Questions:

(1) Which of the following are sets:

$A = \{\text{apple}, 7\}$

$B = \{\text{gold medal winners from Rio 2016}\}$

$C = \{\text{good movies in theaters now}\}$

(2) Which of the following sets are the same:

$D = \{1, 2, 3\}$

$E = \{\text{one, two, three}\}$

$F = \{\sum_{i=1}^{n} 1 : n \in D\}$

(3) (Hammack 1.1:15) Write out elements of $\{5a + 2b : a, b \in \mathbb{Z}\}$

(4) (Hammack 1.1:19) Write $\{\ldots, -3, 0, 3, 6, 9, \ldots\}$ in set-builder notation.

(5) (Hammack 1.1:29) Find $|\{\{1\}, \{2, \{3, 4\}\}, \emptyset\}|$.

(6) (Hammack 1.2:9) Sketch in $\mathbb{R}^2$: $\{1, 2, 3\} \times \{-1, 0, 1\}$.

(7) (Hammack 1.2:13) Sketch in $\mathbb{R}^2$: $\{1, 1.5, 2\} \times [1, 2]$.

Takeaways:

- even things written in mathematical notation can be ambiguous
- goal for this course is to *understand* mathematics, not get answers to random problems

Questions:

(0) What is the difference between $x \in A$ and $x \subseteq A$? Can both be true?

(1) (Hammack 1.3: 5, 7) List all subsets of $\{\emptyset\}$ and $\{\mathbb{R}, \{\mathbb{Q}, \mathbb{N}\}\}$.

(2) (Hammack 1.3: 14) True or false: $\mathbb{R}^2 \subset \mathbb{R}^3$. Explain.

(3) Prove or disprove: $\emptyset \subseteq A$ for any set A.

(4) Prove or disprove: $\emptyset \in A$ for any set A.

(5) (Hammack 1.4: 17) Say $|A| = m$. Find $|\{X \in \mathcal{P}(A) : |X| \leq 1\}|$.

(6) Prove $|\mathcal{P}(A)| = 2^{|A|}$ for a finite set A (Fact 1.4).

Takeaways:

- for proofs, start with what you know, which at the beginning is just definitions
- to disprove something, a single counterexample suffices (and is usually required), though an example will not suffice for a proof (but it may give you the idea for a proof)
- details are important because many similar looking objects and notation have differences that matter (these differences are especially important when coding)

Questions:

(1) Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$. Determine the following sets: $A \cap B$, $A \cup B$, $(A \times B) \cap (B \times A)$, $A^2 \cup B^2$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$.

(2) (Hammack 1.5: 9) Is $(\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$? What about $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$?

(3) Let A be a subset of a set X, and $\overline{A}$ the complement in X. Prove that $A \cup \overline{A} = X$.

(4) Draw a Venn diagram for $(A \cup B) \cap (A \cup C)$. Find another way to describe this set in terms of unions and intersections.

(5) Let A be a finite set with $|A| = n$. Find the cardinality of $\{X \in \mathcal{P}(A) : |X| \geq n - 1\}$.

(6) Let A, B be finite sets. What relations can you determine between $|A|$, $|B|$, $|A \cap B|$ and $|A \cup B|$?

Selected Writing Tips from Houston, Chapter 3:

- Write in simple, punctuated sentences
- Keep it simple
- Say what you mean
- Use words over symbols, but use symbols properly
- Proofread, and reflect

Questions: (Write solutions as clearly as possible.)

(1) Find the minimum of $x^3 - 2x^2 + x + 1$ on $[0, 2]$.

(2) Let A be a subset of a set X, and $\overline{A}$ the complement in X. Prove that $A \cup \overline{A} = X$.

(3) Roll two fair six-sided dice. What is the probability their sum is 7?

(4) This class has 36 students. How many ways can we divide the class into 9 groups of 4 students each?

(5) Prove that the square of an odd number is odd.

The Cardano model

Let S be a finite set, which we think of as a set of possible *outcomes* of an experiment (e.g., rolling 3 dice). We call a subset $A \subseteq S$ an **event**, and define its **probability** to be $P(A) := \frac{|A|}{|S|}$. For two events $A, B \subset S$ we also define the **conditional probability** $P(A|B) := \frac{P(A \cap B)}{P(B)}$, read "the probability of A given B."

Questions: For each of these problems, formulate them in terms of events A, B and probabilities $P(A)$ or $P(A|B)$. (Answer them too.)

(A) What is the probability a coin flip will come up heads given that the previous 5 flips were heads?

(B) Roll two fair dice. What is the probability the sum is 7? What is the probability the sum is 6 or 8?

