MATH 2433 Homework 1

1. The sequence (a_i) is defined recursively by

$$a_1 = 4$$
$$a_{i+1} = 3a_i$$

find a closed formula for a_i in terms of i.

2. In class we showed that the Fibonacci sequence (a_i) defined by $a_i = a_{i-1} + a_{i-2}$ satisfies

$$a_i - \frac{1+\sqrt{5}}{2}a_{i-1} = \frac{1-\sqrt{5}}{2}(a_{i-1} - \frac{1+\sqrt{5}}{2}a_{i-2})$$

Please argue that

$$a_i - \frac{1+\sqrt{5}}{2}a_{i-1} = (\frac{1-\sqrt{5}}{2})^{i-2}(a_2 - \frac{1+\sqrt{5}}{2}a_1)$$

(Optional for 1 bonus point): suppose you also know that

$$a_i - \frac{1-\sqrt{5}}{2}a_{i-1} = (\frac{1+\sqrt{5}}{2})^{i-2}(a_2 - \frac{1-\sqrt{5}}{2}a_1)$$

argue that the close formula for each term is given by

$$a_i = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^i - (\frac{1-\sqrt{5}}{2})^i)$$

3. Argue that for the sequence (a_i) given by

$$a_i = (-1)^i\frac{1}{i}$$

there is a tail of the sequence in the 0.000345-neighbourhood of 0 (please give the starting term of such a tail.)

4. Argue further that the sequence in Problem 3 has limit 0 by using the definition of limits.

5. Show that the following sequence has limit 0, without using the definition of limits.

$$a_n = \sqrt{n^2+1} - n$$

6. Let (a_i) be the sequence given by

$$a_1 = 1$$
$$a_i = \sqrt{a_{i-1}+2}$$

Argue that $\lim_{i\to\infty} a_i = 2$ without using the definition of limits.

7. Let $\sum_{i=1}^{\infty} a_i$ be the geometric series where the terms are given as follows:

$$(a_i) = (9, 3, \frac{1}{3}, \frac{1}{9}, \dots)$$

Show that $\sum_{i=1}^{\infty} a_i = \frac{27}{2}$, WITHOUT using the formula $\frac{a_1}{1-r}$ we gave in class.

(The homework is complete, it is due Friday, June 9th.)

MATH 2433 Homework 2

1. Show that the following series is divergent

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

2. Show that the following series is convergent and find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

3. Show that the following series is convergent and find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

4. Using the integral test (without using the p-series test,) show that the following series is divergent

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$$

5. Using the integral test, show that the following series is convergent

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

6. Using the integral test, show that the following series is convergent

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$$

7. Using the comparison test, show that the following series is convergent

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3+2n}$$

8. Using the limit comparison test, show that the following series is divergent

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}(n+2)}{\sqrt{n^4+8}}$$

9. Using the limit comparison test, show that the following series is divergent

$$\sum_{n=1}^{\infty} \frac{7^n + n^7}{\sqrt{16^n+3}}$$

10. Let (a_i) and (b_i) be two sequences, where $\lim_{i\to\infty} a_i$ does not exist, $b_i > 0$ for all i's and $\lim_{i\to\infty} b_i = 3$. Argue that the sequence $(c_i) = (a_i b_i)$ does not converge.

(The homework is closed, it is due Thusday, June 15th.)

MATH 2433 Homework 3

1. Test if the following series is convergent

$$\sum_{n=1}^{\infty} \frac{n^{2n} \ln n}{(2n)!}$$

2. Find the interval of convergence for the following power series in x

$$\sum_{n=1}^{\infty} \frac{x^{3n}\sqrt{n+1}}{4^n + 5^n}$$

3. Find the interval of convergence for the following power series in x

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{2n}}{(n^2+3)\ln n}$$

4. Let $\sum_{n=1}^{\infty} c_n x^n$ be the series with

$$c_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

show that the series is convergent at $x = \frac{3}{4}$ and divergent at $x = \frac{5}{4}$.

5. Show that the following series is convergent only at $x = 0$, that is, it is divergent everywhere else.

$$\sum_{n=1}^{\infty} n^n x^n$$

6. Show that the following series is convergent for all real values of x.

$$\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

(The homework is closed. It is due Friday, June 23th.)

