V)$ *be the 3-dimensional vector space generated by the symmetric algebra* $e_1 \otimes e_1$*,* $e_1 \otimes e_2 = e_2 \otimes e_1$*,* $e_2 \otimes e_2$*. The symmetric square representation* $\mathrm{Sym}^2(\pi) : G \to \mathrm{GL}_3(F) = \mathrm{GL}(\mathrm{Sym}^2(v))$ *is given by*

$$\mathrm{Sym}^2(\pi)(g)(e_i \otimes e_j) = g \cdot e_i \otimes g \cdot e_j.$$

More explicitly, using the above ordered basis for $\mathrm{Sym}^2(V)$*, we have*

$$\mathrm{Sym}^2(\pi) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

This is irreducible (except in the degenerate case when $F = \mathbb{F}_2$ *and* $G \simeq S_3$*).*

1.2.1 Induction and restriction

Let H be a subgroup of G such that $H \neq G$. For simplicity, we assume G is finite in this section.

Induction and restriction allow us to transfer representations from H to G and vice versa. Restriction is trivial.

Definition 1.2.15. *Let π be a representation of G. The* **restriction** *π_H of π to H is simply the representation defined by*

$$\pi_H(h) = \pi(h).$$

Example 1.2.16. *Let $\pi : G \to \mathrm{GL}_n(F)$. Since $\pi(1) = I_n$, the restriction $\pi_{\{1\}}$ of π to the trivial subgroup decomposes as a direct sum n-copies of the trivial representation. (The only irreducible representation of $\{1\}$ is the trivial one.)*

Conversely, given a representation (ρ, W) of H, one might ask to construct a representation π of G such that $\pi_H = \rho$. This would mean that one could extend the homomorphism $\rho : H \to \mathrm{GL}(W)$ to a homomorphism $\pi : G \to \mathrm{GL}(W)$, which is not always possible. However, what we can always do is extend ρ to a homomorphism $\pi : G \to \mathrm{GL}(V)$ where V is some superspace of W. In particular, one can formally take $V = \bigoplus_{g \in G/H} gW$, where $gW = \{gw : w \in W\} \simeq W$. Then one defines the action $\pi(g)$ on V by

$$\pi(g)g'w = g''\rho(h)w \in g''W, \quad \text{where} gg' = g''h.$$

Exercise 1.2.17. *Check that π is a well defined representation of G, called the representation induced from ρ.*

In particular, if we restrict π to H and $g' = 1$, then we see

$$\pi(h)w = \rho(h)w$$

so the restriction π_H contains ρ as a subrepresentation. However, by definition π_H acts on a larger space V, so $\pi_H \neq \rho$, i.e., the quotient of π_H by ρ is nontrivial. In this sense, we see induction is a converse construction to restriction.

There is another way to define induction, which is more suitable for our point of view.

Definition 1.2.18. *Let (ρ, W) be a representation of H. The* **induction** *of ρ from H to G is the representation $(\mathrm{Ind}_H^G(\rho), V)$ where*

$$V = \{f : G \to W | f(hg) = \rho(h)f(g)\}$$

and

$$\mathrm{Ind}_H^G(\rho)(g)f(x) = f(xg).$$

In general, one might want to restrict to functions $f : G \to W$ of a certain type (e.g., continuous or smooth), but these restrictions are vacuous for finite groups.

Example 1.2.19. *Consider C_3 as a subgroup of the symmetric (or dihedral if you prefer) group S_3 of order 6. Write $S_3 = \langle \sigma, \tau | \sigma^2 = \tau^3 = 1, \sigma\tau\sigma = \tau^{-1} \rangle$ so $C_3 = \langle \tau \rangle$. Let (ρ, W) be the 1-dimensional representation of C_3 over $\mathbb{C}$ given by $W = \mathbb{C}$ and*

$$\rho(\tau^j) = \zeta^j, \quad \zeta = e^{2\pi i/3}.$$

Set

$$V = \{f : S_3 \to W | f(hg) = \rho(h)f(g)\}.$$

There are two cosets in $C_3 \backslash S_3$*, represented by* 1 *and* σ*. Hence* $f \in V$ *means*

$$f(\tau^j) = \rho(\tau^j)f(1) = \zeta^j f(1)$$

and

$$f(\tau^j \sigma) = \zeta^j f(\sigma).$$

Conversely, these conditions imply $f \in V$*, whence* $f \in V$ *is determined by* $f(1)$ *and* $f(\tau)$*. In particular,* $\dim V = 2$*.*

We can use this to write $\pi = \mathrm{Ind}_{C_3}^{S_3}(\rho)$ *in matrix form. Namely, let* $f_1, f_2 \in V$ *be defined by* $f_1(1) = f_2(\sigma) = 1$*,* $f_1(\sigma) = f_2(1) = 0$*. Then*

$$\pi(\sigma)f_1(1) = f_1(\sigma) = 0, \quad \pi(\sigma)f_1(\sigma) = f_1(\sigma^2) = f_1(1) = 1,$$

which implies

$$\pi(\sigma)f_1 = 0 \cdot f_1 + 1 \cdot f_2.$$

A similar computation shows

$$\pi(\sigma)f_2 = 1 \cdot f_1 + 0 \cdot f_2.$$

Consequently, with respect to the ordered basis f_1, f_2*,*

$$\pi(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Next note

$$\pi(\tau)f_1(1) = f_1(\tau) = \zeta, \quad \pi(\tau)f_1(\sigma) = f_1(\sigma\tau) = f_1(\tau^2\sigma) = 0,$$

so

$$\pi(\tau)f_1 = \zeta \cdot f_1 + 0 \cdot f_2.$$

Similarly we see

$$\pi(\tau)f_2 = 0 \cdot f_1 + \zeta^2 \cdot f_2.$$

Thus we can write

$$\pi(\tau) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}.$$

Since σ, τ *generate* S_3*, this determines* π *on any element of* S_3*. Observing* π *is* faithful*, i.e., injective, we see* π *gives a* matrix realization *for* S_3 *in* $\mathrm{GL}_2(\mathbb{C})$*.*

Exercise 1.2.20. *With* π *the induced representation of* S_3 *in the previous example, compute* $\mathrm{Hom}(\pi, \pi)$ *and conclude* π *is irreducible.*

Exercise 1.2.21. *Show* $\dim \mathrm{Ind}_H^G(\rho) = |G/H| \cdot \dim \rho$*.*

Exercise 1.2.22. *Compute* $\mathrm{Ind}_{C_3}^{S_3}(1)$ *where* 1 *denotes the (1-dimensional) trivial representation of* C_3*.*

Exercise 1.2.23. *Let* ρ *be a non-trivial representation of the Klein group* V_4*. Compute* $\mathrm{Ind}_{V_4}^{A_4}(\rho)$*.*

1.2.2 Character theory

One of the main tools to study representations, particularly of finite groups, is to look at their characters. Again, we will restrict to the case of finite groups in this section, and all our representations will be finite dimensional.

Definition 1.2.24. *Let (π, V) be a representation of G over F. The* character χ_π *of π is the function $\chi_\pi : G \to F$ defined by*

$$\chi_\pi(g) = \mathrm{tr}\pi(g) = \sum_{e_i} (\pi(g)e_i, e_i),$$

where $(\ ,\ )$ is an inner product on V and e_i an orthonormal basis.

We remark that χ_π does not depend upon on the choice of inner product or basis. In fact, it only depends upon the equivalence class of π, making it a useful invariant.

Since the trace is conjugacy invariant, any character χ_π of G is a *class function* on G, i.e., $\chi_\pi(g)$ only depends upon the conjugacy class of g in G.

Example 1.2.25. *If $G = \mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ and π is the standard representation on F^n, then $\chi_\pi(g) = \mathrm{tr}(g)$.*

Example 1.2.26. *If π is a 1-dimensional representation of G, then $\chi_\pi(g) = \pi(g)$. This explains why a* 1*-dimensional representation is also called a character.*

Example 1.2.27. *Suppose π is a n-dimensional representation. Then $\pi(1)$ is the identity of $\mathrm{GL}_n(F)$, so $\chi_\pi(1) = n = \dim \pi$. We also call $\chi_\pi(1)$ the* **degree** $\deg \chi_\pi$ *of χ_π.*

For the rest of the section, **we assume** $F = \mathbb{C}$.

