All Aboard the Skein Train!

Using the Skein Algebra to Study a Cluster Algebra

Joey Randich

University of Oklahoma

Graduate Algebra Symposium, October 13, 2018

Cluster Algebras are an interesting class of commutative algebras, introduced in a paper from 2002 by Fomin and Zelevinsky, whose development was initially motivated by total positivity and G. Lusztig's theory of canonical bases.

Cluster Algebras occur in the coordinate rings of important spaces such as semisimple Lie groups, Grassmanians, and decorated Teichmüller spaces. They've also found applications in other areas such as topology, discrete dynamical systems, and mathematical physics.

Last Time vs This Time

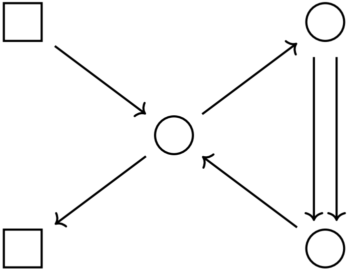
Last time, I defined cluster algebras of geometric type, and we explored some structural properties of such algebras.

This time, we will consider a (seemingly) different algebra whose diagramatics will greatly simplify computations in the cluster algebra.

First, let's begin with a quick review of cluster algebras:

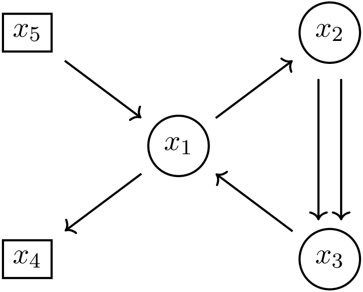
We consider finite quivers with no loops nor oriented 2-cycles. We do allow multiple edges. Moreover, we also distinguish two types of vertices: *frozen* vertices are drawn with squares, and *mutable* vertices are drawn with circles. Such a quiver is called an **Ice quiver**.

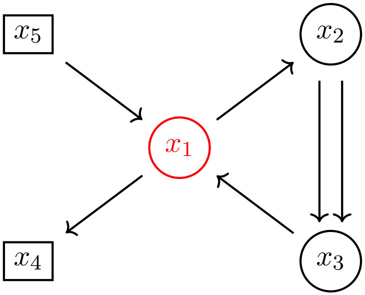
For example:

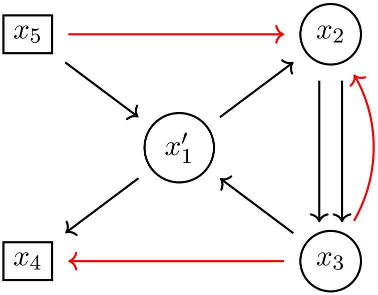


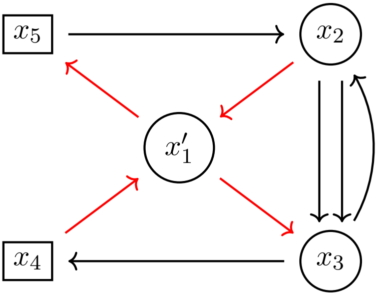
For a given quiver, a labeling of its vertices with independent variables x_i is a **labeled seed** $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$. Let $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m)$.

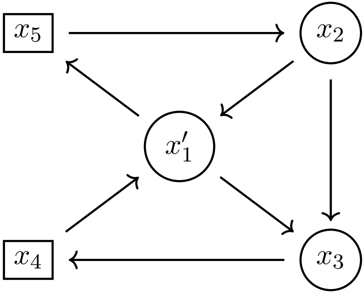
An example of a labeled seed is given below:




x_5
x_2
x_1
x_4
x_3


x_5
x_2
x_1'
x_4
x_3


x_5
x_2
x'_1
x_4
x_3



where $x'_k \in \mathcal{F}$ is given by the following *exchange relation*:

$$x'_k x_k = \prod(\text{upstream}) + \prod(\text{downstream})$$

Definition

Let $(\tilde{\mathbf{x}}_0, Q_0)$ be a given labeled seed. Let $R = \mathbb{Q}[x_{n+1}, \ldots, x_m]$ be the polynomial ring generated by the frozen variables. The **cluster algebra** $\mathcal{A}$ (of geometric type) associated with this seed is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables appearing in seeds which are mutation equivalent to $(\tilde{\mathbf{x}}_0, Q_0)$.

