# PANTS DECOMPOSITIONS OF SURFACES

**A thesis submitted in partial fulfillment of the requirements for the degree**

**MASTER OF SCIENCE**

**in**

**MATHEMATICS**

**by**

**JOSEPH MICHAEL RANDICH**
**APRIL 2015**

**at**

**THE GRADUATE SCHOOL OF THE UNIVERSITY OF CHARLESTON, SOUTH CAROLINA AT THE COLLEGE OF CHARLESTON**

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## ABSTRACT

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In this Master Thesis, we consider pants decompositions of any orientable 2-dimensional surface with any genus $g$. We show that any decomposition compatible with the same zipper system, and which is contractible in the inner handlebody corresponding to the decomposition, may be transformed into any other decomposition satisfying the same conditions via elementary transformations known as zipped flips. This puts us one step closer to showing that the groupoid on double pants decompositions, introduced by Felikson and Natanzon in [5], acts transitively on its objects.

## Acknowledgements

First, I'd like to thank my mother for all the support she's given me over the years. Who knows where I would be without her. Next, I'd like to thank the faculty in the Mathematics Department at the College of Charleston. So many of them have contributed in some way to fostering my mathematical development. Specifically, Dr. Shields, Dr. Ivey, and Dr. Calini for acting as my thesis committee and for the helpful feedback they've offered. Most importantly, I'd like to thank Dr. Andrew Przeworski for pushing me as a student and for agreeing to take on this project; without his persistent guidance, this thesis would not have come to fruition. Also, for the many hours spent in his office discussing math, life, and all other types of nonsense. I would not be the mathematician I am today without him.

Moreover, special thanks go to my fellow graduate students and friends who have worked with me, listened to my ideas (and often times, rants), and have put up with me, in general. Specifically, Stephanie for her encouragement, proofreading, and prop making, and Erin for all the support she provided during the last, and most difficult, months of this thesis. Finally, I would like to thank Robert Vandermolen. He is directly responsible for my pursuance of mathematics. His continuous, and often times emphatic, mentoring has been indispensible. But most importantly, his love for mathematics has been absolutely contagious. Thanks for clearing the fog, buddy.

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# Chapter 0

# Introduction

In this Master Thesis, we study pants decompositions of 2-dimensional surfaces. Our main goal is to fill in holes present in a paper by Felikson and Natanzon [5].

A pants decomposition of a surface is a decomposition that results in pieces that are thrice-punctured spheres. A double pants decomposition is a pair of such decompositions that satisfy certain compatibility conditions. In [5], they introduce a groupoid acting on double pants decompositions generated by elementary transformations called flips and handle twists and attempt to show that this groupoid acts transitively on double pants decompositions corresponding to Heegaard splittings of a 3-dimensional sphere.

Heegaard splittings are certain decompositions of a 3-manifold and are of wide interest in mathematics. Viewing double pants decompositions as Heegaard splittings had been done prior to the work of Felikson and Natanzon, specifically in [7] and [9]. Their claim is that the groupoid they create is a new idea. Furthermore, they extend their result to show that the mapping class group of the surface, which describes certain symmetries, is contained in the groupoid, which is studied in the literature ([1],[8],[10]).

In reading through their paper, however, various inconsistencies were found. Specifically, their Lemma 1.12 incorrectly asserts that a flip $c'$ of a curve $c$ in a pants decomposition $P$ is a zipped flip in respect to some zipper system $Z'$, where a zipped flip is a specific type of flip. This result is an integral piece of the proof of their main theorem, thus justifying the aim of this thesis. Furthermore, much of their work deals with double pants decompositions, but in the proof of their main theorem, single pants decompositions (specifically those which are compatible with a set of curves called a zipper system) play a major role. Therefore, the properties of these types of decompositions explored in this paper are necessary.

In Chapter 1, we define pants decompositions of surfaces, zipper systems, compatibility, and zipped flips. We then introduce a category $\mathfrak{P}_g^z$ of pants decompositions which are compatible with a particular zipper system $Z$ that has zipped flips as morphisms.

In Chapter 2, we discuss how pants decompositions correspond to handlebodies and introduce the notion of a pants decomposition being contractible in the inner handlebody that it corresponds to. We also define what it means for two decompositions to be zipped flip equivalent. The rest of the chapter is devoted to proving transitivity of zipped flips on $\mathfrak{P}_g^z$. The main result comes in the form of a corollary and is stated here:

**Main Result.** *Given a handlebody $\mathfrak{H}$ and any zipper system $Z$, any two pants decompositions contractible in $\mathfrak{H}$ and compatible with $Z$ are zipped flip equivalent.*

In the language of our defined category, this says that our morphisms act transitively on our objects when the objects in question are contractible in a corresponding handlebody. Before proving this result, several lemmas are proven and the notion of a friendly pants decomposition is introduced. An algorithm is provided describing how one may take any friendly pants decomposition to any other via a sequence of zipped flips. Finally, another algorithm is provided for changing the principal zipper (introduced in Chapter 1), which leads to our main corollary.

# Chapter 1

# Pants Decompositions

## 1.1 Preliminaries

**Definition 1.1.1** (Homeomorphism [11])**.** Let $X$ and $Y$ be arbitrary topological spaces, and let $f : X \to Y$ be a continuous bijection. When the inverse of $f$ is also continuous, $f$ is said to be a *homeomorphism*.

**Definition 1.1.2** (Embedding [11])**.** Let $X$ and $Y$ be arbitrary topological spaces, and let $f : X \to Y$ be a continuous injection. The function $f' : X \to f(X)$ obtained by restricting the range of $f$ to its image is bijective. When $f'$ happens to be a homeomorphism, $f'$ is said to be an *embedding*.

**Definition 1.1.3** (Manifold [2])**.** An $n$-dimensional *manifold* is a Hausdorff space $X$ with a countable basis such that each point $x \in X$ has a neighborhood that is homeomorphic with either an open subset of $\mathbb{R}^n$ or of $\mathbb{R}^n_+ = \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n \mid y_n \geq 0\}$.

**Definition 1.1.4** (Closed)**.** A manifold is *closed* when it is compact and contains no boundary, that is, each of its points have an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

**Definition 1.1.5** (Surface)**.** A *surface* $S$ is a 2-dimensional manifold.

**Definition 1.1.6** (Orientable)**.** A surface $S$ is *orientable* if for every $x \in S$, there exists a $\varphi_x : U_x \to \mathbb{R}^n$ (or $\mathbb{R}^n_+$) where $U_x \subset S$ is an open set containing $x$ and $\varphi_x$ is a homeomorphism, and whenever $U_x \cap U_y \neq \varnothing$, $\varphi_y \circ \varphi_x^{-1}$ is orientation preserving (when the domain of $\varphi_x$ is restricted to $U_x \cap U_y$).

From this point on, unless otherwise noted, we consider surfaces $S$ which are connected, closed, and orientable.

**Definition 1.1.7** (Euler Characteristic [4]). Given a surface $S$ and any triangulation of that surface, the surface's *Euler Characteristic* is $\chi(S) = v - e + f$ where $v$ is the number of vertices in the triangulation, $e$ is the number of edges, and $f$ is the number of faces.

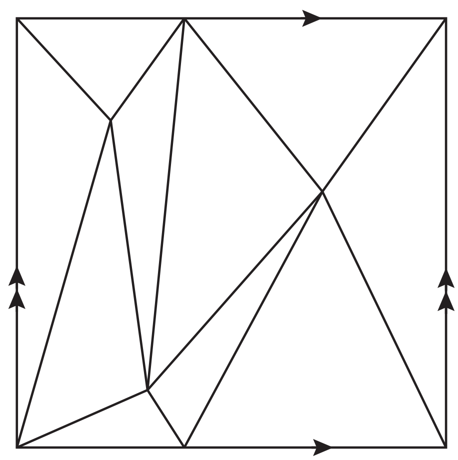


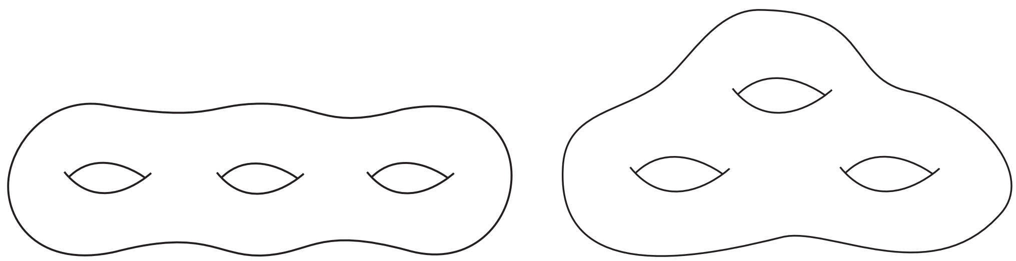
Figure 1.1: A triangulation of the unit square whose edges are identified, or the torus. Notice that there are 5 vertices, 15 edges, and 10 faces (accounting for the identifications) giving an Euler characteristic 0.

**Remark 1.1.8.** The Euler Characteristic may be equivalently defined in terms of polygons rather than triangles. When using polygons, we will refer to it as *cellularization.*

**Theorem 1.1.9** (Classification of Surfaces [4]). *Given two orientable surfaces $S_1$ and $S_2$, $S_1$ is homeomorphic to $S_2$ if and only if $\chi(S_1) = \chi(S_2)$. Furthermore, for every surface $S$, $\chi(S) \leq 2$ and $\chi(S)$ is even.*

**Definition 1.1.10** (Genus). Given a surface $S$, its *genus* is $g = \frac{2-\chi(S)}{2}$.

**Remark 1.1.11.** Throughout this paper, we use two specific embeddings of surfaces in $\mathbb{R}^3$. The first is what we refer to as the lateral embedding which has the holes of the surface arranged in a linear fashion (Figure 1.2a). The second is referred to as the circular embedding which has the holes arranged in a roughly circular fashion (Figure 1.2b). Notice that these embeddings are symmetric with respect to the $xy$-plane.



(a) Lateral Embedding (b) Circular Embedding

Figure 1.2: Examples of circular and lateral embeddings.

In regards to the two embeddings mentioned in Remark 1.1.11, the genus of a surface is equal to the number of 'holes' it possesses.

**Definition 1.1.12** (Homotopy [11])**.** Given a topological space $X$ and two paths $f, f' : I \to X$, where $I = [0, 1]$, $f$ is *path homotopic* to $f'$ when there is a continuous map $F : I \times I \to X$ such that $F(s, 0) = f(s)$ and $F(s, 1) = f'(s)$ for each $s \in I$. We call $F$ a *path homotopy* between $f$ and $f'$.

One may intuitively think of a homotopy of two paths, or curves in a surface, as a 'continuous deformation' of one curve into the other. When discussing path homotopic curves in $S$, we typically just say 'homotopic'.

**Remark 1.1.13** (Path Homotopy Classes [11])**.** Path homotopy defines an equivalence relation on the set of paths in $X$, which verifies easily. Given two paths $f, g$ where $f(1) = g(0)$, one may define composition $f \star g$ of paths to be the path $h$ given by

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The function $h$ is well-defined and continuous, and we think of $h$ as the path whose first half is $f$ and whose second half is $g$. Furthermore, the operation $\star$ is well-defined on path homotopy classes.

**Definition 1.1.14** (Isotopy [13])**.** An *isotopy* is a homotopy $h$ for which every section $h_t$ is a homeomorphism (onto its image). In particular, during an isotopy of a simple closed curve, the image remains simple at every stage.

Notice that isotopy is stronger than homotopy; at a given section of a homotopy, the curve may self-intersect, or backtrack, or act strangely in other ways. Isotopies are better behaved.

However, Baer shows that for closed orientable surfaces with $g > 1$ (the only surfaces we consider), isotopy and homotopy of simple closed unoriented curves are equivalent [3]. Therefore, we may use the two interchangeably.

**Definition 1.1.15** (Contractible Curve)**.** A curve $c$ in $S$ is *contractible* (in $S$) if $c$ is null-homotopic, that is, it is homotopically equivalent to a single point.

From now on, we consider curves $c_i$ in $S$ to be embedded non-contractible loops (with possibly different base points) and consider them up to homotopy, where the basepoints are free to move.

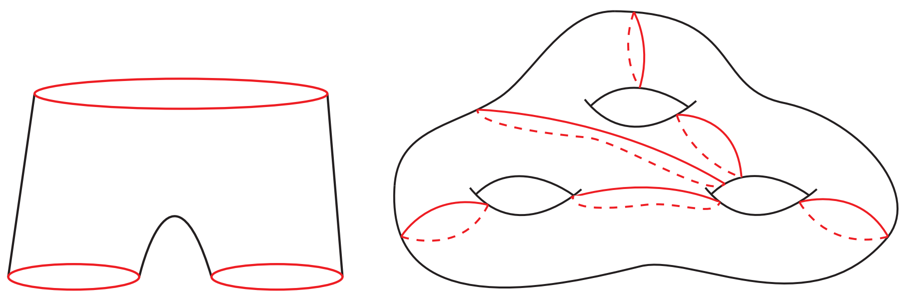
**Definition 1.1.16** (Necessary Intersections [5])**.** Two curves $c_1, c_2$ in $S$ share $k$ *necessary intersections* if $c_1$ and $c_2$ intersect at $k$ points, and there are no homotopically equivalent pairs of curves that intersect each other less than $k$ times.