Theorem 1.2.28. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, i.e., the (finite number of) characters of irreducible representations of G. Then*

(i) (First orthogonality relation) For any i, j,

$$\sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}|G|.$$

(ii) (Second orthogonality relation) For any $g, h \in G$,

$$\sum_{i=1}^{r} \chi_i(g)\chi_i(h^{-1}) = \begin{cases} |C_G(g)| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{else.} \end{cases}$$

(iii) $\chi_1, \ldots, \chi_r$ for a basis for the space of class functions on G. In particular, r is the number of conjugacy classes of G.

(iv)

$$\sum_{i=1}^{r} (\deg \chi_i)^2 = |G|.$$

These results can be found in any standard text on representation theory for finite groups.

All of these results are very useful, but (iii) and (iv) are particularly useful for determining the irreducible representations of G.

Example 1.2.29. *Suppose G is abelian of order n. Then each element of G is its own conjugacy classes, thus there are n irreducible representations by (iii). By (iv), the sum of their dimensions squared must be n, hence they are all 1-dimensional.*

In particular, for $G = C_n = \langle x \rangle$, consider the 1-dimensional representation given by $\pi_1(x) = \zeta = e^{2\pi i/n}$. Then the representations $\pi_j = \pi_i^j$ for $j = 0, 1, \ldots, n-1$ are all inequivalent, and hence are all irreducible representations of G.

Exercise 1.2.30. *Let G be any finite abelian group. Construct all irreducible representations of G.*

Example 1.2.31. *Let's determine all irreducible representations of S_3 over $\mathbb{C}$, and then compute their characters. Let π be as in Example 1.2.19.*

We already know 2 irreducible representations of S_3, namely the trivial one χ_0 and the 2-dimensional induced representation π from Example 1.2.19. By Theorem 1.2.28(iii), there is only 1 more irreducible representation of S_3, and by part (iv) of the same theorem, it must be 1-dimensional. Let us call it ψ.

Since a 1-dimensional representation of S_3 must have abelian image (it lies in $\mathrm{GL}_1(\mathbb{C})$), it must have nontrivial kernel. This kernel is a normal subgroup of S_3, of which the nontrivial ones are C_3 and S_3. If the kernel is S_3, then the representation must be trivial, so ψ must have kernel C_3, i.e., it descends to a representation of the quotient $S_3/C_3 \simeq C_2$. The only irreducible (which in this case is equivalent to 1-dimensional) representations of C_2 are the trivial one and the embedding of C_2 as $\{\pm 1\}$ in $\mathbb{C}^\times \simeq \mathrm{GL}_1(\mathbb{C})$.

Explicitly, we may write ψ as

$$\psi(\tau^j) = 1, \quad \psi(\sigma\tau^j) = -1$$

for any $j = 0, 1, 2$.

Thus χ_0, ψ, π are all irreducible representations of S_3. To compute the character of π, it suffices to determine its value on each of the conjugacy classes of S_3, of which there are 3. We may take for conjugacy class representatives $1, \sigma, \tau$.

It is often convenient to present the irreducible characters of S_3 as a **character table***, which is a table with the rows indexed by the characters and the columns indexed by the conjugacy classes. For S_3, we see it looks like*

	1	σ	τ
χ_0	1	1	1
ψ	1	-1	1
χ_π	2	0	-1

Exercise 1.2.32. *Determine all irreducible representations of A_4 and compute the character table.*

2 Smooth Representations

In this section we will introduce some basic notions and results on representations of certain topological groups, including $\mathrm{GL}_n(\mathbb{Q}_p)$. The reason for this is because (1) to study representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, we will need to look at representations of and restrictions to certain subgroups, and (2) not much is gained by restricting to $\mathrm{GL}_2(\mathbb{Q}_p)$ at this stage.

Much of this chapter is based on Fiona Murnaghan's notes "Representations of reductive p-adic groups." The notes by Prasad and Raghuram entitled "Representation theory of $\mathrm{GL}(n)$ over non-archimedean local fields" contain similar material.

2.1 Some topological groups

Definition 2.1.1. *A* **topological group** *G is a group endowed with a topology such that the map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ is continuous.*

Here, of course, $G \times G$ is given the product topology. An alternative way to define topological group is to require that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

Proposition 2.1.2. *Let $n \in \mathbb{N}$. Let $F = \mathbb{Q}_p$ or $\mathbb{R}$, or more generally, a ring which is a metric space. The the additive group of $n \times n$ matrices $M_n(F)$ over F is a topological group.*

Here, viewing $M_n(F) \simeq F^{n^2}$, we give $M_n(F)$ the product topology. Note that this is also a metric space.

Proof. Let U be an open set in $M_n(F)$, and let $V = \{(A, B) \in M_n(F) \times M_n(F) : A - B \in U\}$. We want to show V is open. Let $(A, B) \in V$. It suffices to show there there are open neighborhoods V_1 of A and V_2 of B such that $V_1 \times V_2 \subseteq V$.

Since U is open, there exist ϵ such that the ball $B_\epsilon(A - B) \subseteq U$. Taking $V_1 = B_{\epsilon/2}(A)$ and $V_2 = B_{\epsilon/2}(B)$ gives the claim. □

Proposition 2.1.3. *Let $F = \mathbb{Q}_p$ or R. Then $\mathrm{GL}_n(F) = \{A \in M_n(F) : \det(A) \neq 0\}$ is a topological group.*

Here we give $\mathrm{GL}_n(F)$ the subspace topology of $M_n(F)$.

Proof. First note that the coordinate maps $A = (a_{ij}) \mapsto a_{i_0,j_0}$ are continuous. Since multiplication and addition are continuous maps from $F \times F \to F$, a composition of these maps shows any polynomial in the coefficients of A is a continuous map.

Write $B = (b_{ij})$ and $AB^{-1} = C = (c_{ij})$. Each c_{ij} is a (well-defined) rational function of the a_{ij}'s and b_{ij}'s, and therefore $(A, B) \mapsto c_{ij}$ for any ij is a continuous map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to F$. Consequently $(A, B) \mapsto AB^{-1}$ is a continuous map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to M_n(F)$, whose image lies in the open set $\mathrm{GL}_n(F)$. Therefore the corresponding map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is continuous. □

Proposition 2.1.4. *Let G be a topological group and H a subgroup. Then H is also a topological group.*

Proof. Let U is a open set in G. Let V be the (open) preimage in $G \times G$ under the map $f(x, y) = xy^{-1}$. Then $V \cap H \times H$ is the preimage of $U \cap H$ under the restriction of f to $f_H : H \times H \to H$. Consequently f_H is continuous. □

Example 2.1.5. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$. Since* $\det : M_n(F) \to F$ *is given by a polynomial in the coordinates of $M_n(F)$, it is a continuous map. Consequently* $\mathrm{SL}_n(F) = \{A \in \mathrm{GL}_n(F) : \det(A) = 1\}$ *is an closed subgroup of* $\mathrm{GL}_n(F)$.

Example 2.1.6. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$ and $G = \mathrm{GL}_n(F)$. Let P denote the set of upper triangular matrices in G (the standard Borel subgroup), A the group of diagonal matrices in G and N the set of upper triangular matrices in G with 1's on the diagonal (so $P = AN$). Then P, A, N are closed subgroups of G, since they are defined by equations on the matrix coefficients.*

More generally, one can consider subgroups of $\mathrm{GL}_n(F)$ defined by polynomial equations in the coefficients. Such groups are the prototypical examples of what are called *algebraic groups.* The most famous of these are the **classical groups**. These groups come in 4 types: linear, orthogonal, symplectic and unitary.

The linear groups are $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$, along with their projective versions: $\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/Z(\mathrm{GL}_n(F))$ and $\mathrm{PSL}_n(F) = \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$. Here $Z(G)$ denotes the center of G. Note $\mathrm{PSL}_n(F) \simeq \mathrm{PGL}_n(F)$ if F is algebraically closed, but not in general.

