Linear Algebra (MATH 3333) Spring 2011 Section 4
Midterm Practice Problem Solutions

Throughout this set, V denotes a vector space. All of the exercises are fundamental, though for preparation purposes, I've included many more than I will be able to fit on an exam. I've starred a few which I think you should pay special attention to.

1. Go over the problems on your quiz and previous homeworks, and make sure you can do them correctly.

Done.

True/False

Circle T or F.

2. T F Any two nonzero vectors in $\mathbb{R}^3$ are linearly independent.

False. They could be scalar multiples of each other.

3. T F A minimal spanning set for V is a basis for V.

True. A minimal spanning set means it is LI.

4. T F Any subspace of $\mathbb{R}^2$ is either a line through the origin or $\mathbb{R}^2$.

False. These are all the $1-$ and $2-$ dimensional subspaces, but there is also the 0-dimensional subspace, i.e., just the origin.

5. T F The span of two nonzero vectors is either a line through the origin or a plane through the origin.

True. The span must either be a 1- or 2- dimensional subspace of $\mathbb{R}^2$.

6. T F The set of polynomials in x of degree at most 5 form a vector space.

True. This example was in the text and in lecture.

7. T F There is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ whose image is the cone $x^2 + y^2 = z^2$.

False. The image is not a subspace of $\mathbb{R}^3$.

8. T F There is a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ whose image is the line $y = x + 1$.

False. The image is not a subspace of $\mathbb{R}^3$.

9. T F There is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ whose image is the plane $z = x + y$.

True. For example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ works, since the columns form a basis for this plane.

10. T F There is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ whose image is $\mathbb{R}^3$.

False. The rank (dimension of the image = 3 here) must be at most the dimension of the domain (2 here).

Questions

11. If $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, define span$(S)$.

Answer in words: the set of all linear combinations of $v_1, \ldots, v_k$.
Answer in symbols: $\text{span}(S) = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_1, \ldots, a_k \in \mathbb{R}\}$.

12. With S as above, define what it means for S to be a basis of V.

$\mathrm{span}(S) = V$ and S is LI.

13. With S as above, define what it means for S to be linearly independent.

The equations

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

has only the trivial solution

$$a_1 = a_2 = \cdots = a_k = 0.$$

14. Find two different bases for $\mathbb{R}^2$ (no proof needed).

For example $S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17026 \\ 1 \end{pmatrix} \right\}$.

15. Consider the basis $S = \{t^2 + 1, t + 1, 3t^2 - t\}$ for the space of polynomials of degree at most 2. If $[v]_S = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, find v.

$v = 3(t^2 + 1) + 2(t + 1) - 1(3t^2 - t) = 3t + 6.$

16. Consider the basis $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$. If $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, find $[v]_S$.

We want to write $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, i.e., solve $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 6 \end{array}\right)$. Solving this, we see $a = \frac{8}{3}$ and $b = \frac{2}{3}$, so $[v]_S = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix}$.

Problems

Show your work.

17. Show that the set of vectors of the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ in $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$. Find a basis for this space (no proof needed). What is it's dimension? Describe this space geometrically.

Let $W = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} \right\}$. Let $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ a'+b' \end{pmatrix} \in W$ and $c \in \mathbb{R}$ Then

$$\begin{pmatrix} a \\ b \\ a+b \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ a'+b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \\ (a+a')+(b+b') \end{pmatrix} \in W$$

and

$$c \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ ca+cb \end{pmatrix} \in W,$$

i.e., W is closed under addition and scalar multiplication, and therefore a subspace of $\mathbb{R}^3$.

Alternatively, observe $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, i.e., $W = \mathrm{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ and therefore is a subpace of $\mathbb{R}^3$. (Recall $\mathrm{span}(S)$ is always a subspace.)

The above shows $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is a basis for W. Its dimension is 2, and geometrically it is the plane in $\mathbb{R}^3$ determined by $\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\end{pmatrix}$, and $\begin{pmatrix}0\\1\\1\end{pmatrix}$. A nicer way to describe geometrically is simply as the plane $z = x + y$ (W is the set of vectors whose 3rd coordinate is the sum of the first 2.)

18. Do the same as the previous problem for the subset $\{(x, y, z) : x + y + z = 0\}$ of $\mathbb{R}^3$.

Similar to above, I'll omit the solution, and just observe you can write this as the space $W = \left\{\begin{pmatrix}a\\b\\-a-b\end{pmatrix}\right\}$.