For a pair of curves, $c_1, c_2 \in S$, we will denote their number of necessary intersections by $|c_1 \cap c_2|$. When dealing with a curve $c_i$ which intersects a collection of curves $\{c_j\}$, we again denote the total number of necessary intersections $c_i$ shares with all of the $c_j$ by $|c_i \cap (\bigcup c_j)|$. The situation should be clear by context.

## 1.2 Pants and Zippers

**Definition 1.2.1** (Pair of Pants)**.** A *pair of pants* is a surface that is homeomorphic to a thrice-punctured sphere, that is, to a sphere with three boundary components. (Note that this surface is also homeomorphic to a disk with two punctures).

**Definition 1.2.2** (Pants Decompositions)**.** A *pants decomposition* $P = \{c_1, \ldots, c_n\}$ of a surface $S$ is a set of non-oriented, mutually-disjoint curves that decompose $S$ into pairs of pants.



(a) A pair of pants. (b) A pants decomposition.

Figure 1.3: A pair of pants and an example of a pants decomposition of a genus 3 surface.

**Definition 1.2.3** (Self-folding)**.** Given a pants decomposition $P$ of a surface $S$, a curve $c_i \in P$ is said to be *self-folding* if it accounts for two of the three boundary components of a pair of pants.

Noting that we allow self-folding curves in $P$, the following Proposition should be clear by inspection. Self-folding curves play a major role in our decompositions, as will be seen in the following sections.

**Proposition 1.2.4.** *Given a pants decomposition $P$ of a surface $S$ with genus $g$, $P$ decomposes $S$ into $2g-2$ pairs of pants.*

*Proof.* Note that a pair of pants has Euler Characteristic $-1$.[1] Given a pants decomposition $P$ of a surface $S$, since $S$ is closed, and since the edges of pairs of pants that are glued to each other are circles (and therefore contribute zero to the Euler characteristic), there are no double counting issues. Thus, each pair of pants in the decomposition will contribute exactly $-1$ to $\chi(S)$. So, for $n(P) =$ the number of pairs of pants of $P$, we have $n(P) = -\chi(S)$. Recall from Definition 1.1.10 that $g = \frac{2-\chi(S)}{2}$. It follows that $n(P) = 2g - 2$. ∎

[1] The reader is encouraged to pick their favorite triangulation of a pair of pants and verify the claim.

**Proposition 1.2.5.** *Given a pants decomposition $P$ of a surface $S$, $P$ contains $3g-3$ curves.*

*Proof.* Given a pants decomposition $P$ of a surface $S$, by Proposition 1.2.4, $P$ yields $2g-2$ pairs of pants. By definition, there are 3 curves of $P$ corresponding to each pair of pants. So there are $6g-6$ total curves. Noticing that each curve in a pair of pants gets identified with another curve (possibly one from the same pair), we avoid double-counting by dividing this number by 2, and we have our result. ◻

**Definition 1.2.6** (Handle)**.** A *handle* is a self-folding pair of pants, that is, a pair of pants that has a self-folding curve account for two of its three boundary components.

For example, the pants decomposition in Figure 1.4 yields three handles.

Given some pants decomposition $P$ of a surface $S$, the curves of $P$ may be classified in the following way: Curves in $P$ that bound handles are *principal curves*. Curves contained completely within handles are referred to as the *self-folding curves* of $P$. If $P$ yields $g$ handles, then curves which lie outside of any handle and which are not principal curves are referred to as the *middle curves* of $P$.

**Definition 1.2.7** (Standard)**.** A pants decomposition $P$ of a surface $S$ is *standard* if it cuts $g$ handles out of $S$.

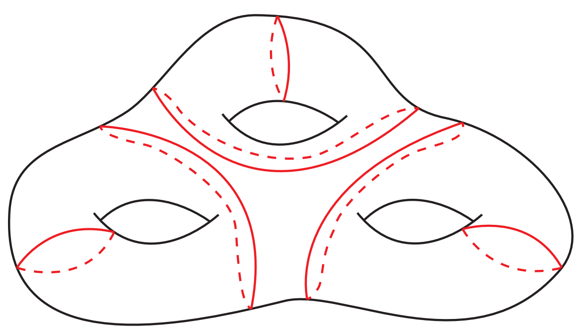


Figure 1.4: A standard pants decomposition of a genus 3 surface. Notice that the decomposition in Figure 1.3b is non-standard.

**Definition 1.2.8** (Zipper System [5])**.** A *zipper system* $Z = \{z_1, \ldots, z_{g+1}\}$ of a surface $S$ is a set of $g+1$ mutually disjoint curves that decompose $S$ into two separate disks, each of which has $g$ punctures. We name these two disks $S_+$ and $S_-$; one may think of these as corresponding to the 'top' and 'bottom' of $S$ with respect to the zipper system.

**Definition 1.2.9** (Equatorial Zipper System)**.** Given a surface $S$ along with either the lateral or circular embedding mentioned in Remark 1.1.11, the *equatorial zipper system* $Z_E$ is the set of curves resulting from intersecting $S$ with the $xy-$plane.

**Definition 1.2.10** (Compatibility [5])**.** We say that a pants decomposition $P = \{c_1, \ldots, c_{3g-3}\}$ is *compatible* with a zipper system $Z = \{z_1, \ldots, z_{g+1}\}$ if $\left| c_i \cap \left( \bigcup_{j=1}^{g+1} z_j \right) \right| = 2$ for each $1 \leq i \leq 3g-3$. This means that each curve of $P$ shares exactly two necessary intersections somewhere in the zipper system.

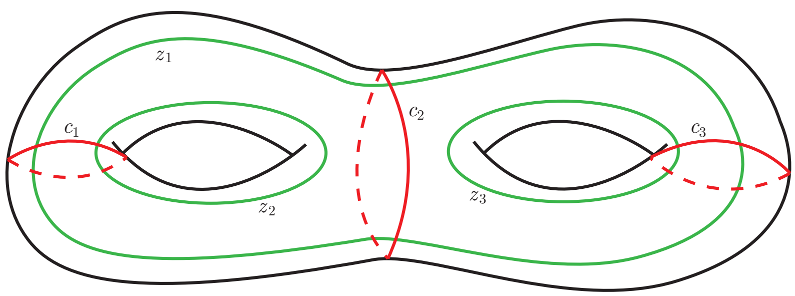
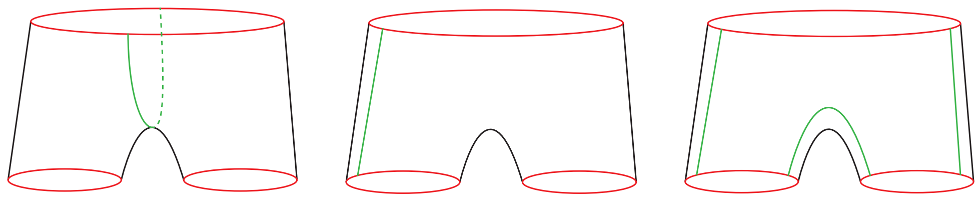



Figure 1.5: A pants decomposition of a genus 2 surface compatible with the equatorial zipper system. The $c_i$ make up the pants decomposition and the $z_i$ make up the zipper system.

**Lemma 1.2.11** (Hexagons [5])**.** *If $P$ is a pants decomposition compatible with a zipper system $Z$, then the curves of $Z$ decompose each pair of pants from $P$ into two hexagons.*

*Proof.* Suppose a curve $z_j$ intersects a curve $c_i$ in the boundary of a pair of pants $\mathfrak{p}_1$. The curves of a pants decomposition cut $z_j$ into segments. Let $l$ be a segment of $z_j$ contained in $\mathfrak{p}_1$. Then, since curves do not have unnecessary intersections, $l$ may travel through $\mathfrak{p}_1$ in two ways, shown in Figure 1.6a and Figure 1.6b. If $l$ intersects the same boundary curve twice, as it does in Figure 1.6a, when considering the entire zipper system, for that boundary curve of $\mathfrak{p}_1$, the $\left| c_i \cap \left( \bigcup_{j=1}^{g+1} z_j \right) \right| = 2$ condition is broken. This happens because the zipper segments that intersect the other boundary curves must travel through the pair of pants in a non-trivial way, so they must also intersect the boundary curve in question.

Therefore, $l$ must intersect each boundary curve only once, as it does in Figure 1.6b. So, considering all of $Z$, $\mathfrak{p}_1$ looks as it does in Figure 1.6c and it is clear that $\mathfrak{p}_1$ is decomposed into two hexagons. ◻



(a) (b) (c)

Figure 1.6: For a pants decomposition compatible with a zipper system $Z$, the curves of $Z$ decompose a pair of pants from $P$ into two hexagons.

**Remark 1.2.12.** If $P$ is a pants decomposition compatible with a zipper system $Z$, then there exists an orientation-reversing involution $\tau$ of $S$ such that $\tau$ preserves $Z$ pointwise and such that $\tau c_i = c_i$ for each $c_i \in P$. To build this involution, one needs only to switch the pairs of hexagons in each pair of pants described in Lemma 1.2.11 [5].

A particularly important involution that will be used throughout the paper is the north/south reflection. This involution arranges the two hexagons of a pair of pants so that, in respect to an embedding of the pants in $\mathbb{R}^3$, they are symmetric about the $xy$-plane. This induces an involution of $S$ so that the curves of $P$ are symmetric about the $xy$-plane.

The following proposition sheds light on how a zipper system $Z$ interacts with a surface $S$. By definition, $Z$ decomposes $S$ into two $g$-punctured disks. But from this definition, it is not immediately clear how this is done. Specifically, we may construct a set of curves that decompose $S$ in the appropriate way where certain curves are boundary components of only one of the two resulting disks. However, we see that zipper systems that are compatible with pants decompositions, and are therefore the only systems we wish to consider, cannot behave in this way.

**Proposition 1.2.13.** *For a surface $S$, let $Z$ be some zipper system such that there exists a pants decomposition $P$ of $S$ where $P$ is compatible with $Z$. Then $Z$ must decompose $S$ into two separate $g$-punctured 2-disks such that each $z_i \in Z$ is a boundary component of each 2-disk.*

*Proof.* Since $P$ is compatible with $Z$, by definition, $Z$ decomposes $S$ into two separate connected, $g$-punctured disks $S_+$ and $S_-$. Choose any $z_j \in Z$. Without loss of generality, suppose $z_j$ is a boundary component of only $S_+$. By Remark 1.2.12, there exists an involution $\tau$ that preserves the curves in $Z$ pointwise and which sends $S_+$ to $S_-$ and vice versa. Since $\tau$ preserves $z_j$ pointwise, this means $z_j$ is contained in $S_-$, contradicting our assumption. Thus, $z_j$ must be a boundary component of both $S_+$ and $S_-$. ∎

**Proposition 1.2.14** ([5])**.** *Let $P$ be a standard pants decomposition compatible with some zipper system $Z$. Then*

- *each handle $\mathfrak{h}_i$ of $P$ completely contains exactly one curve of $Z$, and*
- *there exists exactly one $z_0 \in Z$ that visits each of $g$ handles exactly once.*

*Proof.* To see the first claim, consider a handle $\mathfrak{h}_i$ of $P$. By Lemma 1.2.11, the curves of $Z$ decompose $\mathfrak{h}_i$ into two hexagons. Let $p_i$ and $s_i$ be the principal and self-folding curves, respectively, of $\mathfrak{h}_i$. Then since two of the boundary components of $\mathfrak{h}_i$ come from $s_i$, and are therefore identified, the zipper segment running between them also gets identified, and is therefore the same curve, as seen in Figure 1.7. Therefore, each handle contains exactly one curve of $Z$. Since $P$ is standard, it yields $g$ handles, so $g$ of the $g+1$ curves of $Z$ are contained in them. Therefore, the curve of $Z$ that intersects each principal curve must be the same, and since it may only intersect each principal curve exactly twice, it visits each handle exactly once, so we have our second claim. ∎

**Definition 1.2.15** (Principal Zipper)**.** Given a standard pants decomposition $P$ compatible with a zipper system $Z$, suppose $z_0 \in Z$ is the curve that visits each handle of $P$ mentioned in Proposition 1.2.14. Then $z_0$ is said to be the *principal zipper*.

The $z_1$ curve in Figure 1.5 is an example of a principal zipper.

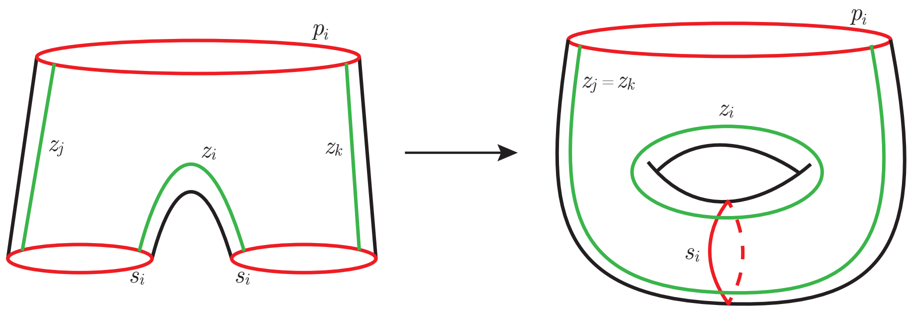



Figure 1.7: A handle $\mathfrak{h}_i$ completely contains a curve $z_i \in Z$. Here, $z_j$ is actually the principal zipper $z_0$ defined below in Definition 1.2.15.