MATH 2433 Homework 4

1. Show that if the series

$$\sum_{n=0}^{\infty} c_n x^n$$

satisfies

$$\lim \frac{c_{n+1}}{c_n} = 4$$

Then the series

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} c_n x^n$$

is convergent on the interval $(-\frac{1}{4}, \frac{1}{4})$.

2. Find the power series representation for

$$\frac{1}{(1+x)^3}$$

3. Find the power series representation for

$$\ln|1+x^2|$$

4. Find the power series representation for

$$(x+1)\arctan x$$

5. Derive the Maclaurin series expansion for $f(x) = \cos x$ and write it in a condensed form.

6. Derive the Maclaurin series expansion for $f(x) = e^x \sin x$ and write it in a condensed form.

7. Use the Maclaurin series expansion for $\sin x$, calculate the Maclaurin series expansion for the following function.

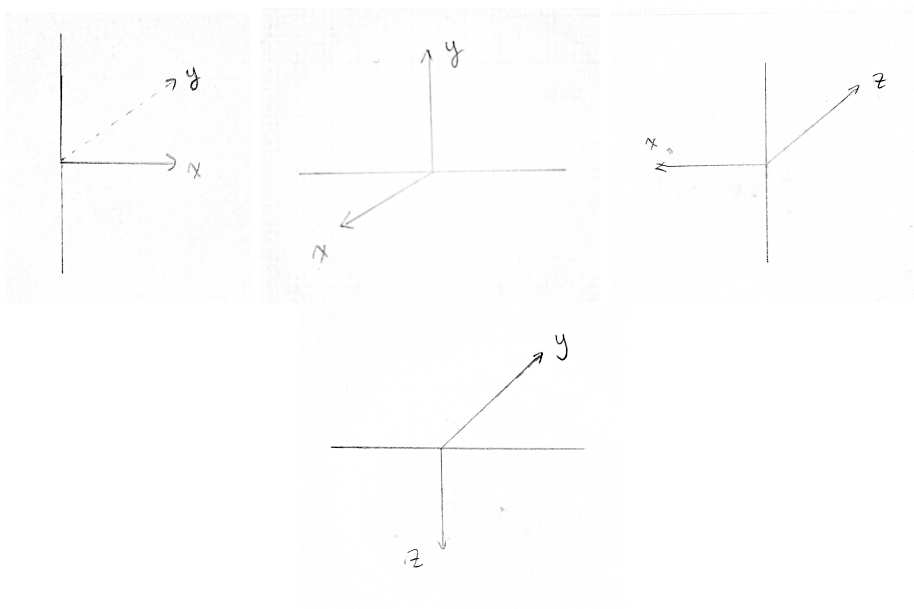
$$\int \sin(x^3)\mathrm{d}x$$

8. If $y = \sum_{n=1}^{\infty} c_n x^n$ is the Maclaurin series expansion for the solution of the following equation

$$y' + y = 1$$

Find c_0, c_1, c_2 and guess a formula for c_n.

9. For the following four coordinate systems, indicate the positive direction of the missing axis that obeys the right hand rule.



(the y-axis is pointing inwards in the last picture.)

10. Draw the point $(1, 3, -2)$ in the three dimensional coordinate system, together with all the lines parallel to the axes that indicate the projections.

11. Express the vector $\mathbf{v} = (-2, -3, 11.5)$ as a linear combination of the following vectors

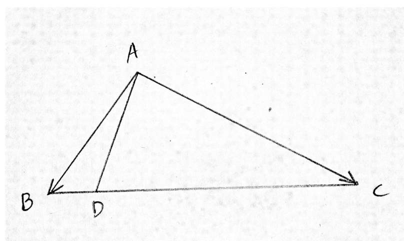
$$\mathbf{u} = (4, -10, 3)$$
$$\mathbf{w} = (-2, 1, 5)$$

12. Express the vector $\mathbf{v} = (4, 9, 11)$ as a linear combination of the following vectors

$$\begin{aligned} \mathbf{u} &= (1, 3, 5) \\ \mathbf{w} &= (2, 8, 4) \\ \mathbf{t} &= (-1, 2, -2) \end{aligned}$$

12. In the triangle below, $|DC| = 9|BD|$. Find real numbers α and β such that

$$\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$$



13. Use the algebraic formula for the dot product of vectors, Show that for vectors $\mathbf{u}$, $\mathbf{s}$, $\mathbf{t}$,

$$\mathbf{u} \cdot (\mathbf{s} + \mathbf{t}) = \mathbf{u} \cdot \mathbf{s} + \mathbf{u} \cdot \mathbf{t}$$

(Note: this is similar but different from the distribution law between real numbers.)