(C) Say you have a box with 5 red balls and 3 black balls, and you remove two balls without looking. What is the probability these balls have the same color? Different color?

(D) Repeat the above question with the assumptions that you first draw one ball, replace it, and then draw the second ball.

(E) Say there are 32 people in the class, comprising 24 males and 8 females, and I randomly assign you into 8 groups of 4. In any given group what is the most likely male:female ratio? If one group has 1 male and 3 females, what is the most likely male:female ratio in any other group?

Think about the suggestions for problem solving in Chapter 5 of Houston, and apply them to the following:

Let $\mathcal{P}_{fin}(X)$ denote the set of finite subsets of X. Let $A, B \in \mathcal{P}_{fin}(\mathbb{Z})$. Which of the following imply $A = B$? Which do not?

(A) $A \times B = B \times A$

(B) $A - B = B - A$

(C) $A^2 = B^2$

(D) $A \subset B$ and $\mathcal{P}(B) \subset \mathcal{P}(A)$

(E) $A \subset B$ and $|B| \leq |A|$

(F) $A \cup C = B \cup C$ for some $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(G) $A \cup C = B \cup C$ for infinitely many $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(H) $A \cup C = B \cup C$ for all $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(I) $A - C = B - C$ for some $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(J) $A - C = B - C$ for infinitely many $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(K) $A - C = B - C$ for all $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(L) $A \times C = B \times C$ for some $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(M) $A \times C = B \times C$ for infinitely many $C \in \mathcal{P}_{fin}(\mathbb{Z})$

(N) $A \times C = B \times C$ for all $C \in \mathcal{P}_{fin}(\mathbb{Z})$

Instructions: Circle T (true) or F (false) for each question. No work is needed. In this section, A, B denote sets and P, Q denote statements, x is a real number and $f : \mathbb{R} \to \mathbb{R}$ is a function.

1. T F $\exists A, \emptyset \in A$
2. T F $\forall A, \emptyset \in A$
3. T F $\exists A, \emptyset \subseteq A$
4. T F $\forall A, \emptyset \subseteq A$
5. T F $A \subseteq B \iff A \in \mathcal{P}(B)$
6. T F $A \times B = B \times A$
7. T F $A \cup (B \cup C) = (A \cup B) \cup C$
8. T F $\mathbb{R} \subseteq \mathbb{R}^2$
9. T F $\mathbb{R} \in \mathbb{R}^2$
10. T F $(\mathbb{Z} \times \mathbb{Z}) - (\mathbb{N} \times \mathbb{N}) = (\mathbb{Z} - \mathbb{N}) \times (\mathbb{Z} - \mathbb{N})$
11. T F $\sim (P \implies Q) \ = \ (\sim P \implies \sim Q)$
12. T F $P \implies Q \ = \ (\sim P) \vee Q$
13. T F $(P \implies Q) \ = \ (\sim Q \implies \sim P)$
14. T F $(P \wedge Q) \implies \sim (P \implies Q)$
15. T F $(P \iff Q) \implies (Q \implies P)$
16. T F $\sim (P \wedge Q) \ = \ (\sim P) \wedge (\sim Q)$
17. T F $\sim (\forall x > 0, f(x) > 0) = \forall x > 0, f(x) \leq 0$
18. T F $\sim (\exists x > 0, f(x) \in \mathbb{Q}) = \forall x > 0, f(x) \notin \mathbb{Q}$

Questions:

1. Let's say a restaurant has 7 main courses: 3 fish dishes and 4 steak dishes. They also have 9 wines: 4 whites and 3 reds. How main ways can you pair a wine with a main course? What if you require that a white wine goes with a fish and a red wine goes with a steak?

2. Let's say you have a date at the restaurant in the previous problem. How many ways can you and your date each select one wine and one main such that neither of you select any items in common?

3. How many ways can you order the letters ABCDEF such that A always comes before F?

4. How many ways are there to order the letters ABCDE such that

   (a) the two vowels are not adjacent? or

   (b) all three consonants are not in a row?

5. Compute $\binom{n}{k}/\binom{n}{k-1}$ for $1 \leq k \leq n$.

6. Let $n, k \in \mathbb{Z}$. What does the statement $\binom{n}{k} = \binom{n}{n-k}$ mean? Explain why it's true.

7. Let $n, k \in \mathbb{Z}$. What does the statement $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ mean? Explain why it's true.

8. Find a formula for the alternating sum of the n-th row of Pascal's triangle. Justify your formula.

9. What is the coefficient of $x^{98}y^2$ in $(x+y)^{100}$?