**Definition 1.2.16** (Cyclic Order [5])**.** Given a standard pants decomposition $P$ compatible with a zipper system $Z$, suppose $z_0 \in Z$ is the principal zipper. Then the *cyclic order* of $Z$ is $[z_1, z_2 \ldots, z_g]$ if an orientation of $z_0$ goes from $\mathfrak{h}_i$ to $\mathfrak{h}_{i+1}$, where $\mathfrak{h}_i$ is the handle containing $z_i$.

For consistency's sake, we always define our cyclic order by using a positive orientation on $z_0$ in respect to our choice of $S_+$.

**Definition 1.2.17** (Boring)**.** Given a pants decomposition $P$ compatible with the equatorial zipper system $Z_E$, $P$ is said to be *boring* if $\tau P|_{S_+} = P|_{S_-}$ where $\tau$ is the north/south reflection. This means that $P|_{S_+}$ and $P|_{S_-}$ are symmetric.

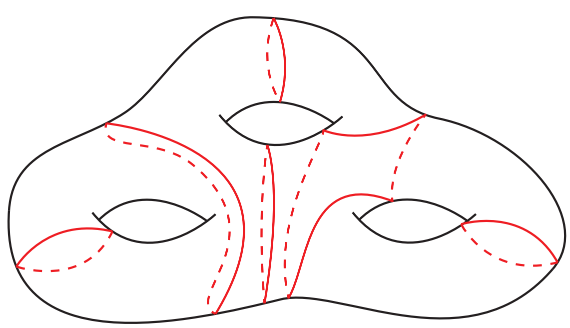


Figure 1.8: A pants decomposition of a genus 3 surface that is not boring. The decompositions shown in Figure 1.3b and in Figure 1.4 are boring.

We now introduce the graph representation of a pants decomposition, $P$. Given any pants decomposition $P$, construct a graph where each vertex corresponds to a pair of pants in $P$ and each edge corresponds to a $c_i \in P$. See Figure 1.9. Then an edge between two vertices corresponds to the curve in $P$ that the two pairs of pants share. Since self-folding curves are allowed, it follows that we must allow loops in our graph; a loop corresponds to a handle in the decomposition.

It is easy to see that we always have tri-valent graphs since each pair of pants is bounded by three curves. Furthermore, these graphs are always connected. In the case of a standard pants decomposition, its corresponding graph is a tree, with one loop attached to each vertex corresponding to a handle.[2]

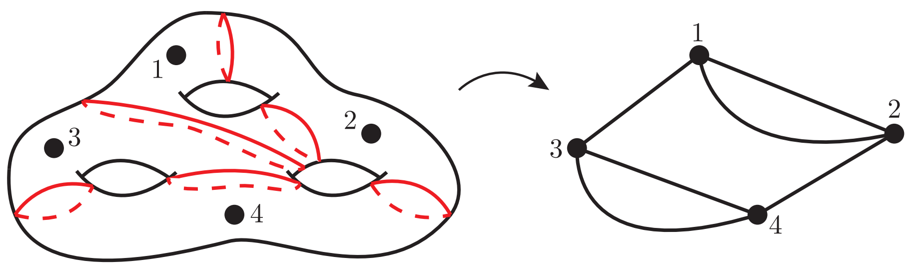



Figure 1.9: A pants decomposition of a genus 3 surface and its corresponding graph.

**Proposition 1.2.18.** *Given a pants decomposition $P$ of a surface $S$, there exists a zipper system $Z$ for which $P$ is compatible if and only if its corresponding graph is planar.*

*Proof.* First, suppose $P$ is compatible with some zipper system $Z$. Then $Z$ decomposes $S$ into two connected pieces, $S_+$ and $S_-$. It follows from Proposition 1.2.13 that there exists a homeomorphism of $S$ that takes $Z$ to $Z_E$ and $P$ to some $P'$ compatible with $Z_E$ such that the north/south involution may be used. Then $S'_+$ and $S'_-$ are symmetric, so we need only consider $S'_+$. Since $S'_+$ is a punctured disk, it may be embedded in $\mathbb{R}^2$. Now within each pair of pants, we may draw a node. For each curve of $P'$, we may draw an edge which intersects the curve exactly once and connects two nodes. If the curve is self-folding (and hence corresponds to only one pair of pants), we may draw the edge intersecting it as a loop, which corresponds to a 1-cycle in the graph. Thus we have our planar graph. See Figure 1.10.

Next, suppose that $P$ has a planar corresponding graph. Consider the graph in $\mathbb{R}^2$.

[2]The skeptical reader will be pleased to find a proof of this in Corollary 1.2.19.

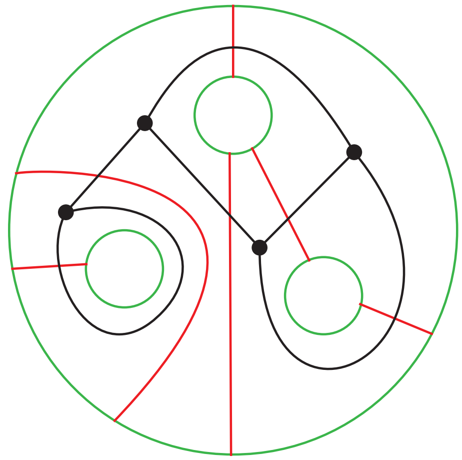


Figure 1.10: For a genus 3 surface and some pants decomposition $P$, the punctured disk $S'_+$ is drawn with circular boundary components from $Z_E$. The curves of $P'$ and the corresponding planar graph are shown.

We wish to trace $g+1$ curves, $z_i$, around the graph such that $\mathbb{R}^2 \setminus \bigcup z_i$ has a connected component that completely contains the graph. To see that there are, indeed, $g+1$ curves, consider the graph as the 1-skeleton of a cellularization of the 2-sphere. Then each $z_i$ corresponds to a face of this cellularization. See Figure 1.11 for an example. Now since each vertex of our graph corresponds to a pair of pants, by Proposition 1.2.4, there are $2g-2$ vertices. Since these graphs are always tri-valent, there are $3g-3$ edges (an edge for each pants curve). So, since the Euler characteristic of a sphere is 2 [4], we have $2g-2-(3g-3)+f=2$ where $f$ is the number of faces. Thus, we have $f=g+1$, so there are $g+1$ of the $z_i$.

Now, in this connected component, for each edge, $e_r$, draw a simple curve $c_r$ that intersects $e_r$ and the two curves $z_j$ and $z_k$ on either side of $e_r$. Glue this component to a copy of itself along the $z_i$. This results in a surface $S'$ that has $\bigcup\{c_i\} = P'$ as a pants decomposition compatible with $Z_E$. Then there exists a homeomorphism that takes $P'$ to $P$ and $Z_E$ to $Z$, where $P$ is compatible with $Z$. See Figure 1.12 for an example. ∎

**Corollary 1.2.19.** *For any standard pants decomposition, there exists a zipper system for which it is compatible.*

*Proof.* Consider a standard pants decomposition $P$ of a surface $S$. Then $P$ cuts $g$ handles

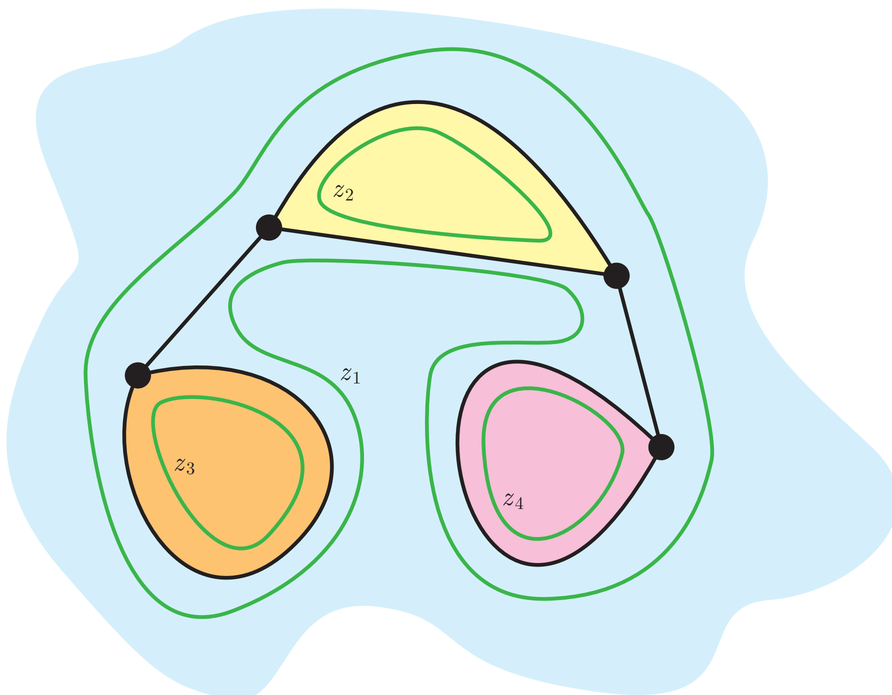
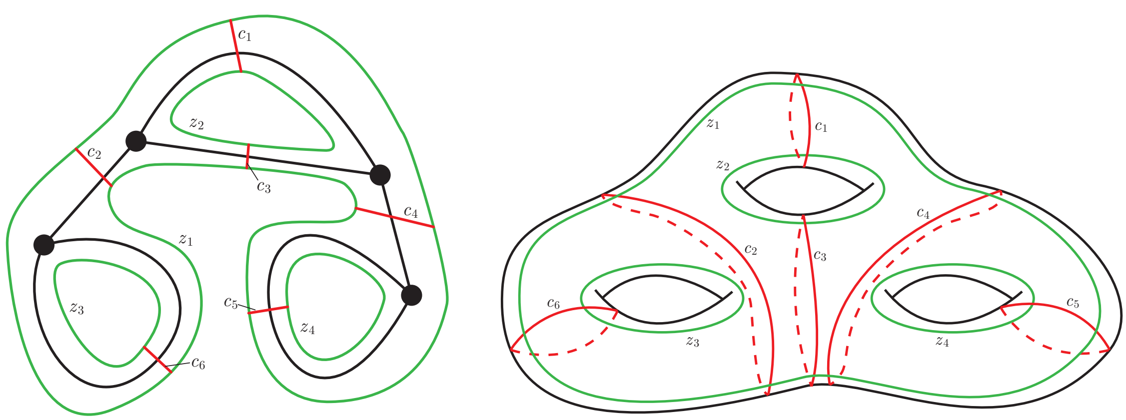



Figure 1.11: A planar graph corresponding to a pants decomposition of a genus 3 surface is shown as a cellularization of the 2-shpere. Notice that each $z_i$ sits within a colored face.




(a) (b)

Figure 1.12: A pants decomposition is created in (a) by drawing a curve $c_r$ for each edge of the planar graph from Figure 1.11. When the two connected components are glued along the $z_i$, the surface in (b) is created, after isotopy.

from $S$. This means the graph $G$ corresponding to $P$ has $g$ nodes with 1-cycles, $c_i$, attached. Apart from these 1-cycles, the claim is that $G$ is a tree. For if it was not, then there would be at least one cycle elsewhere in $G$. Consider a regular neighborhood of $G \setminus \bigcup c_i$. Since the regular neighborhood of $G$ is homeomorphic to $S$, this extra cycle, along with the $g$ 1-cycles, corresponds to a surface with genus at least $g+1$ which contradicts our assumption that $S$ was of genus $g$. Therefore, $G$ cannot contain cycles other than the 1-cycles corresponding to the self-folding curves of $P$, and is therefore a tree, so is compatible with some zipper system. ⧉

## 1.3 Zipped Flips

**Definition 1.3.1** (Flip)**.** Given a pants decomposition $P$, a *flip* $\varphi_i$ of $P$ is a replacement of the curve $c_i \in P$ with a new curve $c_i'$ such that $c_i$ is not self-folding and that the following hold:

- $c_i'$ is not homotopically equivalent to any curve in $P$.
- $|c_i' \cap c_i| = 2$.
- $|c_i' \cap c_j| = 0$ for all $j \neq i$.

Two examples of flips are shown in Figure 1.13. Notice that flips may not be performed in handles, but may be applied to principal curves. See Figure 1.14.

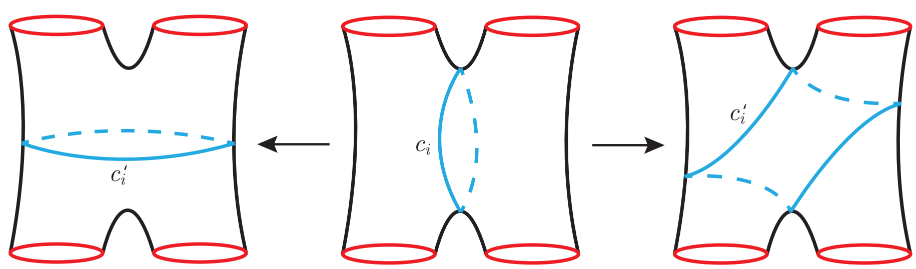



Figure 1.13: Two possible flips $c_i'$ of a curve $c_i$ in $P$.

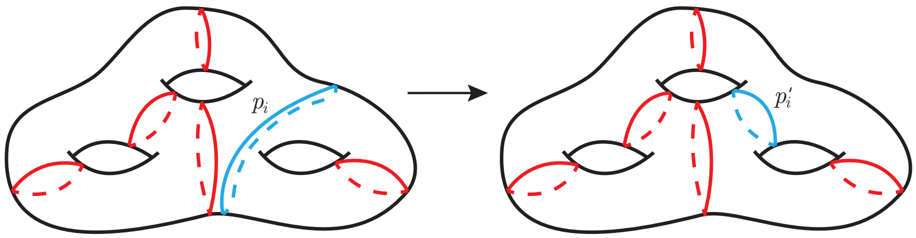



Figure 1.14: The principal curve $p_i$ is flipped to $p_i'$.

