Linear Algebra (MATH 3333 – 04) Spring 2011
Final Exam Practice Problems

Instructions: Try the following on your own, then use the book and notes where you need help. Afterwards, check your solutions with mine online. For Sections 1 and 2, no explanations are necessary. For all other problems, justify your work.

I highly recommend you make sure you can do all of these problems, as well as the Exam 1, Exam 2, and practice midterm problems, *on your own* before the final exam.

Note: Not every problem on the practice sheet is modeled off of one of your problems for homework. However, you can figure out how to do them with an understanding of the basic concepts from the course. They are designed to help piece together your understanding of the course material. There may be questions.

1. Go over your old exams, Homeworks 9–11, and the practice midterm problems.

1 True/False

In this section A is an $n \times n$ matrix.

2. T F Two vectors are linearly independent if one is not a scalar multiple of the other.

3. T F Every 2×2 matrix is diagonalizable.

4. T F If A is diagonalizable, then there is a basis of eigenvectors of A.

5. T F If λ is an eigenvalue for A, then the eigenvectors with eigenvalue λ are scalar multiples of each other.

6. T F If $A = PDP^{-1}$, then $A^3 = P^3D^3P^{-3}$.

7. T F $A = P_{T \leftarrow S}[A]_T P_{S \leftarrow T}$ where S is the standard basis for R^n.

8. T F If A does not have n distinct eigenvalues, then A is not diagonalizable.

9. T F There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is the same as its kernel.

10. T F The set of all even degree polynomials (including 0) is a vector space.

11. T F If $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$ and $\text{span}(S) = \mathbb{R}^n$ then S is a basis for $\mathbb{R}^n$.

12. T F A is invertible if and only if $\det(A) \neq 0$.

13. T F If A is diagonalizable, then A is invertible.

14. T F If $\text{rank}(A) = n$, then A is invertible.

15. T F if $Av = cv$ for some $c \in \mathbb{R}$, then v is an eigenvector for A.

2 Short Answer

16. State the definition for a subset $S = \{v_1, \ldots, v_k\}$ of a vector space to be linearly independent.

17. State the definition of a basis for a vector space V.

18. State the definition of an eigenvalue and an eigenvector for an $n \times n$ matrix A.

19. State the Rank–Nullity Theorem.

20. State three things linear algebra has applications to.

21. Find a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^3$ whose image is the plane $x + 2y + 3z = 0$.

22. What is the geometric significance of $\det(A)$ for a 2×2 matrix A?

23. Why might you want to exponentiate a matrix?

24. If A is a 2×2 matrix such that $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, what is A?

25. Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

26. Give an example of a 3×3 matrix with only 2 eigenvalues which is not diagonalizable.

27. Give an example of a 3×3 matrix with 2 eigenvalues which is diagonalizable.

3 Problems

28. Find all solutions to the following system of equations:

$$\begin{aligned} -2x + y + z &= 1 \\ x + z &= 0 \\ x + y - 2z &= -1. \end{aligned}$$

29. Is $\{(x, y, z) : 2x - 3y = 1 + z\}$ a subspace of $\mathbb{R}^3$. Justify your answer?

30. (i) Show $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

    (ii) If $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, find $[v]_T$.

    (iii) If $[v]_T = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, find v.

    (iv) If S is the standard basis, find the transition matrices $P_{S \leftarrow T}$ and $P_{T \leftarrow S}$.

31. Let $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}$.

(i) FInd a basis for the image of A.

(ii) Find a basis for the kernel of A.

(iii) Find $\text{rank}(A)$ and $\text{nullity}(A)$.

32. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Show $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A.

33. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$.

(i) Find the eigenvectors and eigenvalues of A.

(ii) Diagonalize A, i.e., write $A = PDP^{-1}$ for some diagonal matrix D.

34. Construct a matrix A which acts as reflection about the plane $x + y + z = 0$ in $\mathbb{R}^3$.

35. Suppose you have a (discrete) dynamical system given by

$$\begin{aligned} x(t+1) &= x(t) + 2y(t) \\ y(t+1) &= 4x(t) + 3y(t), \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = 1$. Find explicit formulas for $x(t)$ and $y(t)$.

36. Suppose v is an eigenvector for an $n \times n$ matrix A with eigenvalue λ.

(i) Show cv is also an eigenvector with eigenvalue λ for any $c \neq 0$.

(ii) Show v is also eigenvector for A^2 with eigenvalue λ^2.