Soup or Algebras?

Why Your Answer Should Be "Superalgebras"

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- Soup is tasty.

Overview

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- Superalgebras are more interesting.

- Soup is tasty.
- Algebras are interesting.
- Superalgebras are more interesting.
- Graded algebras appear naturally in many areas of math (think cohomology rings).

Definition

An (unital) **algebra** A over a field $\Bbbk$ is a vector space A along with a bilinear map

$$m : A \times A \to A$$

and a distinguished identity element (unit) 1_A such that $m(1_A, a) = a = m(a, 1_A)$ for all $a \in A$.

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Morally, A is a vector space in which we also know how to multiply two vectors to produce a new vector, and this multiplication is compatible with the existing structures.

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- The general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C}) := (M_n(\mathbb{C}), [\ ,\ ])$

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$\mathbb{k}^{m|n}$ is the superspace $\mathbb{k}^{m+n}$ with $\mathbb{k}_0^{m|n} = \mathbb{k}^m$ and $\mathbb{k}_1^{m|n} = \mathbb{k}^n$.

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$\mathbb{k}^{2|3}$ is a subsuperspace of $\mathbb{k}^{3|3}$ but is not a subsuperspace of $\mathbb{k}^{5|1}$.

Definition

A linear map f between superspaces V, W is **homogeneous of degree** i if $f(V_j) \subset W_{i+j}$ for $i, j \in \mathbb{Z}_2$.

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Example

Endow $\mathcal{C}(1)$ with $\mathbb{C}$-superspace structure $\langle 1 \rangle \oplus \langle x \rangle$. We can define a linear map on a basis by $1 \mapsto 1$ and $x \mapsto 1$.
This map is NOT homogeneous. It is the sum

$$\begin{cases} 1 \mapsto 1 \\ x \mapsto 1 \end{cases} = \begin{cases} 1 \mapsto 1 \\ x \mapsto 0 \end{cases} + \begin{cases} 1 \mapsto 0 \\ x \mapsto 1 \end{cases}$$

Notice that the kernel is $\langle 1 - x \rangle$ which is NOT a subsuperspace!

Definition

A **superalgebra** over a field $\Bbbk$ is a superspace $B = B_0 \oplus B_1$ along with a bilinear map

$$m : B \times B \to B$$

such that $m(B_i \times B_j) \subset B_{i+j}$ for $i, j \in \mathbb{Z}_2$.

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Definition

A nonzero module is **simple** if it has no proper submodules.

Definition

A **module homomorphism** between two A-modules V, W is a $\mathbb{k}$-linear map

$$\varphi : V \to W$$

such that $\varphi(av) = a\varphi(v)$ for all $a \in A$ and $v \in V$.

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Example

- Any algebra A is a module over itself (via left multiplication).
- The kernel of any module homomorphism is a submodule.

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A nonzero supermodule is **simple** if it has no proper subsupermodules.

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A **supermodule homomorphism** between two B-supermodules M, N is a (not necessarily homogeneous) $\Bbbk$-linear map

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Note that since the kernel of a linear map between superspaces may not be a subsuperspace, the kernel of a supermodule homomorphism need not be a subsupermodule in general.

A Closer Look at $\mathcal{C}(1)$

We claim that $\mathcal{C}(1)$ is simple as a supermodule over itself.

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- Any proper subsupermodule is 1-dimensional, so looks like $\langle \alpha 1 + \beta x \rangle$.
- Being closed under the action of $\mathcal{C}(1) \Rightarrow \langle 1 + x \rangle$ or $\langle 1 - x \rangle$.
- Neither of these spaces is a subsuperspace, hence cannot be a subsupermodule.

- Forgetting the $\mathbb{Z}_2$-grading on $\mathcal{C}(1)$ gives an algebra.

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- $\langle 1 + x \rangle$ and $\langle 1 - x \rangle$ are both closed under multiplication in $\mathcal{C}(1)$.
- Hence in the non-super case, $\mathcal{C}(1)$ is NOT simple as a module over itself.

In fact, back in the super setting, $\mathrm{End}_{\mathcal{C}(1)}(\mathcal{C}(1)) \cong \mathcal{C}(1)$ which is 2-dimensional as a usual vector space over $\mathbb{C}$!

This endomorphism space is spanned by

$$\mathrm{id} : \begin{cases} 1 \mapsto 1 \\ x \mapsto x \end{cases} \qquad \text{and} \qquad J : \begin{cases} 1 \mapsto x \\ x \mapsto -1 \end{cases}$$

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Theorem (Schur's Lemma)

Let M be a finite-dimensional simple B-supermodule over an algebraically closed field. Then

$$\mathrm{End}_B(M) = \begin{cases} \mathrm{span}\{\mathrm{id}_M\} & \textit{if } M \textit{ is absolutely irreducible} \\ \mathrm{span}\{\mathrm{id}_M, J_M\} & \textit{if } M \textit{ is self-associate} \end{cases}$$

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- Superspaces have a richer structure than ordinary vector spaces.
- Superalgebras have a strictly richer representation theory than algebras.
- Superalgebras and graded algebras not only are interesting objects to study on their own, they also appear naturally in many areas of math.

THANK YOU!