**Definition 1.3.2** (Zipped Flip)**.** Given a pants decomposition $P$ compatible with a zipper system $Z$, a *zipped flip* $\zeta_i$ of $P$ with respect to $Z$ is a replacement of the curve $c_i \in P$ with a new curve $c_i'$ such that $c_i$ is not self-folding and that the following hold:

- $c_i'$ is not homotopically equivalent to any curve in $P$.
- $c_i'$ shares exactly two necessary intersections with $Z$. That is, $\left| c_i' \cap \left( \bigcup_{j=1}^{g+1} z_j \right) \right| = 2.$
- $|c_i' \cap c_j| = 0$ for all $j \neq i$.

By inspection, we see that $\zeta_i(P)$ is a new pants decomposition that is compatible with $Z$. It should also be noted that there is a unique choice for $\zeta_i$, specifically, it is obvious that $\zeta_i \circ \zeta_i(P) = P$. This also implies that a zipped flip is reversible.

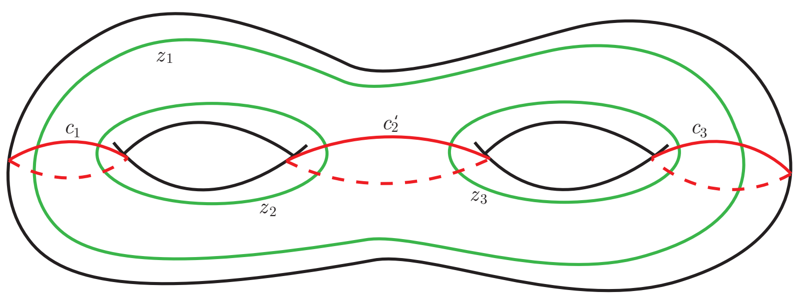



Figure 1.15: The curve $c_2'$ is a zipped flip of the curve $c_2$ seen in Figure 1.5.

The curve $c_2'$ in Figure 1.15 is a zipped flip of $c_2$ from Figure 1.5. Notice that zipped flips are just special types of flips. We see, however, that not every flip is a zipped flip in the following proposition.

**Proposition 1.3.3.** *Not every flip $c_i'$ of a curve $c_i$ in a pants decomposition is a zipped flip.*

*Proof.* Consider the pants decomposition $P$ of the genus 4 surface shown in Figure 1.16. It should be obvious that $P$ has a planar corresponding graph. Now, the curve $c$ may be flipped to the curve $c' \in P'$ as shown in Figure 1.16. Figure 1.17 shows the graph representation of $P'$. Upon inspection of this graph, we see that it is graph isomorphic to a $K_{3,3}$ graph, which is known to be non-planar [12]. Therefore, by Proposition 1.2.18, $P'$ is not compatible with any zipper system, and thus $c'_i$ is not a zipped flip. ∎

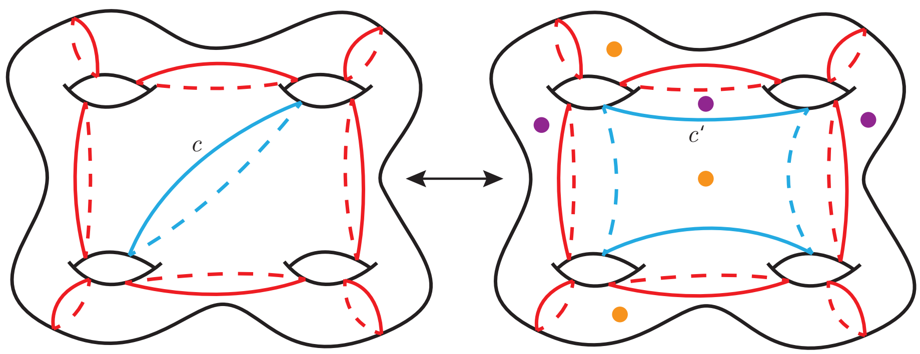



Figure 1.16: A pants decomposition $P$ of a genus 4 surface is shown on the left. A flip $c'$ of the curve $c$ gives way to a pants decomposition $P'$ shown on the right. Colored nodes are placed within each pair of pants corresponding to a graph representation of $P'$. Figure 1.17 shows that this graph is not planar. Note that $Z_E$ is not drawn for clarity's sake.

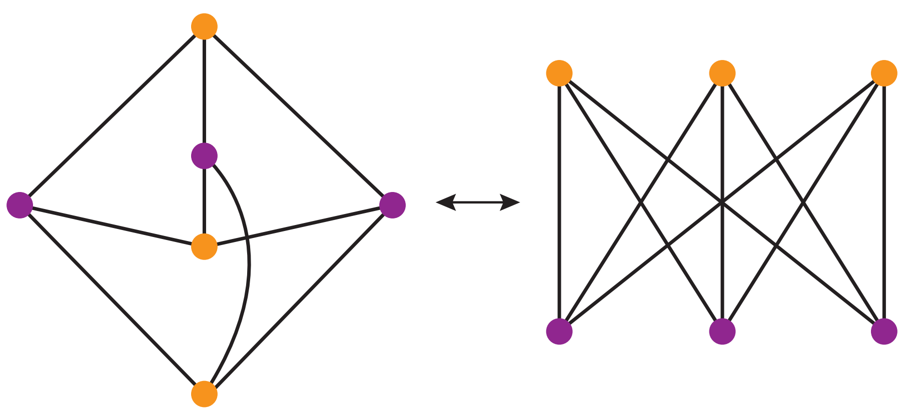


Figure 1.17: The graph corresponding to $P'$ seen in Figure 1.16 is shown on the left. It is obvious that this is graph isomorphic to the $K_{3,3}$ graph shown on the right.

As mentioned in the Introduction, Felikson and Natanzon relied heavily on a result (their Lemma 1.12) which claims that any flip of a curve $c$ in a given pants decomposition compatible with a zipper system $Z$ is a zipped flip in respect to some zipper system $Z'$. Our Proposition 1.3.3 contradicts this, telling us that we must find a workaround if zipped flips are to be used to prove the main theorem of [5].

**Definition 1.3.4** (Category of Compatible Pants Decompositions)**.** Let $\mathfrak{P}_g^z$ be the *category of pants decompositions of a genus g surface that are compatible with a particular zipper system* $Z$. The objects of this category are the pants decompositions, and the elementary morphisms are zipped flips. All other morphisms are compositions of zipped flips. Associativity of the morphisms is immediate.

# Chapter 2

# Transitivity of Zipped Flips

In this chapter, we work towards our Main Result, which is that given any pants decompositions $P_a, P_b$ that are compatible with the same zipper system and contractible in the same handlebody, $P_a$ may be taken to $P_b$ via a sequence of zipped flips. We begin by proving some results regarding standard pants decompositions that are boring and contractible in the inner handlebody corresponding to $P$.

## 2.1 Contractible Decompositions

**Definition 2.1.1** (Handlebody)**.** A *handlebody* is a 3-dimensional ball which has several handles attached, where a handle is a solid cylinder, with each handle being attached to the ball along two 2-disks.

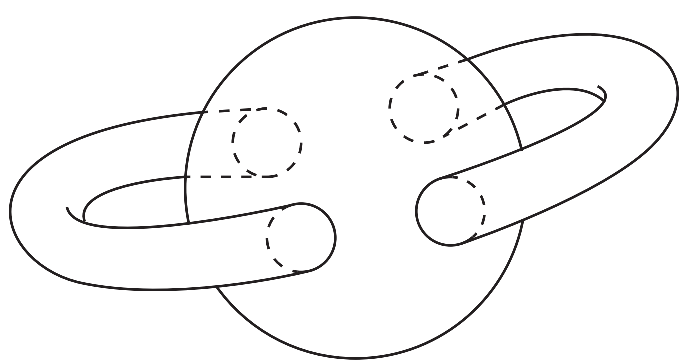


Figure 2.1: An example of a handlebody.

**Remark 2.1.2.** Given a pants decomposition $P$ of a surface $S$, one may construct a handlebody $\mathfrak{H}$ in the following way: First, for each pair of pants $\mathfrak{p}_i \in P$, consider a 3-disk $D_i$ that contains three 2-disks in its boundary - one disk for each curve that bounds $\mathfrak{p}_i$. Then for each adjacent pair of pants $\mathfrak{p}_i, \mathfrak{p}_j$ from $P$, attach $D_i$ to $D_j$ along the appropriate disks. One obtains a handlebody $\mathfrak{H}$ (denoted $\mathfrak{H}(P)$ when speaking specifically of the handlebody generated by $P$) whose frontier coincides with $S$ and which maintains the structure of $P$ [5]. Therefore, when considering pants decompositions of surfaces, we may speak of pants decompositions of corresponding handlebodies.

Note that the genus of $\mathfrak{H}$ is the genus of its frontier.

**Definition 2.1.3** (Contractible - Pants Decompositions)**.** Given a handlebody $\mathfrak{H}$, a pants decomposition $P$ of $\mathrm{fr}(\mathfrak{H})$ is *contractible* in $\mathfrak{H}$ whenever each $c_i \in P$ bounds an embedded disk in $\mathfrak{H}$.

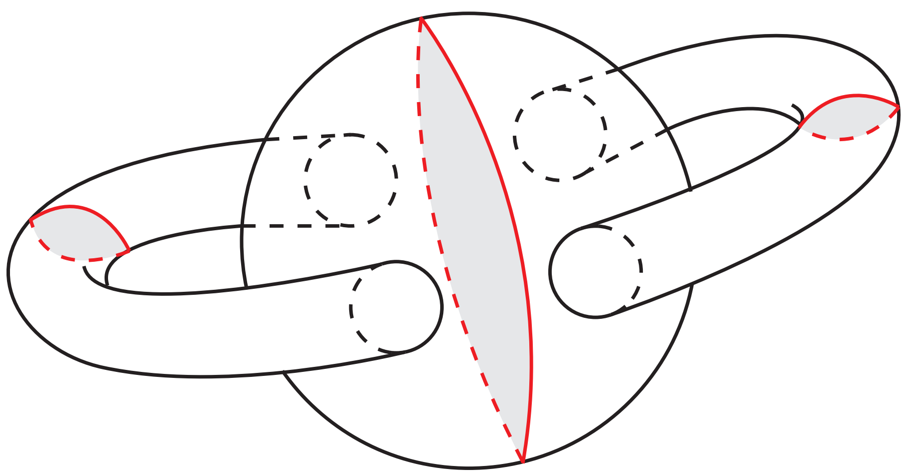


Figure 2.2: An example of a contractible pants decomposition of a handlebody's frontier. The curves of the pants decomposition bound shaded disks in the handlebody.

We mention that contractibility is usually defined in terms of curves bounding disks that are not necessarily embedded. However, by Dehn's Lemma, a contractible curve bounds an embedded disk in $\mathfrak{H}$ [6]. Thus, our definition is suitable.

**Lemma 2.1.4** (Boring iff Contractible)**.** *A pants decomposition $P$ of a surface $S$ compatible with $Z_E$ is boring if and only if it is contractible in the handlebody $\mathfrak{H}$ in $\mathbb{R}^3$ with $fr(\mathfrak{H}) = S$.*

*Proof.* To see the first direction, note that if $P$ is boring, each curve $c_i \in P$ intersects the equator twice, and $c_{i_+} = c_{i_-}$ where $c_{i_+} = c_i \cap S_+$ and $c_{i_-} = c_i \cap S_-$. Therefore, $c_i$ bounds a disk in $\mathfrak{H}$, so is contractible.

To see the reverse direction, since $P$ is compatible with $Z_E$, again note that each $c_i \in P$ has only two equatorial crossings. Then we may orient $c_{i_+}$ and $c_{i_-}$ so that $c_i = c_{i_+} \star c_{i_-}$ is defined. Notice that it is a loop in $\mathfrak{H}$, and since it is contractible, it is null-homotopic. Since $c_{i_+}$ and $c_{i_-}$ are both connected paths, it must be that they are inverses of each other. Therefore, they may be isotoped so that they travel through the same points, just in an opposite orientation. Since this holds for all $c_i \in P$, we have that $\tau P|_{S_+} = P|_{S_-}$ where $\tau$ is the north/south reflection, and thus $P$ is boring. ∎

**Definition 2.1.5** (Zipped Flip Equivalent)**.** Given two pants decompositions $P_a, P_b$, objects in $\mathfrak{P}_g^z$, $P_a$ is said to be *zipped flip equivalent* to $P_b$ if there exists a morphism carrying one to the other.

**Lemma 2.1.6** (Middles Lemma)**.** *Given any standard boring pants decomposition $P$ of a surface with genus greater than 3, any arrangement of its middle curves $M$ is zipped flip equivalent to any other.*

*Proof.* We prove the claim via induction:

Our base case, $g = 4$, is trivial. In this case, by Proposition 1.2.5, there are 9 curves in $P$. Since $P$ is standard, there are 4 principal curves and 4 self-folding curves, leaving only one middle curve. Since $P$ is boring, it follows that there are only two choices for this curve, and performing a zipped flip to either choice yields the other. See Figure 2.3.