14. Find the angle between the following two vectors

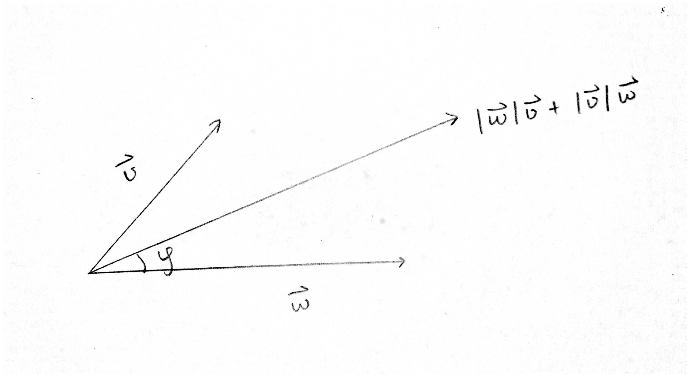
$$\mathbf{v} = (2, 5, 9) \qquad \mathbf{w} = (3, 1, 0)$$

15. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with

$$\begin{aligned} |\mathbf{u}| &= 3 \\ |\mathbf{v}| &= 5 \\ \mathbf{u} \cdot \mathbf{v} &= 10 \end{aligned}$$

Find the value of $|2\mathbf{u} - 3\mathbf{v}|$.

16. For two given vectors $\mathbf{v}$ and $\mathbf{w}$, find $\cos\phi$, where ϕ is the angle between the vectors $|\mathbf{w}|\mathbf{v} + |\mathbf{v}|\mathbf{w}$ and $\mathbf{w}$ as indicated in the following picture.



(The homework is now closed. It is due Thursday, June 29th.)

MATH 2433 Homework 5

1. Determine if the following notations are ambiguous (i.e. whether parentheses are needed to specify the order of operations.) Here, α is a real number and bold fonted letters represent vectors.

$$\alpha\mathbf{u}\cdot\mathbf{w}$$
$$\mathbf{u}\cdot\mathbf{w}\times\mathbf{t}$$
$$\mathbf{u}\times\mathbf{w}\times\mathbf{t}$$

2. Calculate

$$\mathbf{u}\cdot\mathbf{u}\times\mathbf{t}$$

3. Find the area of the triangle formed by the vectors

$$\mathbf{u} = (2, 5, 10)$$
$$\mathbf{v} = (-1, 8, 9)$$

4. Let

$$A = (3, 9, 10)$$
$$B = (-1, 2, 4)$$
$$C = (3, -2, 3)$$

be three points in the space.

a) Find the area of the triangle ABC.

b) Suppose we change the coordinate system in the following way: we regard the old x-axis as the new z'-axis, the old y-axis as the new x'-axis, the old z-axis as the new y'-axis (notice how they still obey the "right hand rule.") Assume the points A, B, C remain at the same position in the space, write down the coordinates for each point under the new $x'y'z'$-

coordinate system and calculate the area of the triangle ABC using the new coordinates.

5. Determine if the following four points lie on the same plane.

$$
\begin{aligned}
P &= (1, 4, 3) \\
Q &= (0, -3, 2) \\
R &= (3, -2, -4) \\
S &= (7, -6, 2)
\end{aligned}
$$

6. For vectors

$$
\begin{aligned}
\mathbf{u} &= (1, 2, 3) \\
\mathbf{s} &= (-1, 1, 0) \\
\mathbf{t} &= (-2, 0, -1)
\end{aligned}
$$

calculate

$$\mathbf{u} \times (\mathbf{s} \times \mathbf{t}) + \mathbf{s} \times (\mathbf{t} \times \mathbf{u}) + \mathbf{t} \times (\mathbf{u} \times \mathbf{s})$$

7. Find the vector $\mathbf{w}$ that is in the opposite direction of

$$\mathbf{v} = (2, 9, -4)$$

with magnitude 100. In addition, write $\mathbf{w}$ as a scalar multiple of $\mathbf{v}$. That is, find the real number λ such that $\mathbf{w} = \lambda \mathbf{v}$.