We will not say precisely what orthogonal, symplectic and unitary mean in general, but essentially they are subgroups of linear groups which preserve symmetric bilinear, skew-symmetric bilinear, and Hermitian forms, respectively. In any dimension, there are various (non-isomorphic) examples of each kind of group according to the classification of the appropriate types of forms on F^n. Here are a couple examples.

Example 2.1.7. *The* **special orthogonal group**

$$\mathrm{SO}_n(F) = \left\{ g \in \mathrm{SL}_n(F) : {}^t g g = I_n \right\}.$$

The **symplectic group**

$$\mathrm{Sp}_{2n}(F) = \left\{ g \in \mathrm{SL}_{2n}(F) : {}^t g J g = J \right\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. *Some authors denote* Sp_{2n} *by* Sp_n.

In Oklahoma, there is another algebraic group you should know, though it does not fall under the heading of "classical groups."

Example 2.1.8. *The* **symplectic similitude group**

$$\mathrm{GSp}_4(F) = \left\{ g \in \mathrm{GL}_4(F) : {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \text{ for some } \lambda(g) \in F \right\}.$$

This of course contains both $\mathrm{Sp}_4(F)$ *and the group* $F^\times$ *embedded diagonally in* $\mathrm{GL}_4(F)$, *but in general more than this.*

The most natural generalization of classical (elliptic) modular forms to higher dimensions is the notion of *Siegel modular forms*, which correspond to automorphic forms or representations on the symplectic groups Sp_{2n} and GSp_{2n}.

In low rank (when n is small) there are some coincidences among these algebraic groups, called *accidental isomorphisms.* We just mention a couple. The group $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$ for any field F. Over $\mathbb{C}$, we have $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{C})$ and $\mathrm{PGSp}_4(\mathbb{C}) := \mathrm{GSp}_4(\mathbb{C})/Z(\mathrm{GSp}_4(\mathbb{C})) \simeq \mathrm{SO}_5(\mathbb{C})$. Over other fields, these latter isomorphisms aren't exactly true, but these groups are still closely related, a fact which is often exploited in number theory and automorphic representations.

Example 2.1.9. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$, and E/F a quadratic field extension. One can define a topology on E by realizing $E \simeq F^2$ as a vector space over F. Then $M_2(E)$ and $\mathrm{GL}_2(E)$ are topological groups. Let σ be the nontrivial Galois automorphism of E/F and $\epsilon \in F^\times$. We define a* **quaternion algebra** *over F by*

$$D(F) = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \in M_2(E) : a, b, \in E \right\}.$$

This is a 4-dimensional algebra over F, whose center is isomorphic to $F^\times$, and it will be a division algebra if and only if ϵ is not a square in F. If $F = \mathbb{R}$, then either $D(\mathbb{R}) \simeq M_2(\mathbb{R})$ if $\epsilon > 0$ or $D(\mathbb{R}) \simeq \mathbb{H}$, Hamilton's quaternions, if $\epsilon < 0$.

The multiplicative group of $D(F)$ is

$$D^\times(F) = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \in \mathrm{GL}_2(E) : a, b, \in E \right\}.$$

This construction of quaternion algebras works more generally (for instance, over $\mathbb{Q}$). Quaternion algebras have many applications to number theory and automorphic forms. One amazing theorem is the local Jacquet–Langlands correspondence, which provides a correspondence between (finite-dimensional) representations of $D^\times(\mathbb{Q}_p)$ and the (infinite-dimensional) "discrete series" representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

For fun, here are a couple simple exercises about subgroup and quotients of topological groups.

Exercise 2.1.10. *Let H be a subgroup of a topological group G. Show the closure of H is also a subgroup.*

Exercise 2.1.11. *Let H be an open subgroup of a topological groups G. Show H is also closed.*

Given a subgroup H of G, one often considers the quotient space G/H (with the quotient topology).

Exercise 2.1.12. *Let H be a subgroup of G. Then G/H the projection map $G \to G/H$ is open. If H is closed, then the singleton sets of G/H are closed.*

Exercise 2.1.13. *Let H be a normal subgroup of G. Then G/H is a topological group.*

2.2 l-groups

The topology of $\mathbb{Q}_p$, and therefore that of $\mathrm{GL}_2(\mathbb{Q}_p)$, is very special. In this section we define l-groups, which will be topological groups with similar topological properties.

Definition 2.2.1. *Let X be a topological space. We say $Y \subseteq X$ is* **connected** *if it is not a disjoint union of proper open subsets of Y (in the subspace topology). We say X is* **totally disconnected** *if no nonempty subsets are connected except singleton sets.*

Lemma 2.2.2. *$\mathbb{Q}_p$ is totally disconnected.*

Proof. Suppose $X \subseteq \mathbb{Q}_p$ is connected and contains more than one element. Let $x \in X$. For some $r > 0$, the ball $B_r(x)$ of radius r about x does not contain X. Since $B_r(x)$ is both open and closed, we can partition X into 2 nonempty disjoint open subsets $U = B_r(x) \cap X$ and $V = B_r(x)^c \cap X$. $\square$

Exercise 2.2.3. *Suppose X and Y are totally disconnected. Show $X \times Y$ is also.*

Exercise 2.2.4. *Let Y be a subspace of X. If X is totally disconnected, so is Y.*

Corollary 2.2.5. *The groups $M_n(\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$, as well as their subgroups, are totally disconnected.*

To get a better sense of the topology of these groups, let's determine bases for the topologies of $M_n(\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$.

First, note that if G is a topological group, then for any $g \in G$, the map given by left multiplication by g is a homeomorphism. Consequently, to determine a basis of open sets for G, it suffices to determine a basis of open neighborhoods around the identity.

Now let's determine a basis of open neighborhoods around 0 for $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$. Recall a basis of open (and also closed) neighborhoods around 0 in $\mathbb{Q}_p$ are the sets $p^j\mathbb{Z}_p = \overline{B}_{p^{-j}}(0)$ for $j \in \mathbb{Z}$. Consequently, a basis of open neighborhoods of 0 in $M_n(\mathbb{Q}_p)$ will be sets of the form $p^{j_1}\mathbb{Z}_p \times p^{j_2}\mathbb{Z}_p \times \cdots \times p^{j_{n^2}}Z_p$ for $j_1, \dots, j_{n^2} \in \mathbb{Z}$. It is clear that each set of this form is both contained in and contains a set of the form $p^j M_n(\mathbb{Z}_p) = p^j\mathbb{Z}_p \times \cdots p^j\mathbb{Z}_p$. Hence a simpler basis of open neighborhoods of any $A \in M_n(\mathbb{Q}_p)$ is $\left\{A + p^j M_n(\mathbb{Z}_p) : j \in \mathbb{Z}\right\}$. Note each of these sets are also closed.

Next, let's determine a basis of open neighborhoods around 1 for $\mathrm{GL}_n(\mathbb{Q}_p)$. By the above remarks, one basis is

$$\left\{(1 + p^j M_n(\mathbb{Z}_p)) \cap \mathrm{GL}_n(\mathbb{Q}_p) : j \in \mathbb{Z}\right\}.$$

In fact, it suffices to restrict j to be sufficiently large, say $j \geq 1$. For $j \geq 1$, put

$$K_j = (1 + p^j M_n(\mathbb{Z}_p)) \cap \mathrm{GL}_n(\mathbb{Q}_p).$$

Then $\{K_j\}$ is a basis of open (and closed) neighborhoods of 1 in $\mathrm{GL}_n(\mathbb{Q}_p)$

It is a general fact that if a space has a basis of open neighborhoods which are also closed, then the space is totally disconnected.

In fact, these sets K_j have more structure.

Exercise 2.2.6. *Show K_j is an open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Further show, that*

$$K_j \subseteq K_0 := \mathrm{GL}_2(\mathbb{Z}_p) = \{g \in \mathrm{GL}_n(\mathbb{Z}_p) \cap M_n(\mathbb{Z}_p) : |\det g|_p = 1.\}.$$

Note that K_0 is an open compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. To see that it is open (and closed), note that $M_n(\mathbb{Z}_p)$ is open (and closed) in $M_n(\mathbb{Q}_p)$, so its restriction to $\mathrm{GL}_n(\mathbb{Q}_p)$ is also. Next observe that the preimage of $(1/p, p)$ under the map $\det : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}^\times$ is the open (and closed) set $\{g \in GL_n(\mathbb{Z}_p) : |\det g|_p = 1\}$. Thus K_0 is open (and closed).