Assume the claim holds for $g = n$. Now consider $g = n + 1$. The middle curves are embedded in a $(n + 1)$-punctured sphere. Each puncture corresponds to a principal curve, $p_i \in P$. Suppose the principal curves are labelled in ascending order with respect to the cyclic order of $Z_E$. By Proposition 1.2.5, every pants decomposition contains $3g - 3$ curves. It is obvious that for $g > 2$, standard pants decompositions have $2g$ curves which make up handles ($g$ principal curves and $g$ self-folding curves), leaving $g - 3$ middle curves. Therefore, we have $n - 2$ middle curves.

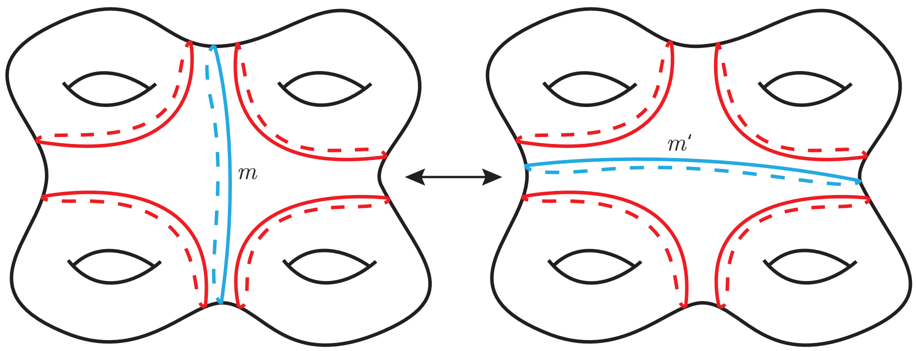



Figure 2.3: There are only two possible middle curves for a standard boring pants decomposition of a genus 4 surface. It is obvious that $m$ may be zipped flipped to $m'$ and vice versa. Note that the self-folding curves are not drawn to avoid clutter.

The principal zipper $z_0 \in Z_E$ contains $g = n + 1$ connected components $\{z_0^k \mid 1 \leq k \leq g\}$ in the sphere, where $z_0^j$ is the section of $z_0$ travelling between $\mathfrak{h}_j$ and $\mathfrak{h}_{j+1}$. Now consider $z_0^2$. If there are middle curves that intersect $z_0^2$, zipped flip them so this is not the case. The claim is that if $z_0^2$ has no intersections with middle curves, $m_i$, then there is some middle curve $m_v$ which bounds a pair of pants with $p_2$ and $p_3$ as the other boundary components. Specifically, this would mean that the curve $m_v$ intersects $z_0^1$ and $z_0^3$.

To see that this is true, suppose that there is no middle curve intersecting $z_0^3$. Then there are no curves separating $p_2, p_3$, and $p_4$ from each other, so they must bound a pair of pants. But the only way a pair of pants can be bounded by three principal curves is in the $g = 3$ case. So, there must be a middle curve, say $m_h$, that intersects $z_0^3$. The same argument applies for $z_0^1$, so there is a middle curve $m_k$ that intersects $z_0^1$. Since no curve intersects $z_0^2$, and since pairs of pants are bound by only three curves from $P$, either $m_h = m_k$ or there is another curve $m_v$ that intersects both $z_0^1$ and $z_0^3$. See Figure 2.4.

So, let $\mathfrak{p}_\mathfrak{i}$ be the pair of pants bounded by $m_v, p_2, p_3$. Consider the complement of $\mathfrak{p}_\mathfrak{i} \cup \mathfrak{h}_2 \cup \mathfrak{h}_3$ in the surface. It is precisely the $g = n$ case. By induction, we may rearrange the middle curves in the surface in any boring way we please, via zipped flips. Arrange them so that they all emanate from $z_0^1$. See Figure 2.5.

Since any arrangement of middle curves may be put into this form, and since zipped flips are invertible, we have our result. □

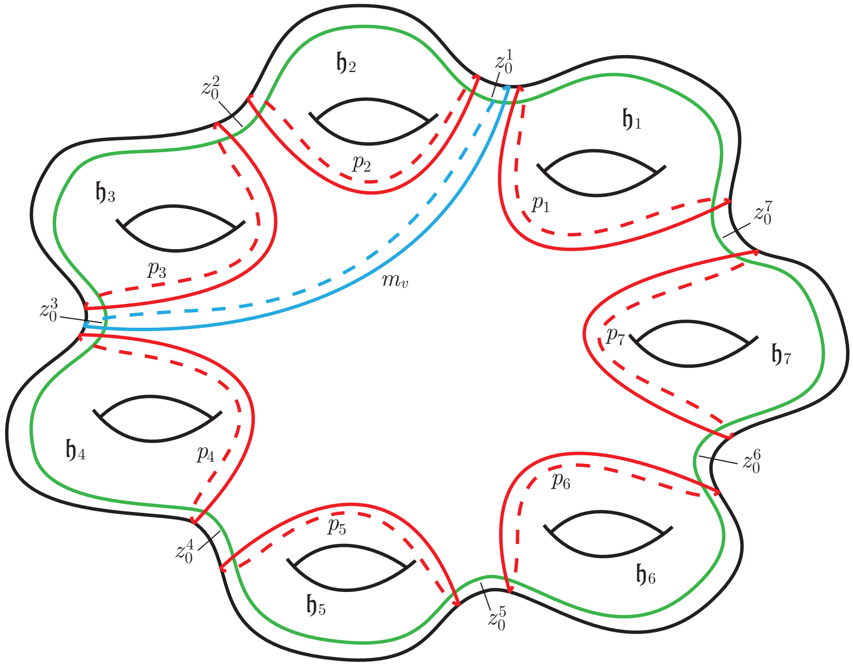



Figure 2.4: If no middle curve intersects $z_0^2$, then there must be a middle curve $m_v$ that intersects both $z_0^1$ and $z_0^3$, isolating $\mathfrak{h}_2$ and $\mathfrak{h}_3$. Note that the self-folding curves, zippers other than $z_0$, and middle curves other than $m_v$ are not drawn to avoid clutter.

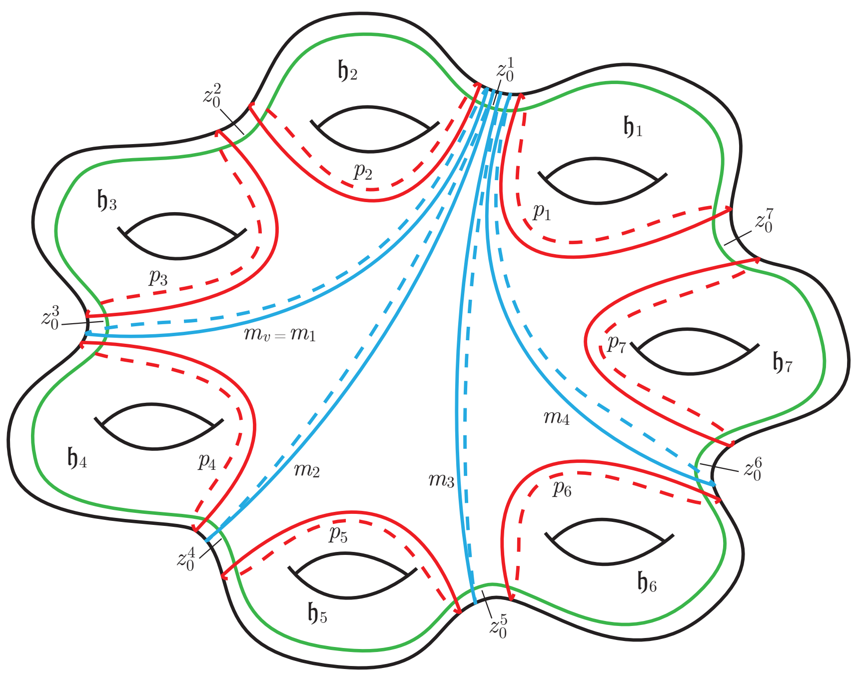



Figure 2.5: The middle curves emanate from $z_0^1$. Here, the $m_1$ curve is $m_v$ from Figure 2.4.

**Lemma 2.1.7** (Permutation of Cyclic Order)**.** *Given any standard boring pants decomposition $P$ and a zipper system $Z$, any two zippers $z_i, z_{i+1}$ may be transposed in the cyclic order via zipped flips.*

*Proof.* Without loss of generality, the handles of $P$ are labelled in ascending order with respect to an orientation of $z_0$. The case of $g = 2$ is trivial. For $g > 2$, consider the handles $\mathfrak{h}_\mathsf{i}, \mathfrak{h}_{\mathsf{i+1}}$ which correspond to $p_i$ and $s_i$, and $p_{i+1}$ and $s_{i+1}$, respectively, where $p_i, p_{i+1}$ are principal curves and $s_i, s_{i+1}$ are self-folding curves. If $g = 3$, then $\mathfrak{h}_\mathsf{i}, \mathfrak{h}_{\mathsf{i+1}}$ are separated from the third handle by its principal curve. If $g > 3$, then by Lemma 2.1.6, we may arrange the middle curves of $P$ is such a way that $\mathfrak{h}_\mathsf{i}, \mathfrak{h}_{\mathsf{i+1}}$ are separated from the rest of the decomposition by a middle curve. Then as seen in Figure 2.6, we may transpose $z_{i+1}$ and $z_i$ in the cyclic order in the following way:

First, zipped flip $p_{i+1}$. Then zipped flip $s_{i+1}$. We now have the desired result. Since the choice of $z_i, z_{i+1}$ were arbitrary, the process may be repeated so that any permutation of the cyclic order is achievable. □

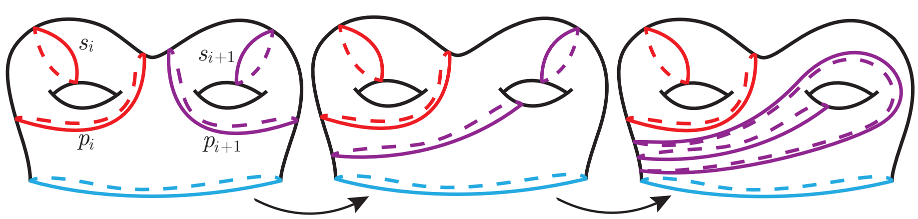



Figure 2.6: Two handles may be transposed in the cyclic order by zipped flipping $p_{i+1}$ followed by $s_{i+1}$. Note that the equatorial zippers are not drawn to avoid clutter.

Notice that zipped flipping $p_i$ followed by $s_i$ would result in a different pants decomposition, but would have the same effect of $z_{i+1}$ and $z_i$ being transposed in the cyclic order. Later in Theorem 2.1.9, when speaking of 'pushing one handle past another', we are referring to zipped flipping one handle's principal and self-folding curves and leaving the other's alone.

Before proving the following Theorem, we begin by introducing some notation: For the next theorems, with respect to the specific embeddings used, we refer to the *outermost zipper* as the outermost curve, geometrically, in $Z_E$. We refer to an *inner zipper* as any curve in $Z_E$ which lies inside of the outermost zipper, geometrically. See Figure 1.15; here, $z_1$ is the outermost zipper while $z_2$ and $z_3$ are inner zippers.

**Definition 2.1.8** (Friendly Pants Decomposition)**.** A pants decomposition $P$ that is compatible with the equatorial zipper system $Z_E$ is said to be *friendly* if it is

- boring,
- standard, and
- the outermost zipper is the principal zipper.

Note that the pants decomposition shown in Figure 1.4 is friendly.

Given a friendly pants decomposition $P$, we can arrange the curves of $P$ in the following way:

First, considering the lateral embedding of $S$, since $P$ is friendly, we need only to consider $P|_{S_+}$ in view of symmetry. Choose a basepoint $x_0 \in z_0 \subset S_+$ in the center of the bottom of $S_+$. Give each self-folding curve of $P$ a positive orientation from endpoints on $z_0$ to endpoints on some other $z_i$. Isotope all curves so that they begin in some small neighborhood of $x_0$, their cyclic order is preserved, and they contain no unnecessary intersections. Furthermore, isotope the curves so that they share no unnecessary intersections with the vertical lines which pass through the centers of each puncture of $S_+$. We will refer to these vertical lines $v_i$ as the *vertical(s)* of $S_+$, or just the *vertical(s)*. See Figure 2.7. Notice that at this point, each principal curve follows almost the same path as its corresponding self-folding curve, up to the puncture the self-folding curve ends at. Once the principal curve reaches this puncture, it wraps around it and follows almost the same path back to the neighborhood of $x_0$. Therefore, a principal curve is determined by its corresponding self-folding curve, so we may focus only on the self-folding curves of $P$.

Next, we will define a curve's *complexity* to be the number of necessary intersections it shares with the upper verticals of $S_+$ (the sections of the verticals between the top of the puncture and the outer boundary of $S_+$). Then *simplifying* a curve amounts to reducing this number of intersections. The claim is that this is accomplished via zipped flips.