8. Find the parametric and symmetric equations of the line l passing through the point $P(3, -2, 1)$ with directional vector $\mathbf{v} = (2, 0, 9)$.

9. Let l be the line given by the following equations

$$
\begin{aligned}
x &= 2 + \lambda \\
y &= 2\lambda \\
z &= -1 - \lambda
\end{aligned}
$$

The values $\lambda = -1,\ 0,\ 1$ determine three points in space. Draw these points in the xyz-coordinate system, putting each of the point on a vertex of some box whose sides are parallel to the coordinate axes.

10. Find the parametric and symmetric equations of the line l passing through the points $P(0, 3, 4)$ and $Q(-1, 1, -2)$.

11. Determine if the following lines are skew or intersecting:

$$\begin{aligned} l_1 : \quad & \frac{x-1}{2} = \frac{y+2}{3} = z \\ l_2 : \quad & \frac{x+3}{4} = \frac{y}{4} = \frac{z-1}{2} \end{aligned}$$

12. Let l be the line given by

$$\begin{aligned} x &= 2 + \lambda \\ y &= 1 - \lambda \\ z &= 4 + 2\lambda \end{aligned}$$

and $P(3, 2, 1)$ is a point in space.

a) Find the distance between P and Line l.

b) If l' is the line going across P that is perpendicular to l, find the equations (either parametric or symmetric) of l'.

13. Let l be the line given in Problem 12, $Q(x_Q, y_Q, z_Q)$ be some fixed point in space (here x_Q, y_Q and z_Q are some fixed constants.) Find the distance between Q and l. Of course, your answer is allowed to contain x_Q, y_Q and z_Q.

(The homework is now closed. It is due Monday, July 10th.)

MATH 2433 Homework 6

1. Produce (any) two points from the plane

$$2x + 3y - z = 3$$

2. Check that the line l:

$$\frac{x-3}{2} = \frac{y+1}{3} = z - 4$$

is contained in the plane p:

$$4x + y - 11z = -33$$

3. Produce any line that is contained in the plane

$$3x + y - z = 10$$

4. Given a point $P(2, 3, -1)$ and a line l

$$\frac{x-8}{10} = \frac{y+1}{2} = \frac{z}{3}$$

where the plane p is perpendicular to l and passes through P.
a) Draw all the above objects in one picture (without the coordinate axes.)
b) Find the equation of the plane p.

5. Given a point $P(3, 0, 1)$ and a line l.

$$\frac{x}{3} = \frac{y-2}{-1} = \frac{z+4}{2}$$

the plane p contains both the point P and the line l.
a) Draw all the above objects in one picture (without the coordinate axes.)
b) Find the equation of the plane p.

6. Given two skew lines l_1, l_2:

$$\frac{x-2}{3} = \frac{y+5}{2} = \frac{z-7}{-1}$$
$$\frac{x-1}{1} = \frac{y}{-1} = \frac{z-8}{3}$$

the plane p passes through line l_1 and is parallel to l_2.
a) Draw all the above objects in one picture (without the coordinate axes.)
b) Find the equation of the plane p.

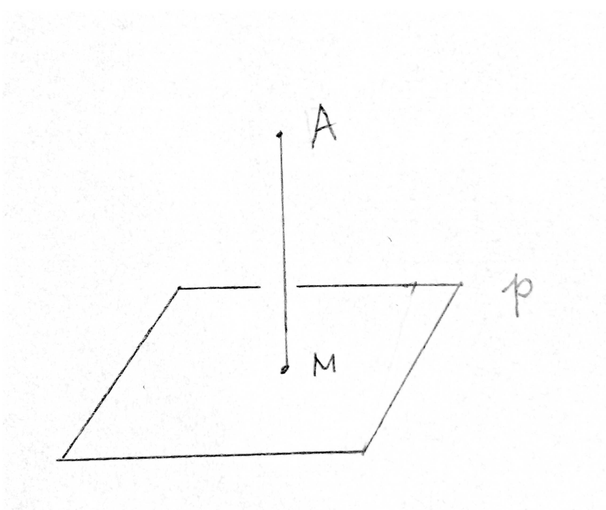
7. Find the parametric and symmetric equations for the line that lies in the intersection of the two planes.