19. Is $\{(x, y, z) : 2x - 3y + z = 1\}$ a subspace of $\mathbb{R}^3$?

No. It does not contain the origin.

20.* Let $A = \begin{pmatrix}1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & 1\end{pmatrix}$.

(a) Find a basis for the image of A.
(b) Find a basis for the kernel of A.
(c) Determine rank A and nullity A.

It row reduces to the identity, so all columns are linearly independent. Recall for this, and subsequent problems, the image of A is just the span of the columns of A.

Therefore

(a) Any basis for $\mathbb{R}^3$ is a basis for A, e.g., the standard basis, or $\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$.

(b) By the rank nullity theorem (or the row reduced form) the kernel is trivial, i.e., $ker A = \{0\}$, so a basis is the empty set.

(c) $\text{rank}(A) = 3$ and $\text{nullity}(A) = 0$.

Note: I believe we didn't actually say the basis or $\{0\}$ is the empty set in class, so it probably wouldn't be on the exam like this. I meant to give a matrix of rank 2. So on the exam, I might either ask just for the kernel (not a basis) or make sure the kernel is nontrivial.

21. Do the same as the previous problem for $A = \begin{pmatrix}1 & 1 & 2 & -1\\2 & -2 & 1 & 1\\-2 & -2 & -1 & 1\\3 & 3 & 0 & -1\end{pmatrix}$.

It row reduces to

$$\begin{pmatrix}1 & 1 & 2 & -1\\0 & 1 & \frac{3}{4} & -\frac{3}{4}\\0 & 0 & 1 & -\frac{1}{3}\\0 & 0 & 0 & 0\end{pmatrix}.$$

Therefore

(a) the first three columns of A form a basis for the image;

(b) the kernel is the set of $\begin{pmatrix}x & y & z & w\end{pmatrix}$ with w free, $z = \frac{1}{3}w$, $y = \frac{1}{2}w$ and $x = -\frac{1}{6}w$, i.e., $\ker A = \left\{w\begin{pmatrix}-1/6\\1/2\\1/3\\w\end{pmatrix}\right\}$ so a basis is $\begin{pmatrix}-1/6\\1/2\\1/3\\w\end{pmatrix}$; and

(c) $\text{rank}(A) = 3$ and $\text{nullity}(A) = 1$.

22. Find a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ whose image is the set of vectors of the form $\begin{pmatrix} a \\ 2b-a \\ b \end{pmatrix}$.

Note $\begin{pmatrix} a \\ 2b-a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, i.e., we want the image to be the span of $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. Hence

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$$

does the job.

23. Find a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose image is the plane $z = x + y$.

We need a basis for this plane, so take two vectors in the plane which are not LI, say $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Now we need a 3×3 matrix who columns spans this plane, e.g.,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(Remark: this plane is the space from #17.)

24.* Let $u, v \in V$. Show $\{u, v\}$ is linearly dependent if and only if $u = cv$ or $v = cu$ for some $c \in \mathbb{R}$.

Proof. ($\Rightarrow$) Suppose $\{u, v\}$ is linearly dependent. This means $au + bv = 0$ for a, b not both 0. If $a \neq 0$ then $u = -\frac{b}{a}v$. Otherwise, $a = 0$ so $bv = 0$ with $b \neq 0$, i.e., $v = 0$ so $v = 0 \cdot u$.

($\Leftarrow$) Now suppose $u = cv$ or $v = cu$. If $u = cv$, then $u + (-c)v = 0$, i.e., u and v are linearly dependent. The case $v = cu$ is similar.

25. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose A is a projection (i.e., its image is a line or a point). Show $\det A := ad - bc = 0$.

Proof. If A is a projection, this means its rank is 0 or 1, i.e., the columns of A are linearly dependent. If either column is all zeroes, it is clear $\det A = 0$. Otherwise $\begin{pmatrix} a \\ b \end{pmatrix} = k\begin{pmatrix} c \\ d \end{pmatrix}$ for some k. Then $\det(A) = ad - bc = a(kb) - b(ka) = 0$.

26.* Let $A : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation. Suppose the kernel of A is a plane in $\mathbb{R}^4$. What can you say about the image of A?

$\ker(A)$ being a plane means $\text{nullity}(A) = 2$, so by the Rank-Nullity Theorem, $\text{rank}(A) = 2$, i.e., the image of A is a plane (through the origin) in $\mathbb{R}^3$.