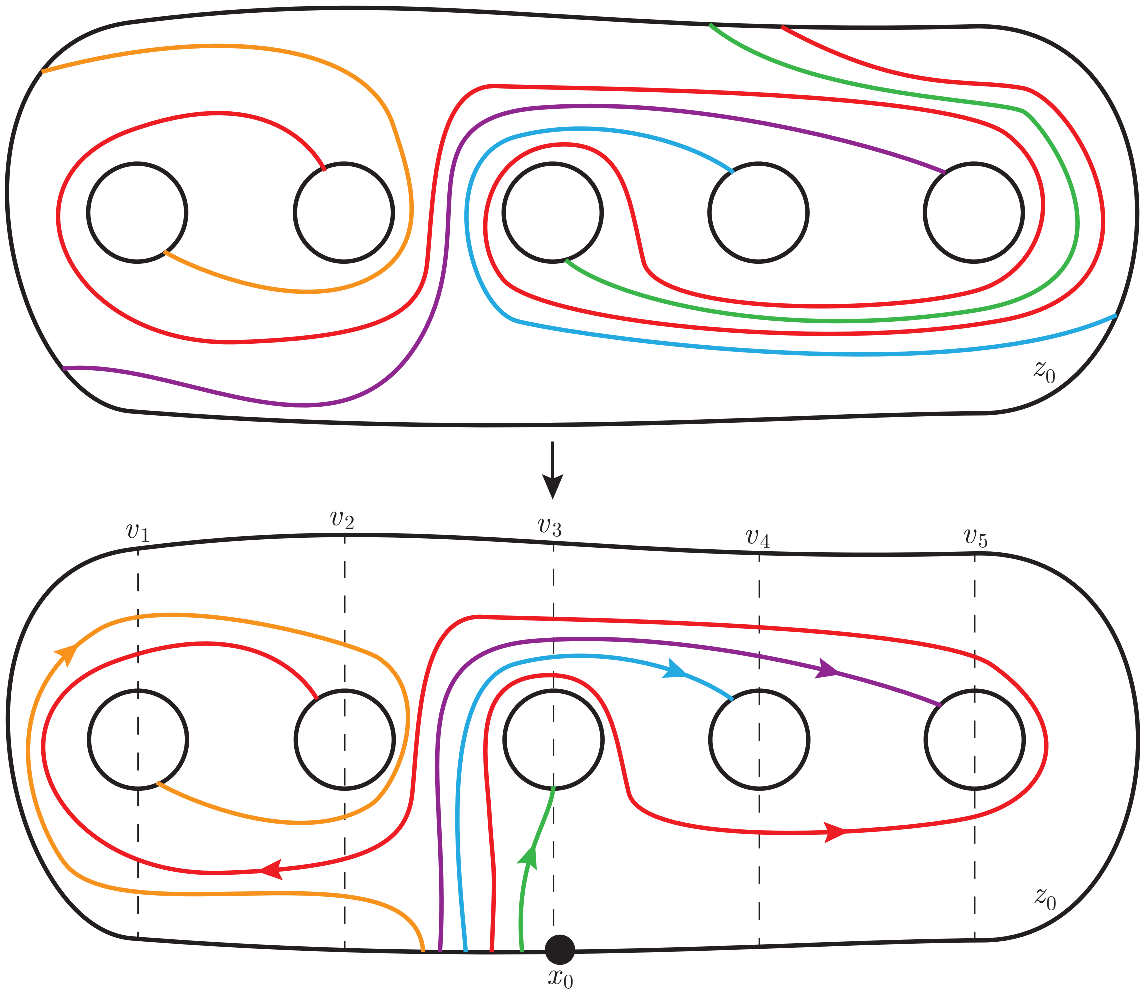



Figure 2.7: Curves of a friendly pants decomposition of a genus 5 surface are isotoped so that they begin in a neighborhood of $x_0$ and share no unnecessary intersections with other curves or verticals. Note that only $S_+$ and the self-folding curves are shown.

Finally, we denote by $P_s$ the *simple friendly pants decomposition* seen in Figure 2.8. Specifically, $P_s$ is the friendly decomposition whose self-folding curves have no intersections with the upper verticals and whose principal curves all have one intersection with the vertical corresponding to the puncture it wraps around.

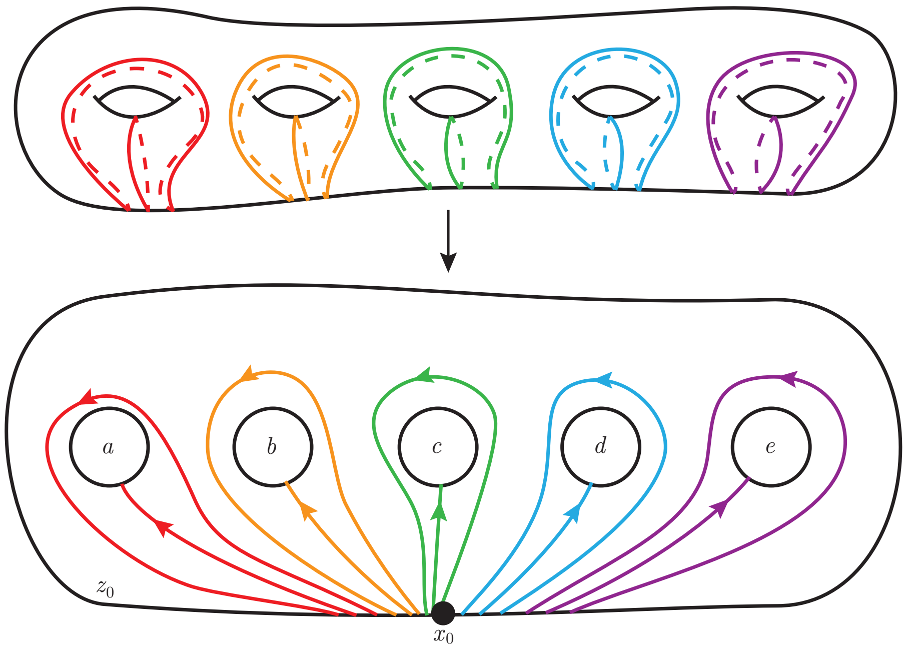



Figure 2.8: A simple friendly pants decomposition of a genus 5 surface. In $P|_{S_+}$, its self-folding curves have no intersections with upper verticals, and its principal curves have one intersection with upper verticals. Note that the middle curves and the zippers are not drawn to avoid clutter.

**Theorem 2.1.9** (The Friendly Theorem)**.** *Given any two friendly pants decompositions $P_a$ and $P_b$, $P_a$ is zipped flip equivalent to $P_b$.*

*Proof.* Using the set-up described above, we first show that given any friendly pants decomposition $P_a$, that $P_a$ is zipped flip equivalent to $P_s$ where $P_s$ is the simple friendly decomposition. Then since $P_a$ was an arbitrary decomposition, and since zipped flips are invertible, it will follow that any friendly $P_a$ is zipped flip equivalent to any friendly $P_b$.

First, consider the base case of $g = 2$. This verifies trivially as, up to isotopy, there is only one possible friendly pants decomposition.

Next, assume the claim holds for $g = n$. Consider the case of $g = n + 1$. Select a self-folding curve, $s_i$, you wish to simplify (obviously this choice excludes curves not sharing intersections with the verticals). The corresponding principal curve $p_i$ bounds a handle $\mathfrak{h}_i$. Consider the complement of $\mathfrak{h}_i$ in $S$. It is a genus $n$ surface with a single puncture. Filling in this puncture with a disk yields a closed genus $n$ surface. Notice that in doing so, any homotopically equivalent curves become identified, and we are left with a friendly pants decomposition of a $g = n$ surface. So, by induction, we may rearrange the curves on this surface in any friendly way we wish, using only zipped flips. We may then use these new curves on our $g = n + 1$ surface.

For $j \neq i$, choose a vertical $v_j$ that shares a necessary intersection with $s_i$. Redraw the self-folding curve $s_j$ that ends at the puncture corresponding to $v_j$ in the following way: Trace the $s_i$ curve exactly until the lowest intersection with the upper vertical. Then isotope $s_j$ slightly so that it contains no intersections with $s_i$. See Figure 2.9. At this point, $s_i$ and its corresponding principal curve $p_i$, as well as the newly drawn $p_j$ and $s_j$, correspond to handles $\mathfrak{h}_i$ and $\mathfrak{h}_j$, respectively. Now by Lemma 2.1.6, we may draw a middle curve $m_i$ so that it isolates $\mathfrak{h}_i$ and $\mathfrak{h}_j$ and so that these handles are next to each other in the cyclic order. See Figure 2.10. Note that in the case of $g = 3$, the third principal curve isolates these handles. Now draw in the remaining curves however you like. Then $\mathfrak{h}_i$ and $\mathfrak{h}_j$ are still next to each other in the cyclic order.

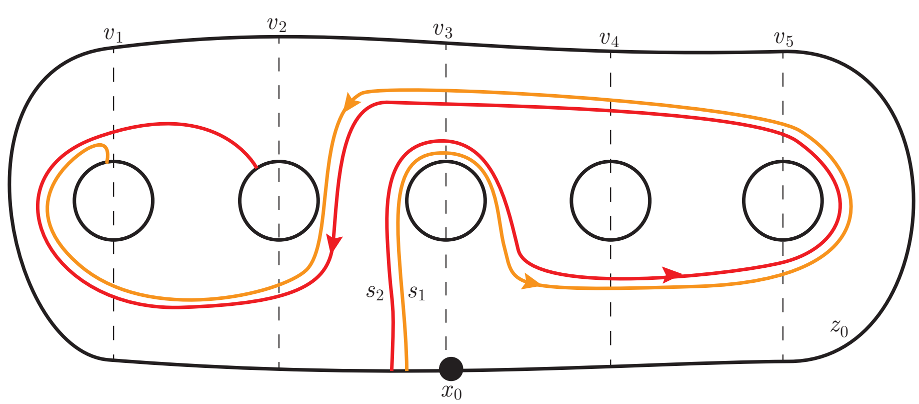



Figure 2.9: Here, we wish to simplify the self-folding curve from Figure 2.7 ending at the puncture that corresponds to $v_2$. This curve is labelled $s_2$. We wish to remove the intersection with $v_1$, so $s_1$ is redrawn so that it follows $s_2$ until the puncture at $v_1$.

Now by Lemma 2.1.7, we may transpose the cyclic order of $\mathfrak{h}_i$ and $\mathfrak{h}_j$. The claim is that this will simplify $s_i$ and $p_i$. To see this, notice that transposing the cyclic order of

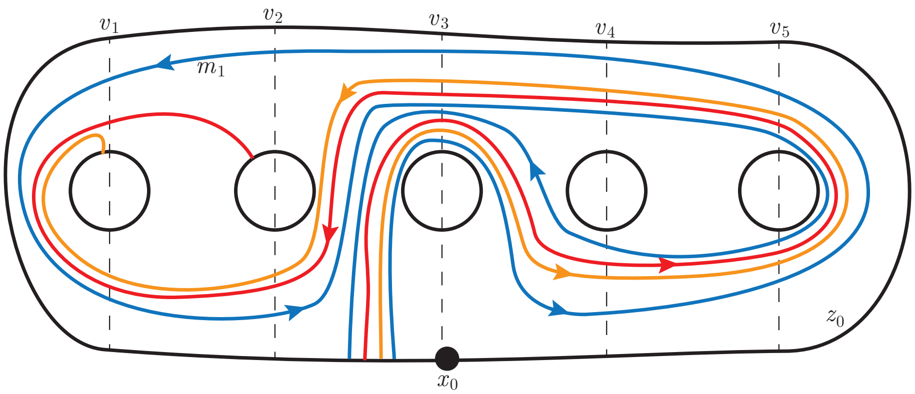



Figure 2.10: A middle curve $m_1$ is drawn so that it isolates $s_1$ and $s_2$ (and therefore $p_1$ and $p_2$) and preserves the cyclic order.

these two handles has the effect of pushing $\mathfrak{h}_i$ past $\mathfrak{h}_j$. See Figure 2.11. In $S_+$, this causes the curves of $\mathfrak{h}_i$ to follow a similar path as before, just on the other side of the curves of $\mathfrak{h}_j$. In particular, this causes the section of $s_i$ that shared the lowest intersection with the upper verticle to now travel below the puncture (and therefore intersect the lower vertical). See Figure 2.12. This causes an intersection with the upper vertical to be removed. Now we may isotope the $\mathfrak{h}_i$ curves so that they share no unnecessary intersections with the verticals. Notice that we have reduced the complexity of these curves. See Figure 2.13.

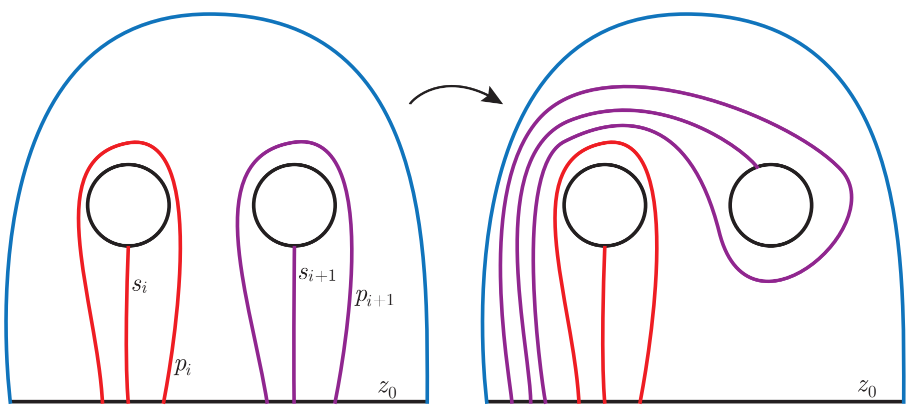



Figure 2.11: An isotopy of Figure 2.6 restricted to $S_+$ is shown. Notice that by transposing the cyclic order, the handle corresponding to the $i+1$ curves is pushed past the other via zipped flips to $p_{i+1}$ and $s_{i+1}$.