$$2x + 3y - 4z = 1$$
$$3x - y + 7z = 2$$

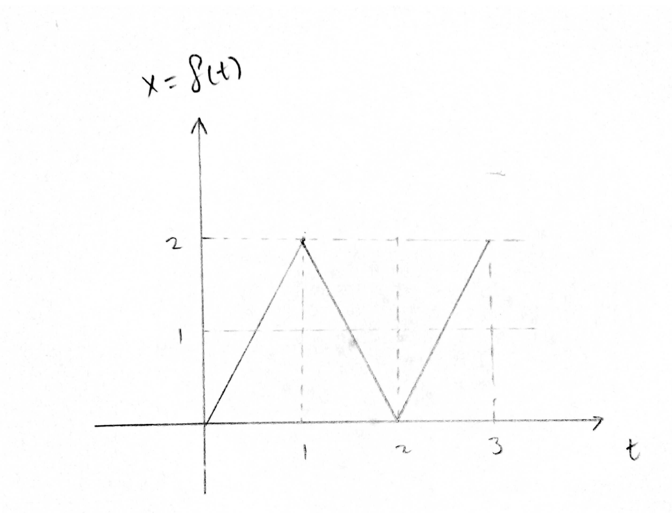
8. Find the distance between the point $A(1,2,3)$ and the plane p

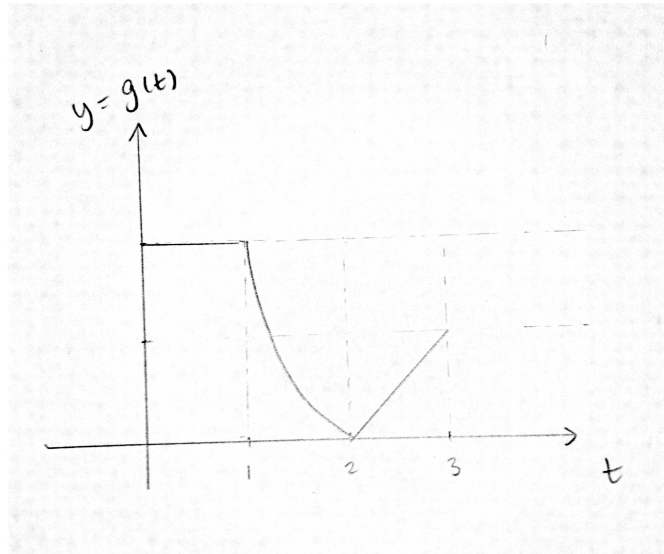
$$8x + 9y + 10z = 10$$

That is, find the length of the line segment AM in the following picture, where M is a point within the plane p and AM is perpendicular to p.



9. A parametric curve is given by $x = f(t)$, $y = f(t)$, $0 \leq t \leq 3$, where the graphs of the function f and g are as follows. Sketch the graph of the curve.





10. A parametric curve is given by

$$x = \sin t \cos t$$
$$y = \sin^2 t$$

a) Write an equation of the same curve with only variables x and y.
b) Use your answer in a) and implicit differentiation, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.
c) Use the parametrization, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, and compare your answer to that in b).

11. A parametric curve is given by

$$x = \sin 2t \sin t$$
$$y = \sin 2t \cos t$$

a) Sketch the curve on the interval $0 \leq t \leq 2\pi$
b) Write an equation of the same curve with only variables x and y.
c) Use the parametrization, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

12. Given the ellipse

$$x = a \cos t$$
$$y = b \sin t$$
$$0 \leq t \leq 2\pi$$

where a and b are some fixed constants.
a) Find an equation containing only x and y whose graph is the same ellipse.
b) Find $\frac{\mathrm{d}}{\mathrm{d}x} y$ in terms of t.

c) Find $\frac{\mathrm{d}^2}{\mathrm{d}x^2}y$ in terms of t.
d) Find the area of the upper half of the ellipse.

13. Given the logarithmic spiral

$$
\begin{aligned}
x &= e^t \cos t \\
y &= e^t \sin t \\
t &\geq 0
\end{aligned}
$$

a) Find the arc length of the segment from $t = 0$ to $t = 2\pi$.
b) Find the area under the the segment of the curve from $t = 0$ to $t = \pi$.

