1. Prove the following “Limit Location Theorem”: Suppose that \( \{a_j\} \) is convergent and bounded above by \( M \). Then the limit \( L \) is less than or equal to \( M \). (In fact, the sequence only needs to be eventually bounded; i.e. there exists \( N \in \mathbb{N} \) such that for all \( j > N \), \( a_j \leq M \).)

2. Suppose the terms of the sequence \( \{a_j\} \) are colored using red, green, and blue, where each sequence element is given exactly one color and each color is used infinitely often. Form the subsequences \( \{a_{jr}\} \) consisting of all the red-colored elements, \( \{a_{jg}\} \) = all the green-colored elements, and \( \{a_{jb}\} \) = all the blue-colored elements. Prove that if each of the subsequences formed by the colors converges to the same limit \( L \), then \( \{a_j\} \) converges to \( L \).

3. Suppose that \( \{a_j\} \) converges to \( L \) and \( a_j \geq 0 \) for all \( j \in \mathbb{N} \). Prove that the sequence \( \{\sqrt{a_j}\} \) converges to \( \sqrt{L} \). (Hint: start by proving the statement for \( L = 0 \).)

4. Suppose that \( a_1 \geq 0 \) and we define the sequence \( \{a_j\} \) recursively by

\[
a_{j+1} = \sqrt{2 + a_j}.
\]

For example, let \( a_1 = 7 \). Then the sequence starts out:

\[
7, 3, \sqrt{5}, \sqrt{2 + \sqrt{5}}, \ldots
\]

Prove that if \( \{a_j\} \) converges, the limit \( L \) must be equal to 2, regardless of the starting value \( a_1 \). (Hint: Don’t use the definition of the limit, use #1, 3 from this worksheet!)