Repeat this process for $s_i$ as many times as necessary until $s_i$ and $p_i$ have no intersections with any vertical except for $v_i$. Since $s_i$ shares no intersections with any

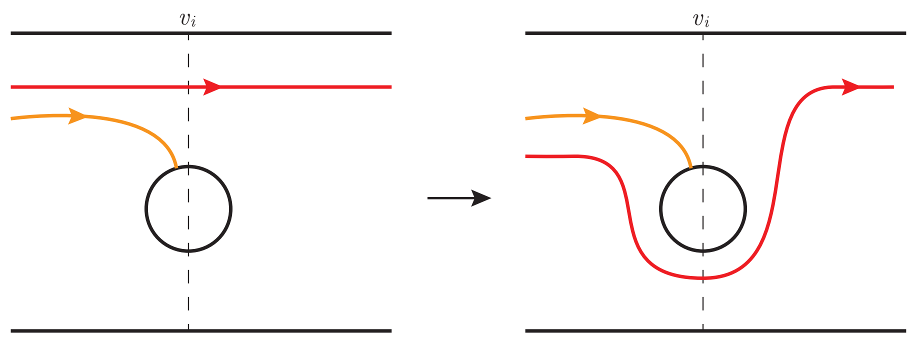



Figure 2.12: In general, pushing a self-folding curve past $s_i$ removes an intersection with upper $v_i$. Note that a similar result holds if the curves are travelling in the opposite direction.

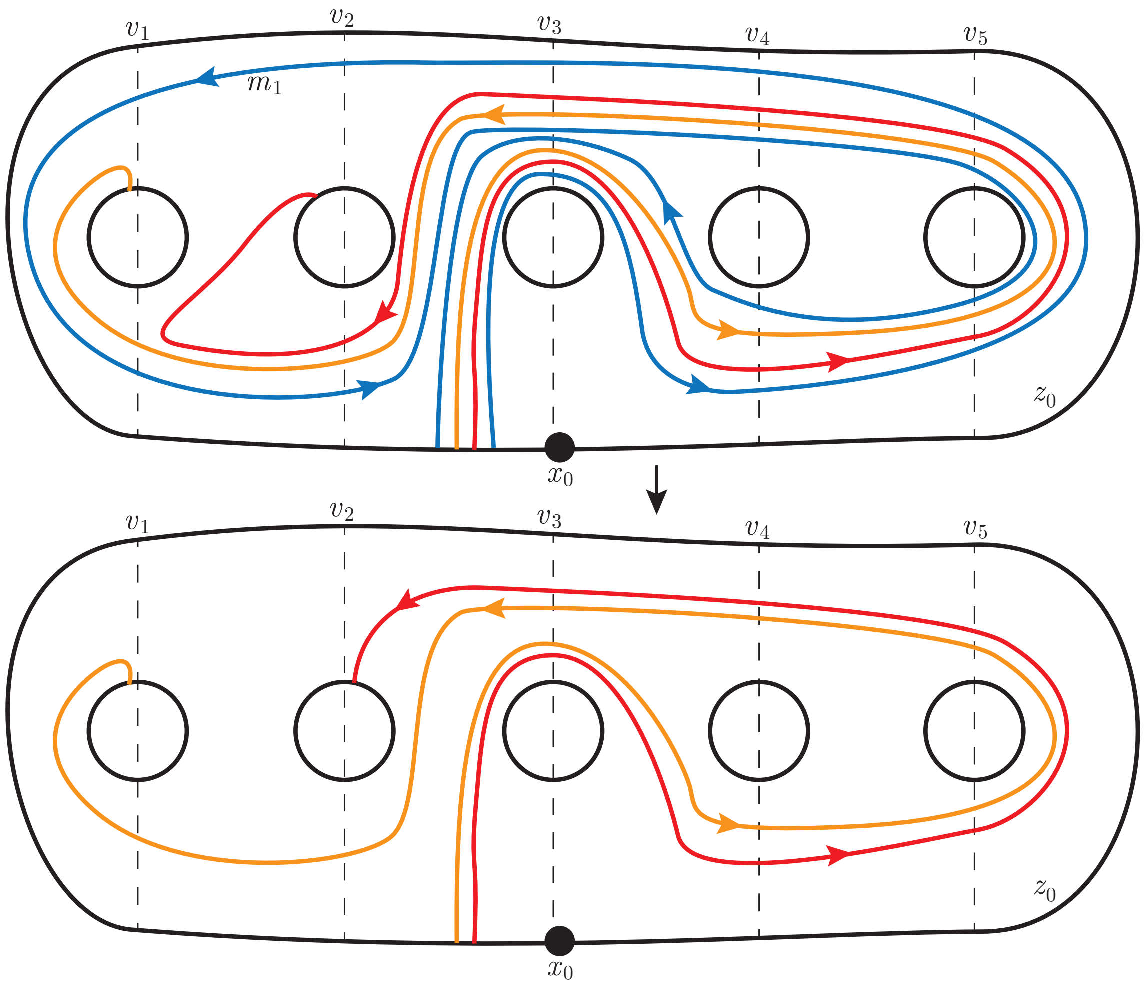



Figure 2.13: The $s_2$ curve from Figure 2.9 is pushed past $s_1$ in the top figure. This new curve may be isotoped to the curve shown in the bottom figure.

other upper vertical or any other curve in $P$, $s_i$ must travel from the neighborhood of $x_0$ to the puncture in a trivial way, specifically, so that $s_i$ may be istoped such that it has no intersections with $v_i$. So, $s_i$ has no intersections with upper verticals, and therefore $p_i$ has one intersection with the upper $v_i$. By induction, it is possible to arrange the rest of the curves in the simple friendly manner. Thus, $P_a$ is zipped flip equivalent to $P_s$. Since $P_a$ was arbitrary, and since zipped flips are invertible, we have our result. ∎

**Lemma 2.1.10** (The Boring Lemma)**.** *Given any boring pants decomposition $P$, $P$ is zipped flip equivalent to a standard boring pants decomposition $P'$.*

*Proof.* Boring pants decompositions have planar corresponding graphs. Suppose $P$ is a boring non-standard pants decomposition. Since it is non-standard, its graph contains at least one cycle of length greater than one. Choose one of these cycles. Choose an edge $e_i$ in this cycle. Then $e_i$ corresponds to a curve $c_i \in P$. Two pairs of pants are adjoined by $c_i$ (since $e_i$ is not a loop, and therefore $c_i$ is not a self-folding curve). Each of these pairs of pants correspond to nodes in the graph, say $n_1$ and $n_2$. For each of these nodes, there are two edges connecting them to two other nodes. Say $n_1$ gets connected to $n_1^1$ and $n_1^2$ and that $n_2$ gets connected to $n_2^1$ and $n_2^2$. See Figure 2.14. Then zipped flipping $c_i$ results in $n_1$, $n_2$ being replaced with new nodes $n_1'$, $n_2'$, respectively, where $n_1'$ is connected to $n_1^1$ and $n_2^1$ and $n_2'$ is connected to $n_1^2$ and $n_2^2$. This has the effect of replacing the edge $e_i$ in the cycle with either the node $n_1'$ or $n_2'$, and so the length of the cycle that $e_i$ was part of is reduced by one. See Figure 2.14.

Note that this process does not affect 1-cycles. To see this, notice that since our graph is trivalent, there is only one edge eligible for this process connected to a node that has a 1-cycle. Therefore, the edge is not part of a larger cycle, and thus will not be flipped leaving the 1-cycle untouched. Now, repeat this process to edges in the cycle, and to each cycle, until only 1-cycles remain. This gives a tree with 1-cycles attached which corresponds to a standard pants decomposition $P'$. Since $P$ was boring, so is $P'$, and we have our result. ∎

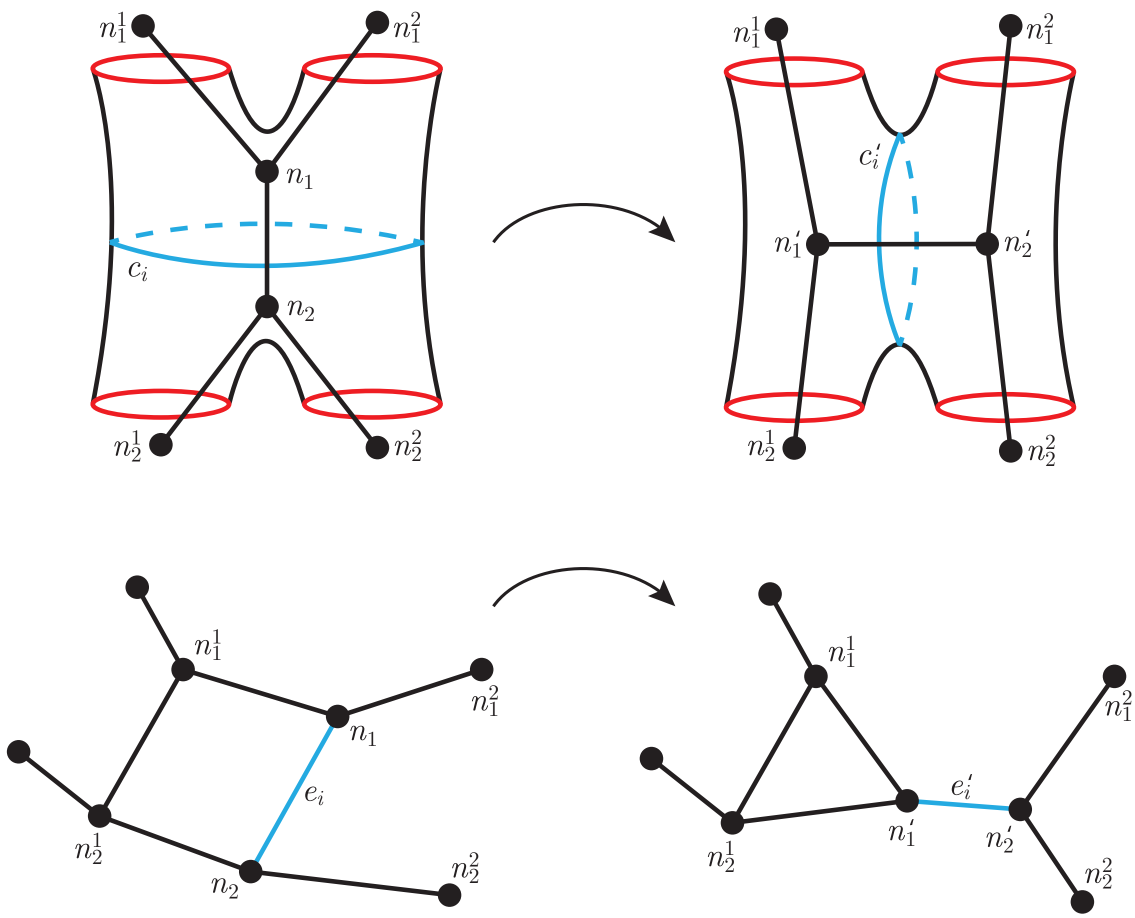



Figure 2.14: The top row illustrates how the pairs of pants joined via $c_i$ and the corresponding graph relate. The bottom shows a section of the graph for some boring pants decomposition. Specifically, it shows a cycle of length 4 that contains the edge $e_i$ and the resulting graph after $c_i$ is zipped flipped.

**Lemma 2.1.11** (The Handle Lemma)**.** *The curves of the pants decomposition $P$ shown in Figure 2.15 are zipped flip equivalent to one another.*

*Proof.* Follow the steps in Figure 2.15. Zipped flip $p_i$, then zipped flip $s_i$. Notice that these two zipped flips took our handle $\mathfrak{h}_i$ to a new handle $\mathfrak{h}_i'$ with principal curve $s_i'$ and self folding curve $p_i'$. ◻

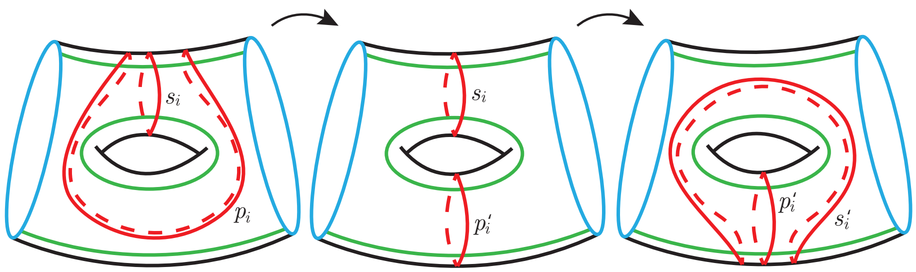



Figure 2.15: The handle that has $s_i$ as a self-folding curve and $p_i$ as a principal curve is contained in a surface bounded by two other curves from $P$. Zipped flip $p_i$ then $s_i$ to change which zipper section the curves intersect.

Before proving the following Theorem, we introduce more notation: Using the circular embedding for a given surface $S$ and a pants decomposition $P$ compatible with the equatorial zipper system $Z_E$, denote by $\langle c_1, c_2, \dots \rangle_{z_i}$ the order in which curves $c_j \in P$ intersect a zipper $z_i \in Z_E$ with respect to a counter-clockwise orientation on $z_i$, when viewing $S$ from above. We refer to this as the *curve order on* $z_i$.

