Let $f : \mathbb{R} \to [0, \infty)$ be a function that satisfies the property that for all $x, y,$

$$f(x + y) = f(x)f(y).$$

1. Prove that for all $n \in \mathbb{N}$ that $f(nx) = [f(x)]^n.$

2. Prove that if there is a single value $x$ for which $f(x) \neq 0$, then $f(0) = 1.$

3. Prove for all $k \in \mathbb{Z}$ that $f(kx) = [f(x)]^k.$

4. Prove for all rational values $r \in \mathbb{Q}$ that $f(rx) = [f(x)]^r.$ Hint: Start with $r = \frac{1}{k}.$

5. Prove that $f$ is continuous at 0 if and only if $f$ is continuous on $\mathbb{R}.$

6. Prove that if $f(x)$ is continuous at 0, then there is some positive real number $a$ such that

$$f(x) = a^x$$

for all $x \in \mathbb{R}.$