(The homework is closed, it is due Monday, July 17th.)

MATH 2433 Homework 7

1. Given the parametric curve

$$x = e^t \cos t$$
$$y = e^t \sin t$$

Let $\theta(t)$ be the inclination of the tangent line (the angle between the tangent line and the x-axis) to the point at t, $s(t)$ be the arc length between the point $t = 0$ and t.

a) Find $\theta'(t)$.

b) Find $s'(t)$.

c) Find curvature $\kappa(t)$ at point t.

2. Given two functions with the same independent variable x,

$$w = \sec x + \cos x$$
$$t = x^3 + x^2 + 3$$

Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ in terms of x.

3. Suppose a curve is given by $y = h(x)$, where h is a differentiable function. Use the other arc length formula, where the arc length $s(x)$ from the point $x = 0$ to an aribituary x is given by.

$$s(x) = \int_0^x \sqrt{1 + (h'(\lambda))^2}\, \mathrm{d}\lambda$$

a) Derive $\frac{\mathrm{d}\theta}{\mathrm{d}x}$ in terms of x, where θ is the inclination of the tangent line.

b) Derive $\frac{\mathrm{d}s}{\mathrm{d}x}$ and further calculate the curvature in terms of x.

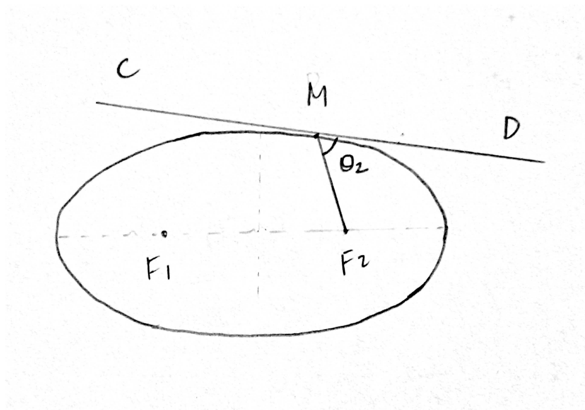
4. For the ellipse

$$x = a \cos t$$
$$y = b \sin t$$
$$(a > b)$$

a) Calculate the combined distance from a point $(x(t), y(t))$ to the two foci $(0, c)$, $(0, -c)$.

b) Show that the combined distance is independent of t.

5.



For the above ellipse with parametric equations

$$x = a\cos t$$
$$y = b\sin t$$

where F_1 are F_2 are the foci, $2c = |F_1F_2|$. Line CD is the tangent line to the ellipse at the point M.

a) Calculate $\cos\theta_2$.

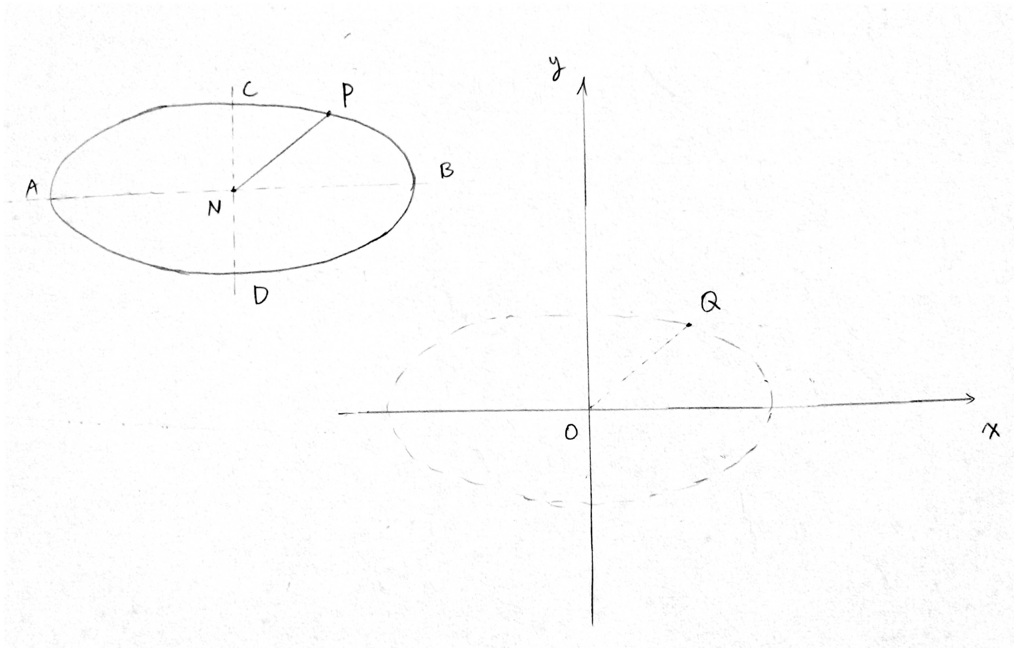
b) Show that your answer in a) is equal to the value of $\cos\theta_1$ we found in class, where θ_1 is the angle $\angle CMF_1$ and