To see that K_0 is compact, observe it is the restriction of the compact subgroup $M_n(\mathbb{Z}_p)$ of $M_n(\mathbb{Q}_p)$ to the closed subset $\{g \in M_n(\mathbb{Q}_p) : |\det g|_p = 1\}$. One can similarly see each K_j is compact, and we have a family of inclusions

$$\cdots \subset K_2 \subset K_1 \subset K_0.$$

It turns out that K_0 is a maximal compact open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Maximal compact open subgroups play an important role in our theory. In the case of $\mathrm{GL}_n(\mathbb{Q}_p)$, there is only one maximal compact open subgroup, up to conjugacy, but this is not true for other algebraic groups. This is one feature that makes the theory simpler for $\mathrm{GL}_n(\mathbb{Q}_p)$.

Definition 2.2.7. *An l-**group** is a locally compact totally disconnected Hausdorff topological group.*

Example 2.2.8. *Since $\mathbb{Q}_p$ is locally compact Hausdorff, $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$ is also. Since an open subset of a locally compact space Hausdorff is locally compact Hausdorff, so is $\mathrm{GL}_n(\mathbb{Q}_p)$. Consequently, $M_n(\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$ are l-groups. Note that $M_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$ are not, since $\mathbb{R}$ is not totally disconnected.*

Exercise 2.2.9. *Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and consider the subgroups $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Show P, A and N are l-groups.*

In fact, all of the matrix groups we considered last section, taken over $F = \mathbb{Q}_p$, are l-groups.

2.3 Smooth representations

The representation $\pi : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{C})$ given by inclusion is a perfectly reasonable one, in that it should be continuous and smooth (as a map of real manifolds). That is, varying $g \in \mathrm{GL}_2(\mathbb{R})$ continuously or smoothly does the same thing to $\pi(g)$. A less trivial example is the symmetric square representation $\mathrm{Sym}^2 : \mathrm{GL}_2(F) \to \mathrm{GL}_3(F)$ given in Example 1.2.14.

On the other hand, one can take the metric completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and this is abstractly isomorphic to $\mathbb{C}$ as a field. Hence (with the axiom of choice), one has an injective ring homomorphism $\iota : \mathbb{Q}_p \to \mathbb{C}$. This extends to an homomorphism $\pi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{C})$, which is algebraically a representation, but not one with any utility, for it completely disregards the metric/topological structures of both $\mathbb{Q}_p$ and $\mathbb{C}$.

Therefore, when dealing with a topological group G, it makes sense to consider **continuous representations.** Namely, suppose V is a finite-dimensional vector space over C. As above, one can give $\mathrm{GL}(V)$ which is isomorphic to $\mathrm{GL}_n(\mathbb{C})$ for some n, a topology and consider continuous homomorphisms $\pi : G \to \mathrm{GL}(V)$.

If G is not compact, then there will be infinite-dimensional irreducible representations of G, so one needs a way to put a topology on $\mathrm{GL}(V)$ for infinite-dimensional V (which one often takes to be Hilbert or at least Banach, and replaces $\mathrm{GL}(V)$ with the group of invertible bounded linear operators). These ideas are at the core of functional analysis.

However, for l-groups, it turns out that the stronger notion of *smooth representations* is easier to define. While it is easy to see what it means for a function of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ to be smooth—namely, being smooth (real or complex) in each coordinate—it is less obvious what smoothness should mean for a function of $\mathbb{Q}_p$, let alone $\mathrm{GL}_2(\mathbb{Q}_p)$. Though not obvious, the correct definition is quite simple.

For the rest of this chapter, unless otherwise stated, **we assume G is an l-group.**

Definition 2.3.1. *Let $G = \mathbb{Q}_p$, or more generally any l-group. Then a function $f : G \to \mathbb{C}^n$ is* **smooth** *if it is locally constant, i.e., given any $x \in G$, there exists an open neighborhood U of x such that $f(y) = f(x)$ for all $y \in U$.*

From the point of view of functional or harmonic analysis, this really is the right analogue of smooth functions on $\mathbb{R}$ or $\mathbb{C}$. While we will not discuss this now, here is another characterization of smoothness.

Proposition 2.3.2. *Let G be an l-group, and $f : G \to \mathbb{C}^n$. Then f is smooth if and only if for any subset $U \subseteq \mathbb{C}^n$, the preimage $f^{-1}(U)$ is open in G.*

Proof. First suppose f is smooth. Consider the case when $U = U_z = \{z\}$ is a singleton set, let $V = f^{-1}(U)$ and take $x \in V$. Then there is an open neighborhood V_x of x such that $f(y) = a$ for all $y \in V_x$. By definition, $V_x \subseteq V$, so V is open.

For an arbitrary (nonempty) U, $f^{-1}(U)$ is a union of preimages $f^{-1}(U_a)$ of singleton sets, which we now know are open.

Now suppose $f^{-1}(U)$ is open for any $U \subseteq \mathbb{C}^n$. In particular, $f^{-1}(U)$ is open when $U = \{z\}$ is a singleton set. This means if $x \in G$ such that $f(z) = x$, then $f^{-1}(U)$ is an open neighborhood of x on which f is constant. Hence f is smooth. $\square$

In particular, this means f is continuous, and we see that being locally constant is much stronger than just being continuous. Of course, on connected sets, such as $\mathbb{R}$ or $\mathbb{C}$, locally constant just means constant. However, on l-groups, there are a plethora of useful locally constant functions.

Example 2.3.3. *The p-adic absolute value $|\cdot|_p : \mathbb{Q}_p^\times \to \mathbb{R}$ is smooth. To see this, consider any $x \in \mathbb{Q}_p$. Write $|x|_p = p^{-n}$ for some $n \in \mathbb{Z}$. Then the preimage of p^{-n} under the absolute value is the open (and closed) set $p^n\mathbb{Z}_p^\times$.*

Note that $|\cdot|_p$ is not smooth on $\mathbb{Q}_p$ because the preimage of 0 is just a single point, i.e., there is no open neighborhood of 0 on which the absolute value is constant. Just to remark on the analogy with the reals, the usual absolute value on $\mathbb{R}$ is smooth (infinitely differentiable) when restricted to $\mathbb{R}^\times$ but of course not at 0.

Exercise 2.3.4. *Show the map $g \mapsto |\det g|_p : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{R}$ is smooth.*

Identifying $M_m(\mathbb{C})$ with $\mathbb{C}^{m^2}$, we have a notion of $f : G \to M_m(\mathbb{C})$ being smooth, where G is an l-group. It is easy to see that $f : G \to M_m(\mathbb{C})$ being smooth is equivalent to each coordinate function being smooth. Given a finite-dimensional vector space V, we can identify $\mathrm{GL}(V)$ with a (topological) subspace of some $M_m(\mathbb{C})$. Hence we may say $f : G \to \mathrm{GL}(V)$ is a smooth function if it is locally constant.

Thus, at least when V is a finite-dimensional vector space over $\mathbb{C}$, the notion of a smooth representation $\pi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(V)$ of an l-group should mean that f is a homomorphism which is smooth, or equivalently each coordinate function is smooth. There is another way to characterize this condition, which will be more useful for us.

Definition 2.3.5. *Let V be a complex vector space. We say a representation $\pi : G \to \mathrm{GL}(V)$ is* **smooth** *if*

$$Stab_G(v) := \{g \in G : \pi(g)v = v\}$$

is an open subgroup of G for all $v \in V$.

Recall that by Exercise 2.1.11, open subgroups of topological groups are also closed.

Definition 2.3.6. *Let $\pi : G \to \mathrm{GL}(V)$ be a representation of G on a Hilbert space V with inner product $\langle \cdot, \cdot \rangle$. A* **matrix coefficient** *of π is a function $f : G \to \mathbb{C}$ of the form*

$$f(g) = \langle \pi(g)v, v' \rangle$$

for some fixed $v, v' \in V$.