**Theorem 2.1.12** (The Boring Theorem)**.** *Given any two boring pants decompositions $P_a, P_b$ both compatible with the equatorial zipper system, $P_a$ is zipped flip equivalent to $P_b$.*

*Proof.* Let $P_a$ be some boring pants decomposition. Lemma 2.1.10 states that $P_a$ is zipped flip equivalent to some $P_a'$ which is standard and boring. The same holds for some $P_b$. Now if $P_a'$ and $P_b'$ have the outermost zipper as their principal zipper, we have nothing more to prove in light of Theorem 2.1.9. So, assume $P_a'$ has one of the inner zippers as its principal zipper. Then showing that the principal zipper can be changed via zipped flips yields our result.

First, using the circular embedding, without loss of generality, we may arrange the curves of $P$ as in Figure 2.16 by Lemma 2.1.6 and by the curve-simplifying method used in Theorem 2.1.9. Specifically, arrange the curves so that $z_0$ is the principal zipper and the curve order on $z_0$ is

$\langle s_1, p_1, p_2, s_2, p_2, m_1, p_3, s_3, p_3, m_2, p_4, s_4, p_4, \ldots, m_{g-3}, p_{g-1}, s_{g-1}, p_{g-1}, p_g, s_g, p_g, m_{g-3},$
$m_{g-4}, \ldots, m_2, m_1, p_1\rangle_{z_0}$.

Next, zipped flip $p_1, m_1, m_2, \ldots, m_{g-3}$ consecutively in that order. By inspection, we have $\langle s_1, p_2, s_2, p_2, p_1', p_3, s_3, p_3, m_1', p_4, s_4, p_4, m_2', \ldots, m_{g-3}', p_g, s_g, p_g\rangle_{z_0}$ and $\langle s_1, p_1', m_1', m_2', \ldots, m_{g-3}'\rangle_{z_1}$. See Figure 2.17.

Now apply Lemma 2.1.11 consecutively to each handle. We get $\langle s_1, p_1', m_1', m_2', \ldots, m_{g-3}'\rangle_{z_0}$ and $\langle s_1, s_2', p_2', s_2', p_1', s_3', p_3', s_3', m_1', s_4', p_4', s_4', m_2', \ldots, m_{g-3}', s_g', p_g', s_g'\rangle_{z_1}$. See Figure 2.18.

Finally, zipped flip $m_{g-3}', m_{g-4}', \ldots, m_2', m_1', p_1'$ consecutively in that order. We see that $z_1$ now has curve order

$\langle s_1, p_1'', s_2', p_2', s_2', m_1'', s_3', p_3', s_3', m_2'', s_4', p_4', s_4', m_3'', \ldots, m_{g-3}'', s_{g-1}', p_{g-1}', s_{g-1}', s_g', p_g', s_g',$
$m_{g-3}'', m_{g-4}'', m_{g-5}'', \ldots, m_2', m_1', p_1''\rangle_{z_1}$, and $z_0$ has curve order $\langle s_1\rangle_{z_0}$. So $z_1$ is now the principal zipper as it visits each handle, and we have our result. See Figure 2.19. ◻

**Corollary 2.1.13.** *Given a handlebody $\mathfrak{H}$ and any zipper system $Z$, any two pants decompositions contractible in $\mathfrak{H}$ and compatible with $Z$ are zipped flip equivalent.*

*Proof.* Let $P$ be some pants decomposition of the frontier of the handlebody $\mathfrak{H}$ that is compatible with $Z$, and that is contractible in $\mathfrak{H}$. Then each curve in $P$ bounds an embedded disk in $\mathfrak{H}$. Cutting along these disks results in pieces that are solid pairs of pants. As in Remark 1.2.12, there exists an involution, $\tau$, of $\mathfrak{H}$ fixing $Z$ and leaving $P$, and the disks, invariant. Then there is a homeomorphism of $\mathfrak{H}$ which carries $Z$ to the equatorial zipper system and $\tau$ to the north/south reflection. Here, by Lemma 2.1.4, a pants decomposition $P'$ that is compatible with the equatorial zipper system is contractible if and only if it is boring. Then Theorem 2.1.12 applies. ◻

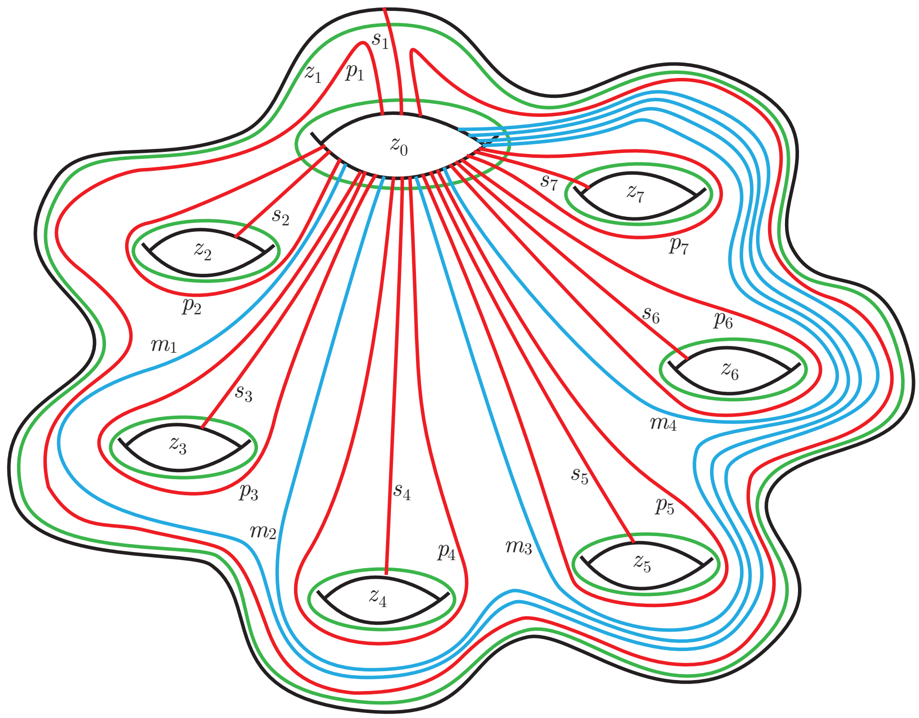



Figure 2.16: The initial arrangement of curves for Theorem 2.1.12 on a genus 7 surface. Note that only the top part of the curves are drawn in light of symmetry.

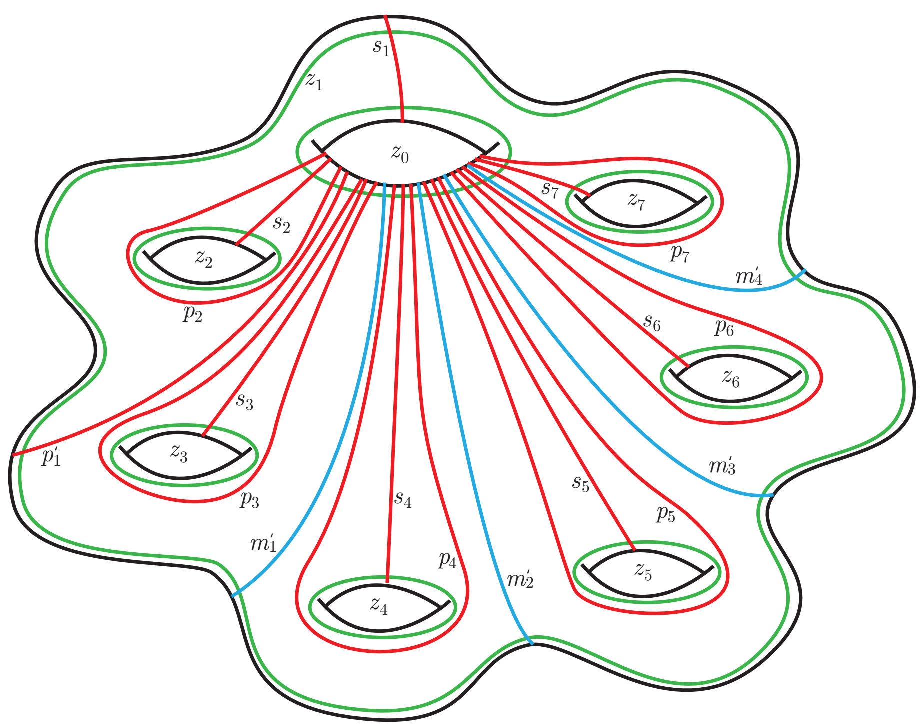



Figure 2.17: The second arrangement of curves for Theorem 2.1.12 on a genus 7 surface. Again, note that only the top part of the curves are drawn in light of symmetry.

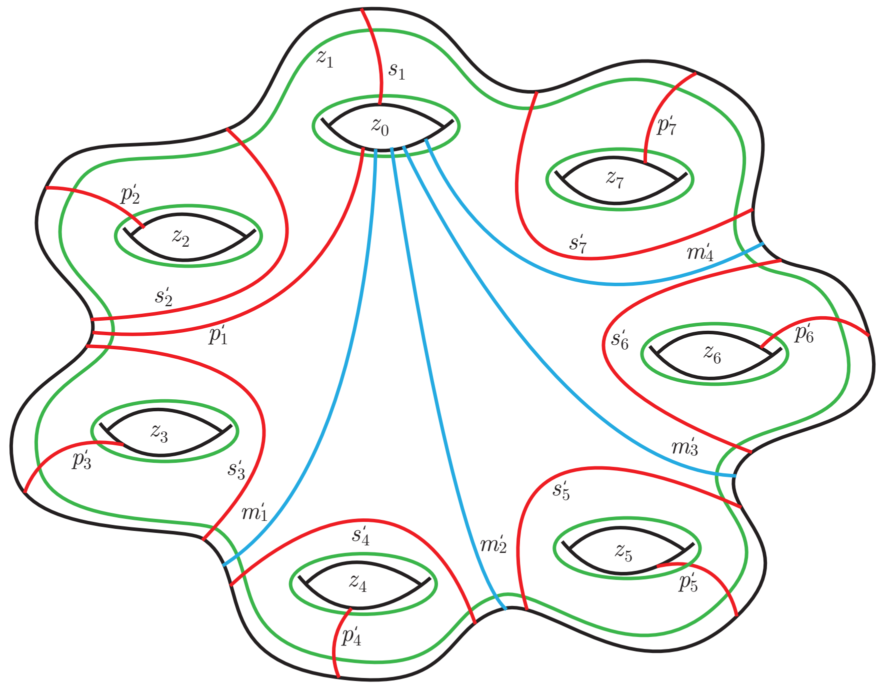



Figure 2.18: The third arrangement of curves for Theorem 2.1.12 on a genus 7 surface. Again, note that only the top part of the curves are drawn in light of symmetry.

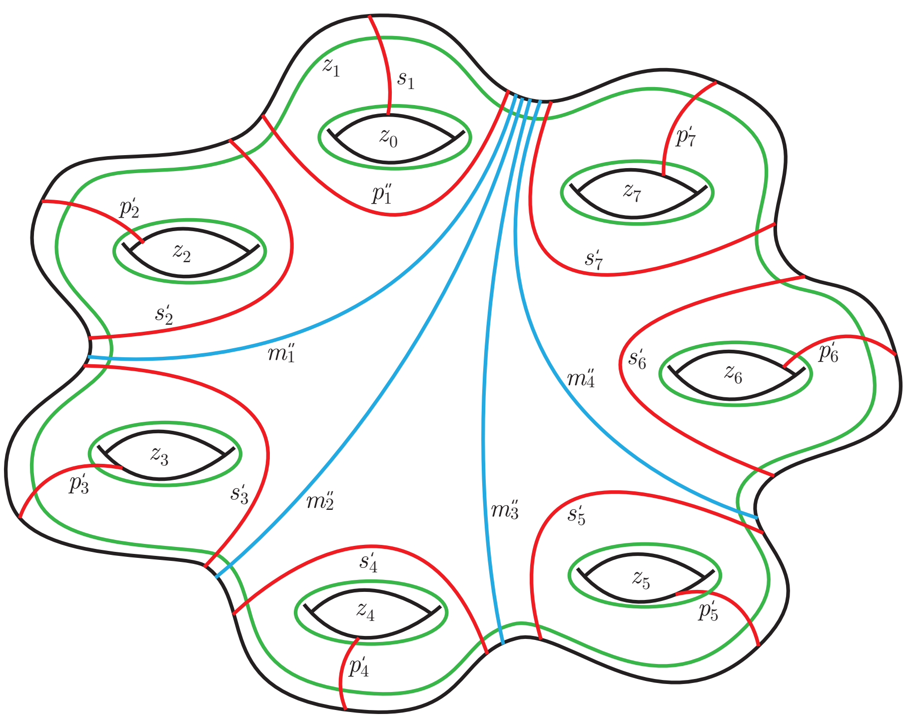



Figure 2.19: The final arrangement of curves for Theorem 2.1.12 on a genus 7 surface. Again, note that only the top part of the curves are drawn in light of symmetry.

# Conclusions and Future Directions

We conclude that, using zipped flips, any two pants decompositions compatible with the same zipper system and which are contractible in the same handlebody may be taken from one to the other. The final goal would be to show that any two pants decompositions contractible in the same handlebody and compatible with potentially different zipper systems are zipped flip equivalent. Due to time constraints, this was not achieved here. However, upon successful completion of this task, a major aspect of the argument used in the main theorem of [5] would be restored.

Even further research would entail addressing other aspects of [5] dealing with double pants decompositions, and potentially working with Felikson and Natanzon to correct any remaining inconsistencies.

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