Chapter 7

Riemann Zeta Function

NOTE: This chapter is just an outline as we didn't have time to cover this this semester.

In 1859, Bernhard Riemann utterly transformed analytic number theory with a 10-page paper on what is now known as the *Riemann zeta function*, his only work in number theory.[1] Here is roughly what I hoped to say about it:

- Explain the definition $\zeta(s) = \sum \frac{1}{n^s}$, which makes sense for complex numbers $s = r + it$ and converges when $r > 1$, but can be extended to an analytic function for $s \neq 1$.
- Explain the *Euler product* $\zeta(s) = \prod \frac{1}{1-p^{-s}}$ (again converging for $r > 1$) and how this is equivalent to the fundamental theorem of arithmetic.
- Explain in more detail Euler's proof of the infinitude of primes, discussed at the end of the introduction, which using the fact that

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

- Explain the *prime number theory*, which tells us about the distribution of primes, and how this is related to $\zeta(s)$. Also explain the relation with the conjectural *Riemann hypothesis*, about where $\zeta(s) = 0$, which is the most famous open problem in number theory now (after the fall of Fermat's last theorem).
- Make some comments about on one hand primes seem to be distributed randomly, but statistically they obey very precise arithmetic laws (e.g., there are asymptotically the same number of primes 1 mod 4 as there are 3 mod 4).

[1] I hope to have a similar impact someday on the world of mathematical comedy. My imagined eulogy: *He invented the 3-minute riff on Pell's equation, and his theoretical work establishing a ring structure on stand-up jokes is now used by lecturers everywhere. He died the way he lived, stabbing by students.*