Example 2.3.7. *Let $V = \mathbb{C}^n$ with the standard inner product and $e_1, \cdots, e_n$ the standard basis. For a representation π of G on V, and $g \in G$, let $A = (a_{ij}) = \pi(g)$. Then the matrix coefficient*

$$f_{ij}(g) = \langle \pi(g)e_i, e_j \rangle$$

is simply given by

$$f_{ij}(g) = a_{ji}.$$

Hence this definition of matrix coefficients generalizes the notion for finite-dimensional representations.

Proposition 2.3.8. *Let π be a smooth representation of G on a Hilbert space V, and $f : G \to \mathbb{C}$ be a matrix coefficient. Then f is a smooth function of G.*

Proof. Write $f(g) = \langle \pi(g)v, v' \rangle$ for some $v, v' \in V$. Fix $g \in G$ and let $K = Stab_G(v)$. Then for any $k \in K$, we have

$$f(gk) = \langle \pi(gk)v, v' \rangle = \langle \pi(g)\pi(k)v, v' \rangle = \langle \pi(g)v, v' \rangle = f(g).$$

Since $1 \in K$, this means gK is an open neighborhood of g on which f is locally constant, i.e., smooth. $\square$

This makes our definition of smooth representation seem more reasonable. In particular, it shows that if V is finite dimensional, the smooth representation π is a smooth (locally constant) function from G into $\mathrm{GL}(V)$. In fact these criteria are equivalent, as the following exercise shows.

Exercise 2.3.9. *Let π be a representation of G on $V = \mathbb{C}^n$. Suppose every matrix coefficient of π is a smooth function. Show π is a smooth representation.*

This is in fact true when V is an arbitrary Hilbert space.

Inter(O)mission

From here we followed Section 3 of Murnaghan's notes "Representations of reductive p-adic groups," then went through a good deal of Chapter 6 of Goldfeld and Hundley's new book "Automorphic representations and L-functions for the general linear group, Vol. I."

3 A summary of local representation theory for $\mathrm{GL}(2)$

All representations from now on will be on a complex vector space. Some references are Gelbart (Automorphic forms on adele groups), Bump (Automorphic forms and representations) and Goldfeld–Hundley (Automorphic representations and L-functions for the general linear group, Vol. I).

3.1 The p-adic case

Fix a prime p. We put $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $Z = \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} \right\}$ the center, A the diagonal subgroup, $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$ the standard unipotent, $B = AN$ the standard Borel and $K = \mathrm{GL}_2(\mathbb{Z}_p)$ the standard maximal compact.

Recall that a representation (π, V) of G is called **admissible** if (i) π is smooth, and (ii) for each compact open subgroup K' of G, the set of K'-fixed vectors,

$$V^{K'} = \left\{ v \in V : \pi(k)v = v \text{ for all } k \in K' \right\},$$

is finite dimensional. The local component of an automorphic representation is admissible, so these are the representations we are interested in classifying.

In fact, any smooth irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ is admissible, so the class of irreducible smooth representations is the same as the class of irreducible admissible representations, but this was not known at the time of the classification of the latter set and is inherently needed in the theory we recall below.

3.1.1 Finite-dimensional representations

The finite-dimensional representations are relatively easy: any irreducible smooth finite-dimensional representation of G is 1-dimensional (Schur's lemma), and these are all of the form $g \mapsto \chi(\det g)$ where χ is a smooth 1-dimensional representations, i.e., characters, of $\mathbb{Q}_p^\times$.

The characters of $\mathbb{Q}_p^\times$ can be described as follows. (See, e.g., Paul Sally's article "An introduction to p-adic fields, harmonic analysis and the representation theory of SL_2.")

We can write any $x \in \mathbb{Q}_p^\times$ uniquely as $p^n u$ where $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_{\mathfrak{p}}^\times$ is a unit. This gives an isomorphism $\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}_p^\times$. The characters of $\mathbb{Z}$ are just given by $n \mapsto e^{sn}$ for $s \in \mathbb{C}$, which for our purposes we will rewrite in the form $p^{-ns'}$ where $s' = -s/\ln p$. Hence we can write any character χ of $\mathbb{Q}_p^\times$ as

$$\chi(x) = p^{-ns}\omega(u) = |x|^s \omega(u), \quad (x = p^n u,\, u \in \mathbb{Z}_p^\times)$$

for some $s \in \mathbb{C}$ and ω a character of $\mathbb{Z}_p^\times$.

Any character ω of $\mathbb{Z}_p^\times$ is unitary (has image in S^1). By smoothness (in fact continuity), ω has some higher unit group $\mathbb{Z}_p^{(n)} = 1 + p^n\mathbb{Z}_p$ $(n > 0)$ or $\mathbb{Z}_p^{(0)} = \mathbb{Z}_p^\times = \mathrm{GL}_1(\mathbb{Z}_p)$ in its kernel. Note these subgroups $\mathbb{Z}_p^{(n)}$ for $n \geq 0$ are open compact subgroups of $\mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$, i.e., they are analogous to the family of compact open subgroups K_n of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ we defined earlier. The quotient $\mathbb{Z}_p^\times/\mathbb{Z}_p^{(n)}$ is a finite abelian group, specifically $\mathbb{Z}_p^\times/\mathbb{Z}_p^{(n)} \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$, which has order $p^{n-1}(p-1)$ if $n \geq 1$ (and order 1 if $n = 0$). Hence ω may be viewed as a character of some finite abelian group $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

We say χ has **conductor** $c(\chi) = n$ if n is minimal such that $\mathbb{Z}_p^{(n)}$ is contained in the kernel of ω (or equivalently, of χ). If $c(\chi) = 0$, i.e., $\omega = 1$, we say χ is **unramified**; otherwise χ is **ramified**.

This means the only unramified characters of $\mathbb{Q}_p^\times$ are $|\cdot|_p^s$, which is unitary if and only if s is purely imaginary.

Further for a given conductor n, there are only finitely many possibilities for ω; to be precise $p-2$ possibilities if $n = 1$ and $p^{n-2}(p-1)^2$ if $n > 1$. Again, $\chi(x) = |x|_p^s \omega(u)$ is unitary if and only if $\mathrm{Re}(s) = 1$.

3.1.2 Principal series representations

Let ω_1 and ω_2 be two normalized unitary characters of $\mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ and $s_1, s_2 \in \mathbb{C}$. Then one can consider the characters χ_1 and χ_2 of $\mathbb{Q}_p^\times$ given by

$$\chi_i(x) = \omega_i(x)|x|_p^{s_i}.$$

Consequently, $\chi = (\chi_1, \chi_2)$ extends to a character of the Borel B by

$$\chi\left[\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right] = \chi_1(a)\chi_2(b).$$

We define the **normalized parabolic induction** of χ to be

$$V(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \text{ smooth} \mid f\left[\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right] = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{\frac{1}{2}} f(g) \right\}.$$

Note Goldfeld and Hundley work with the non-normalized induction

$$\mathcal{V}_{nn}(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \text{ smooth} \mid f\left[\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right] = \chi_1(a)\chi_2(b) f(g) \right\}.$$

It is clear one can go between the two via

$$V(\chi_1, \chi_2) = \mathcal{V}_{nn}(\chi_1|\cdot|^{1/2}, \chi_2|\cdot|^{-1/2}).$$

The normalization factor of $|a/b|^{1/2}$ makes relations among and conditions on these representations, as we will note below. Therefore, we will work with the normalized induction from now on, which is standard.

We call $V(\chi_1, \chi_2)$ the **principal series representation** of G induced from (χ_1, χ_2). Here the action of G on $V(\chi_1, \chi_2)$ is given by right translation, i.e.,

$$g \cdot f(x) = f(xg), \quad \text{for } g, x \in G,\ f \in V(\chi_1, \chi_2).$$

For example, one has

Lemma 3.1.1. *The contragredient of $V(\chi_1, \chi_2)$, denoted $\check{V}(\chi_1, \chi_2)$ or $\widetilde{V}(\chi_1, \chi_2)$ is equivalent to $V(\chi_1^{-1}, \chi_2^{-1})$.*

Recall the following

Theorem 3.1.2. *The principal series $V(\chi_1, \chi_2)$ is admissible. It is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|_p^{\pm 1}$.*

If $\chi_1\chi_2^{-1} = |\cdot|_p$, then $V(\chi_1, \chi_2)$ contains an irreducible admissible subspace of codimension 1, called a **special representation**.