$$\cos\theta_1 = \frac{-(a\cos t + c)b\cos t + ab\sin^2 t}{\sqrt{b^2\cos^2 t + a^2\sin^2 t} \cdot \sqrt{(a\cos t + c)^2 + b^2\sin^2 t}}$$

You may use the identity we showed in class that $c^2 = a^2 - b^2$.

(more problems on the next page)

6.



The above solid ellipse $ADBC$ has a major axis of length 20, a minor axis of length 10 and a center $N(-3,-2)$. The other dotted ellipse is centered at the origin with the same shape and size as the solid ellipse.

a) For an arbiturary point $P(x_P, y_P)$ on the ellispe, find the vector $\overrightarrow{NP}$. If OQ is parallel to NP, find the coordinate of Q.

b) Write down the equation of the dotted ellipse and the condition that Q is on the ellipse.

7. a) Find the equation of the ellipse with foci $F_1(4,8)$ and $F_2(12,14)$, whose major axis is of length 15.

b) Simply your answer in a) so that there are no radical terms in the equation.

8. Given the ellipse

$$x = 2\cos t$$
$$y = \sin t$$

a) Given a point P on the ellipse, let θ be the angle in the polar coordinates associated to P, find the parameter t associated to the point P in terms of θ.

b) Find $r = |\overrightarrow{OP}|$ in terms of θ, then simply your answer so it contains no trigonometric functions.

9. For the polar curve

$$r = \theta$$

Let A be the point $\theta = \frac{3\pi}{4}$ and C be the point $\theta = \frac{\pi}{2}$.

a) Convert the polar equation to parametric equations, and calculate the area under the curve $\overset{\frown}{AC}$ using the definition $\int y \, \mathrm{d}x$.

b) Assume you don't know the formula $\int \frac{1}{2} r^2 \, \mathrm{d}\theta$, use the answer in a), find the enclosed area bounded by the arc $\overset{\frown}{AC}$ and the two lines OA and OC.

10. Construct an explicit real-valued function with vectors as inputs.

11. (For the following problems, bold fonted letters are function names for vector-valued functions.) Find a tangent vector to the following curve at an arbituray point t.

$$\mathbf{f}(t) = (t^3 + 2t, \sec t, 2)$$

12. Let

$$\begin{aligned} \mathbf{f}(t) &= (e^t, \ln t, t^3) \\ g(t) &= \sin t \end{aligned}$$

Calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}(g(t)))$$

and

$$(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f})(g(t)) \cdot g'(t)$$

13. Derive a formula for

$$\frac{\mathrm{d}}{\mathrm{d}t}((\mathbf{f}(t) \times \mathbf{g}(t)) \times (\mathbf{h}(t) \times \mathbf{s}(t))$$

14. For the following curve

$$\mathbf{r}(t) = (t, \ln(\cos t), 3)$$

a) Calculate the curvature at point t using any formula.

b) Calculate the curvature using a different formula than the one you used in part a).

15. Calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{(e^t, t^3, \ln t)}{\sin t}$$

(You won't need it but here is the quotient rule for vector differentiation: For a vector valued function $\mathbf{f}(t)$ and a real valued function $g(t)$,)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{f}(t)}{g(t)} = \frac{g(t)\mathbf{f}'(t) - g'(t)\mathbf{f}(t)}{g(t)^2}$$

16. a) Sketch the curve

$$\mathbf{r}(t) = (\sin t, \cos t, t)$$

b) Find the coordinates of the center of the osculating circle at t.

c) Find the equation of the osculating plane at t.

(The homework is closed, it is due Friday, July 28th.)