If $\chi_1\chi_2^{-1} = |\cdot|_p^{-1}$, then $V(\chi_1, \chi_2)$ contains an invariant 1-dimensional subspace whose quotient is irreducible. This quotient is also called a **special representation**.

(If one works with non-normalized induction for the principal series, the above conditions on $\chi_1\chi_2^{-1}$ become $\chi_1\chi_2^{-1} = |\cdot|_p^2$ and $\chi_1\chi_2^{-1} = 1$.)

Definition 3.1.3. *If $V(\chi_1, \chi_2)$ is irreducible, we write $\pi(\chi_1, \chi_2) = V(\chi_1, \chi_2)$. If $V(\chi_1, \chi_2)$, we denote by $\pi(\chi_1, \chi_2)$ the corresponding special representation.*

Note we can write a special representation in the form $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2})$ for an arbitrary character χ of $\mathbb{Q}_p^\times$. When $\chi = 1$, we call this the **Steinberg representation** St. Then one can identify $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2})$ with a *twisted* Steinberg representation $St \otimes \chi$.

In general, for any representation (π, V) of G and a character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, one can form the twist $(\pi \otimes \chi, V)$ where the action is given by

$$(\pi \otimes \chi)(g)v = \chi(\det g)\pi(g)v.$$

Hence one has that all special representations are obtained as twists of a single one, the Steinberg. Similarly, one has the relation $\pi(\chi_1, \chi_2) \otimes \chi) \sim \pi(\chi_1\chi, \chi_2\chi)$ for twists of principal series, where $\sim$ denotes equivalence.

Theorem 3.1.4. *The irreducible admissible representations $\pi(\chi_1, \chi_2)$ and $\pi(\mu_1, \mu_2)$ (principal series or special) are equivalent if and only if χ_1 and χ_2 equal, in some order, μ_1 and μ_2.*

(This is another statement which is made much nicer by working with normalized induction for the principal series.)

3.1.3 Supercuspidal representations

So now we know three kinds of irreducible admissible representations of G: the 1-dimensionals, the irreducible principal series, and the special representations. There is one more kind: supercuspidal.

To motivate the definition, let us try to imagine proving that all infinite irreducible admissible representations are principal series or special. Let (π, V) be an infinite irreducible admissible representation of G, and consider the subspace

$$V_N = \langle \pi(n)v - v | n \in N, \, v \in V \rangle.$$

It is not hard to see that V_N is invariant under the diagonal subgroup A (in fact, under the Borel). One can then consider the action of B on the quotient

$$V^N = V/V_N,$$

called the **Jacquet module** of V. One can show the Jacquet module is an admissible representation of A, whose dimension is at most 2.

If (π, V) is an irreducible principal series $\pi(\chi_1, \chi_2)$, then the Jacquet module is essentially (χ_1, χ_2). Conversely, whenever the Jacquet module is 2-dimensional, V is a principal series.

If (π, V) is a special representation $\pi(\chi \cdot |\cdot|_p, \chi)$, then the Jacquet module is 1-dimensional and gives back χ. Conversely, whenever the Jacquet module is 1-dimensional, V is a special representation.

There is a third, sneaky possibility—the Jacquet module is *zero*-dimensional!

Definition 3.1.5. *We say an infinite-dimensional irreducible admissible representation (π, V) of G is* **supercuspidal** *if the Jacquet module V^N is 0-dimensional, i.e., if $V_N = V$.*

Exercise 3.1.6. *Let (π, V) be a 1-dimensional representation of G. One can still define the Jacquet module V^N as above. Show V^N is 0-dimensional.*

The Jacquet module, in some sense, gives us the classification of irreducible admissible representations of G (1-dimensional, principal series, special, supercuspidal)—however it may seem unsatisfactory as the supercuspidal guys are essentially *defined* to be the things that aren't one of the types we already know!

The first question to ask would be, do supercuspidal representations exist? The answer is yes, and constructions are known but the theory is more complicated than for principal series. Roughly the idea is that one can induce an irreducible representation of some compact open subgroup K' of G (here one uses "compact induction.") The simplest case comes from taking irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ and lifting them to K via the projection

$$K = \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$$

induced by the isomorphism $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$. These are known as *depth 0* supercuspidal representations.

However, even without knowing the construction of supercuspidal representations (which was not complete at the time of the classification), supercuspidal representations can be shown to have several nice properties (indeed, the classification is not used to show this). For instance, one can put an inner product $\langle \cdot, \cdot \rangle$ on V. Recall a matrix coefficient of (π, V) is a function $f : G \to \mathbb{C}$ given by

$$f(g) = \langle \pi(g)v, v' \rangle$$

for $v, v' \in V$. For supercuspidal π (but not principal series or special representations), the matrix coefficients f have compact support. Further, they are what Harish-Chandra called supercusp forms, i.e.,

$$\int_N f(g_1 n g_2) dn = 0$$

for all $g_1, g_2 \in G$. These turn out to be particularly useful facts, allowing one to prove many things for supercuspidal representations that are not so easy to prove for principal series or special representations.

One should think of supercuspidal representations as the representations that are actually native to GL(2), whereas the principal series and special representations (and 1-dimensionals) all come from representations of GL(1). (Even though supercuspidals can be constructed by induction from subgroups like $K = \mathrm{GL}_2(\mathbb{Z}_p)$, this is still GL(2), just over $\mathbb{Z}_p$ instead of $\mathbb{Q}_p$.)

We remark that this notion still holds when one works with representations of other groups, such as $\mathrm{GL}_n(\mathbb{Q}_p)$: there are "native" representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ which are supercuspidal. Roughly, other representations can be constructed by inducing a representation $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$ of a parabolic $P = MN$, where the Levi subgroup

$$M \simeq \mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \mathrm{GL}_{n_2}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{Q}_p)$$

with $n_1 + n_2 + \cdots + n_k = n$ and ρ_i being a supercuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ (here by a supercuspidal of $\mathrm{GL}_1(\mathbb{Q}_p)$ we just mean a character of $\mathbb{Q}_p^\times$). (To be precise, one should perhaps allow ρ_i to be a *discrete series* representations—which for $\mathrm{GL}_2(\mathbb{Q}_p)$, means supercuspidal or special.)

3.1.4 Classification

Here we summarize the classification. Write $\pi_1 \sim \pi_2$ for π_1 and π_2 being equivalent.

Theorem 3.1.7. *Let π be an irreducible admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Then π is one of the following disjoint types, where χ, χ_1 and χ_2 are arbitrary characters of $\mathbb{Q}_p^\times$.*

(i) irreducible principal series $\pi(\chi_1, \chi_2)$, i.e., $\chi_1\chi_2^{-1} \neq |\cdot|_p^{\pm}$; we have $\pi(\chi_1, \chi_2) \sim \pi(\chi_2, \chi_1)$ and no other equivalences;

(ii) a special representation, which we may write in the form $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2}) = St \otimes \chi$, and $St \otimes \chi \sim St \otimes \chi' \iff \chi = \chi'$;

(iii) a supercuspidal representation;

(iv) 1-dimensional, of the form $\chi \circ \det$.

Recall the **central character** ω_π of π is the character of $Z \simeq \mathbb{Q}_p^\times$ satisfying

$$\pi(zg) = \omega(z)\pi(g), \quad z \in Z,\, g \in G.$$

Because the 1-dimensional representations will not arise as local components of global automorphic representations, we will exclude them in the discussion which follows. One often works with representations of $\mathrm{PGL}_2(\mathbb{Q}_p) = G/Z$. It is easy to see that the irreducible admissible representations of $\mathrm{PGL}_2(\mathbb{Q}_p)$ are same as representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character. We remark that for $\pi = \pi(\chi_1, \chi_2)$ (irreducible principal series or special), $\omega_\pi = \chi_1\chi_2$.

Exercise 3.1.8. *(a) Check that for any representation of G, $\omega_{\pi\otimes\chi} = \chi^2\omega_\pi$.*

(b) Deduce the following corollary.

Corollary 3.1.9. *The irreducible admissible representations of $\mathrm{PGL}_2(\mathbb{Q}_p)$ are of one of the following types*

(i) irreducible principal series $\pi(\chi, \chi^{-1})$ where $\chi \neq |\cdot|_p^{\pm 1/2}$ is an arbitrary character of $\mathbb{Q}_p^\times$;

(ii) a quadratic twist *of Steinberg: $St \otimes \chi$ where $\chi^2 = 1$;*

(iii) a supercuspidal representation of G with trivial central character;

(iv) 1-dimensional, of the form $\chi \circ \det$ where $\chi^2 = 1$.

There is some further classification one can do. For instance, one can consider which representations are unitary.

Definition 3.1.10. *Let $(\pi, V)* be an admissible representation of G. Then π is* **unitary** *(or* **unitarizable***) if there exists a positive-definite invariant Hermitian form on V, i.e., there is a positive-definite Hermitian form $(\,,\,)$ on V such that*

$$(\pi(g)v, \pi(g)w) = (v, w) \quad \textit{for all } g \in G,\, v, w \in V.$$

The above definition also makes sense for representations of $\mathbb{Q}_p^\times$.

Lemma 3.1.11. *A character χ of $\mathbb{Q}_p^\times$ is unitary if and only it is of the form $\omega|\cdot|_p^{ir}$ where $r \in \mathbb{R}$ and ω is a finite order character.*

Proof. First observe a 1-dimensional representation χ is unitary if and only if its image lies in S^1: for $z, w \in \mathbb{C}$, $x \in \mathbb{Q}_p^\times$ and $(\,,\,)$ a Hermitian form on $V = \mathbb{C}$, we have

$$(\chi(x)z, \chi(x)w) = \chi(x)(z, \chi(x)w) = \chi(x)\overline{\chi(x)}(z, w) = |\chi(x)|^2(z, w).$$

Now by the classification of characters of $\mathbb{Q}_p^\times$ given in Section 3.1.1, we can write $\chi = \omega|\cdot|_p^s$ for some ω of finite order and $s \in \mathbb{C}$. Since ω is finite order, it has image in S^1. Therefore χ is unitary if and only if $|\cdot|_p^s$ has image in S^1, which is equivalent to s being purely imaginary. □

Theorem 3.1.12. *Let π be an irreducible admissible representation of G. Then π is unitary if and only if π is one of the following types:*
(i-a) (continuous series) an irreducible principal series $\pi(\chi_1, \chi_2)$ where χ_1, χ_2 are both unitary;
(i-b) (complementary series) an irreducible principal series $\pi(\chi, \overline{\chi}^{-1})$ where $\chi = |\cdot|_p^\sigma$, $0 < \sigma < 1$;
(ii) a special representation with unitary central character; or
(iii) a supercuspidal representation with unitary central character.

Conjecturally, only types (i-a), (ii) and (iii) should occur as local components of automorphic representations. We will say more about this when we move to the global theory.

3.1.5 Conductors

To each type of representation (i)–(iii) in the above theorem is associated some data which is used in connection with the study of modular and automorphic forms.

First we discuss ramification. For $n \geq 0$, let

$$K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in p^n\mathbb{Z}_p \right\}.$$

In particular $K(0) = K$. This is a local (p-adic) analogue of the congruence subgroup $\Gamma_0(N)$, where n is the largest power of p such that $p^n|N$.

Definition 3.1.13. *Let (π, V) be an infinite-dimensional irreducible admissible representation of G. Let $n \geq 0$ be minimal such that $V^{K(n)} \neq \{0\}$. We say the* **conductor** *of π is $c(\pi) = n$. If $c(\pi) = 0$, we say π is* **unramified***; otherwise it is* **ramified**.

The conductor is always finite, and an important fact is that $V^{K(c(\pi))}$ is 1-dimensional. A vector in $V^{K(c(\pi))}$ is called a **new vector** or **new form**, and is analogous to the notion of new forms in the sense of modular forms.

Theorem 3.1.14. *(i) For an irreducible principal series $\pi = \pi(\chi_1, \chi_2)$, $c(\pi) = c(\chi_1) + c(\chi_2)$.*
(ii) For a special representation $St \otimes \chi$, $c(\pi) = 1$ if χ is unramified; otherwise $c(\pi) = 2c(\chi)$.
(iii) If π is supercuspidal, then $c(\pi) \geq 2$.

Corollary 3.1.15. *Let π be an infinite-dimensional irreducible admissible representation. Then π is unramified if and only if π is an* unramified principal series*, i.e., an irreducible principal series $\pi(\chi_1, \chi_2)$ with both χ_1 and χ_2 unramified.*

One reason to understand this is the following: if $f \in S_k(N)$ is a new form, then it gives rise to an irreducible admissible infinite-dimensional local representation π_p for each p. To apply representation theory to modular forms, one wants to understand the representations π_p in the sense of our classification above. For each p, the conductor $c(\pi_p) = n_p$ where n_p is the largest power of p such that $p^{n_p}|N$.

In particular, if $N = 1$, then π_p is an unramified principal series for all p. If $N = p_1 \cdots p_k$ where all the p_j's are distinct, then π_p is either Steinberg or ramified principal series (with conductor 1) for any $p = p_j$, and π_p is an unramified principal series for all other p.

3.1.6 L- and ϵ- factors

To the infinite-dimensional irreducible admissible representations π of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ one can associate certain functions called local L- and ϵ- factors. When patched together these will give global L- and ϵ- factors attached to automorphic representations.

One way to construct the L-factors is as follows. Suppose π has a Kirillov model $\mathcal{K}$. Then for $\phi \in \mathcal{K}$, one can define the **zeta integral**

$$Z(s,\phi) = \int_{\mathbb{Q}_p^\times} \phi(y)|y|_p^{s-1/2} d^\times y.$$

Then the L-factor $L(s,\pi)$ should be defined so it is the "gcd" of the local zeta functions $Z(s,\phi)$ as ϕ ranges over $\mathcal{K}$. More precisely, for each ϕ, there is a polynomial h_ϕ such that $Z(s,\phi) = h_\phi(p^{-s})L(s,\pi)$. In fact, for some ϕ, $h_\phi = 1$. Put another way, for a well chosen ϕ, we have $L(s,\pi) = Z(s,\pi)$.

This is carried out for $\mathrm{GL}_1(\mathbb{Q}_p)$, i.e., for characters of $\mathbb{Q}_p^\times$, in Tate's thesis.

We remark these zeta integrals are analogous to the construction of the completed L-function $\Lambda(s,f)$ of a modular form f via the *Mellin transform*:

$$\Lambda(s,f) = \int_0^\infty f(iy)|y|^s d^\times y.$$

The fact that one needs choose an appropriate $\phi \in \mathcal{K}$ to get the L-function from the zeta integral is analogous to the fact that in the above Mellin transform definition of $\Lambda(s,f)$, one needs to choose f to be, say, a normalized Hecke eigen cusp form to define a nice L-function with an Euler product.

Definition 3.1.16. *For certain irreducible admissible representations π of G, we define the* **local L-factor** $L(s,\pi)$ *as follows.*

(i) For an irreducible principal series $\pi = \pi(\chi_1,\chi_2)$, we set

$$L(s,\pi) = \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})}$$

where $\alpha_i = \chi_i(p)$ if χ_i unramified and $\alpha_i = 0$ if χ_i is ramified.

(ii) For a special representation $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2}) = St \otimes \chi$, we set

$$L(s,\pi) = \frac{1}{1-\alpha p^{-s}}$$

where $\alpha = \chi(p)|p|_p^{1/2} = p^{-1/2}\chi(p)$ if $\chi|\cdot|_p^{1/2}$ is unramified, and $\alpha = 0$ else.

(iii) For π supercuspdial, we set

$$L(s,\pi) = 1.$$

For simplicity, we only define ϵ-factors for $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Definition 3.1.17. *Let ψ be the standard additive character of $\mathbb{Q}_p$, and π be an irreducible admissible representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The* **local ϵ-factor** *$\epsilon(s, \pi, \psi)$ attached to π is*

$$\epsilon(s, \pi, \psi) = \epsilon p^{c(\pi)(1/2-s)}$$

where $\epsilon = \pm 1$. Specifically

(i) if $\pi = \pi(\chi, \chi^{-1})$ is an irreducible principal series, then $\epsilon = \chi(-1)$.

(ii-a) if $\pi = St$, then $\epsilon = -1$.

(ii-b) if $\pi = St \otimes \chi$, χ nontrivial quadratic, then $\epsilon = \chi(-1)$.

3.2 The real case

Now let $G = \mathrm{GL}_2(\mathbb{R})$. Of course, there are no compact open subgroups of G, but a maximal compact subgroup is the orthogonal group $K = \mathrm{O}(2)$. As in the p-adic case, we let $B = AN$ where A is the diagonal subgroup of G and $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \subset G$.

The representation theory for $\mathrm{GL}_2(\mathbb{R})$ (which of course was historically studied before that for $\mathrm{GL}_2(\mathbb{Q}_p)$) largely parallels the representation theory for $\mathrm{GL}_2(\mathbb{Q}_p)$, but the details are quite different due to the very different topologies on these groups. In fact, many problems turn out to be much easier for real groups, whereas others turn out to be much easier for p-adic groups. Nevertheless, Harish-Chandra—who developed much general theory over the reals and p-adics in the 1950's and 1960's—described a philosophy which he called the "Lefschetz principle:" whatever is true for real groups is also be true for p-adic groups, and one should be able to treat them equally. I.e., even though the details are quite different, one should be able to put the theories for $G(\mathbb{Q}_p)$ and $G(\mathbb{R})$ inside a single framework.

In any case, it is not one of our goals to discuss the representation theory for $\mathrm{GL}_2(\mathbb{R})$ in detail. We simply give a summary of facts.

Let (π, V) be a (smooth) representation of G on a Hilbert space V. First we should define admissibility.

In the p-adic case, we defined admissible as the condition that $V^{K'}$ is finite dimensional for any compact open subgroup. Here, we don't have compact open subgroups to work with. Another way to state the p-adic condition is that the restriction $\pi_{K'}$ of π to K' only contains the trivial representation finitely many times. In particular, this means the restriction of π to $\mathrm{GL}_2(\mathbb{Z}_p)$ contains any finite order character $\chi \circ \det$ of $\mathrm{GL}_2(\mathbb{Z}_p)$ at most finitely many times (to see this, restrict further to a compact open subgroup on which $\chi \circ \det$ is trivial).

While one can define admissibility for $G = \mathrm{GL}_2(\mathbb{R})$ in terms of K, it is perhaps simpler to think of it in terms of the compact subgroup

$$K_0 = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < \pi \right\},$$

which has index 2 in $K = \mathrm{O}(2)$. Since $\mathrm{SO}(2)$ is compact and abelian, all its irreducible representations are characters.

Exercise 3.2.1. *Show any continuous character of* $\mathrm{SO}(2)$ *is of the form*

$$\chi_k \left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right] = e^{2\pi i k\theta}$$

for some $k \in \mathbb{Z}$.

Further since $\mathrm{SO}(2)$ is compact, any representation of $\mathrm{SO}(2)$ is semisimple, i.e., decomposes as a direct sum of irreducible representations.

Definition 3.2.2. *We say* π *is* **admissible** *if, for any* $k \in \mathbb{Z}$, *the restriction* π_{K_0} *of* π *to* K_0 *contains* χ_k *with finite multiplicity.*

We first state what the classification looks like. Then we will briefly and informally discuss each type of representation.

Theorem 3.2.3. *Let* π *be an irreducible admissible unitary representation of* $\mathrm{PGL}_2(\mathbb{R})$. *Then* k *is one of the following types*
(i) irreducible principal series;
(ii) an even weight k *discrete series* D_k *for* $k \geq 2$*;*
(iii) finite-dimensional.

One defines principal series as in the p-adic case: begin with two characters χ_1 and χ_2 of $\mathbb{R}$ to give a character of A; extend this to a character of B and induce to G. This is usually irreducible. When it is not, it gives one of the discrete series representations D_k ($k \geq 2$ is an integer). There is also a prinicpal series representation which is in some ways similar to a discrete series—it is called the limit of discrete series and denoted D_1^*. The representations D_k are analogous to the special representations in the p-adic case. (There are no supercuspidal representations in the real case.)

To be a little more precise, there are discrete series of the same weight with different central characters, so one should write something like $D_{k,\omega}$, where ω is the central character. If we fix a central character ω, then the parity of k must be compatible with the central character and there is a unique discrete series $D_{k,\omega}$ of weight k. Specifically, if ω is even ($\omega(-1) = 1$) then k must be even, and if ω is odd ($\omega(-1) = -1$), then k must be odd. This is why one needs the condition of even weight in part (ii) above. The fact that the are no odd weight discrete series with trivial central character is closely related to the fact that that there are the weight of a modular form (with trivial nebentypus character) must be even, which you may recall from last semester.

For simplicity, we describe the discrete series when $\omega = 1$ (so k must be even). Let $\mathcal{H}$ denote the upper half plane. Let V_k be the space of holomorphic square-integrable functions on $\mathcal{H}$ (w.r.t. the hyperbolic measure on $\mathcal{H}$), i.e.,

$$V_k = \left\{ f : \mathcal{H} \to \mathbb{C} \text{ holomorphic} \;\Big|\; \int_{\mathcal{H}} |f(x+iy)|^2 y^k \frac{dx\,dy}{y^2} < \infty \right\}.$$

The discrete series of weight k is the representation (D_k, V_k) given by

$$(D_k(g)f)(z) = \frac{(\det g)^{k/2}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right) \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+.$$

This statement technically needs $g \in \mathrm{GL}_2(\mathbb{R})^+$ (matrices of positive determinant) to make sense as $\mathcal{H}$ is not preserved by the full action of $\mathrm{GL}_2(\mathbb{R})$, but one can extend the above definition to all $g \in \mathrm{GL}_2(\mathbb{R})$ without much difficulty. Observe that when we restrict the discrete series to the subgroup $\mathrm{GL}_2(\mathbb{Z})$, the invariant one dimensional subspaces are the modular forms of weight k. One can show the restriction $D_k|_K$ of the discrete series D_k to $K = \mathrm{SO}(2)$ decomposes as

$$D_k|_K \simeq \bigoplus_{\substack{|j| \geq k \\ j \equiv k \bmod 2}} \chi_j.$$

Finally, we remark that in the real case, irreducible finite-dimensional does not mean 1-dimensional—the standard representation $\rho : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{C})$ is the simplest example. One also has the symmetric powers of the standard representation

$$\mathrm{Sym}^n(\rho) : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_{n+1}(\mathbb{C}).$$

However all irreducible finite-dimensional representations are of the form $\mathrm{Sym}^n(\rho) \otimes (\chi \circ \det)$, where χ is a character of $\mathbb{R}^\times$. (Here $\mathrm{Sym}^1(\rho) = \rho$ and $\mathrm{Sym}^0(\rho)$ is trivial.)

One last thing that we should mention is that one often works with something slightly more general than honest representations of $\mathrm{GL}_2(\mathbb{R})$. Namely, one often works with what are called $(\mathfrak{g}, K)$*-modules*, which are compatible pairs of representations of the Lie algebra $\mathfrak{g}$ of G and of the maximal compact subgroup K of G. Given a unitary representation of $\mathrm{GL}_2(\mathbb{R})$, one naturally gets a pair of representations on $\mathfrak{g}$ and K, which form a $(\mathfrak{g}, K)$-module, though not all $(\mathfrak{g}, K)$-modules are obtained in this way. However the classification for $(\mathfrak{g}, K)$-modules looks the same as the classification for representations of G. Whether one uses $(\mathfrak{g}, K)$-modules or actual representations of G depends upon the model one chooses for automorphic forms. We'll say a little bit about this in the next section.

4 Automorphic Representations

From here, we went through a good part of Gelbart's *Bulletin* article (linked to from the course page).