Definition

Let S be a connected oriented Riemann surface with boundary. Let $M \subset S$ denote a finite set of *marked points* with at least one point on each connected boundary component. The pair (S, M) is referred to as a **marked surface** or just a surface.

Points in the interior of S are called *punctures*. For our purposes, we do not consider pairs (S, M) of the following forms:

- Spheres with 1, 2, or 3 punctures;
- an unpunctured or once-punctured monogon;
- an unpuctured digon;
- an unpunctured triangle.

Definition

An **arc** γ in (S, M) is a curve in S, considered up to isotopy, such that

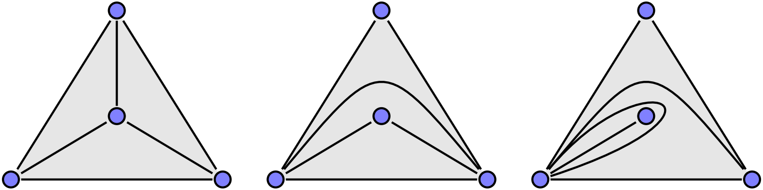
- the endpoints of γ are in M;
- γ does not cross itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from M and from ∂S,
- γ does not cut out an unpunctured monogon or an unpunctured digon (so γ is not contractible into M or onto the boundary of S).

Definition

A curve β between two marked points which lies entirely in the boundary of S and which does not pass through any other marked point is called a **boundary arc**.

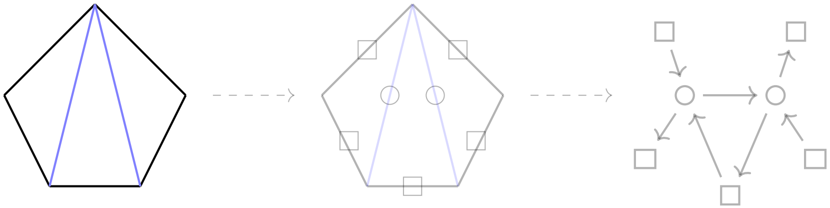
Definition

An **ideal triangulation** is a maximal collection of distinct pairwise compatible arcs $\{\gamma_1, \ldots, \gamma_n\}$, together with all possible boundary arcs $\{\beta_1, \ldots, \beta_m\}$. The arcs of a triangulation cut the surface into *ideal triangles*. Triangles that have only two distinct sides are called *self-folded triangles*.



The Cluster Algebra Associated to a Triangulated Surface

We may associate a quiver to a triangulation as follows:

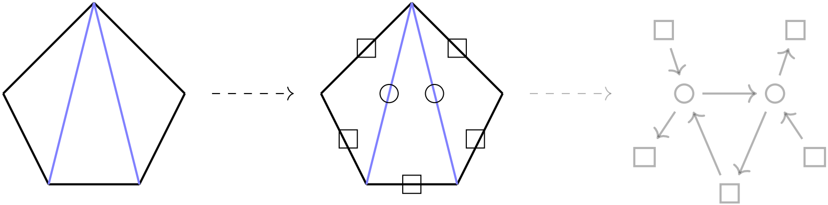


Definition

Given a marked surface (S, M), construct the associated quiver Q_T as above. Assign an indeterminant x_i to each vertex i, so that $\tilde{\mathbf{x}}_T = (x_1, \ldots, x_m)$ is an extended cluster. Then $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ is the **cluster algebra associated to the marked surface** (S, M).

The Cluster Algebra Associated to a Triangulated Surface

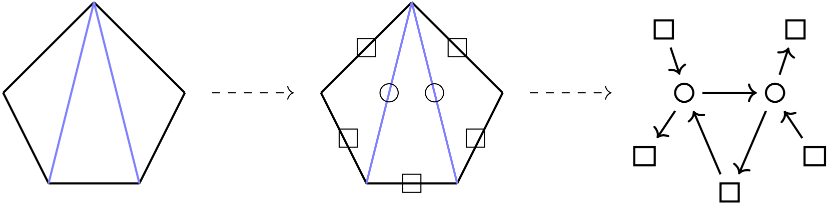
We may associate a quiver to a triangulation as follows:



Definition

Given a marked surface (S, M), construct the associated quiver Q_T as above. Assign an indeterminant x_i to each vertex i, so that $\tilde{\mathbf{x}}_T = (x_1, \ldots, x_m)$ is an extended cluster. Then $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ is the **cluster algebra associated to the marked surface** (S, M).

We may associate a quiver to a triangulation as follows:

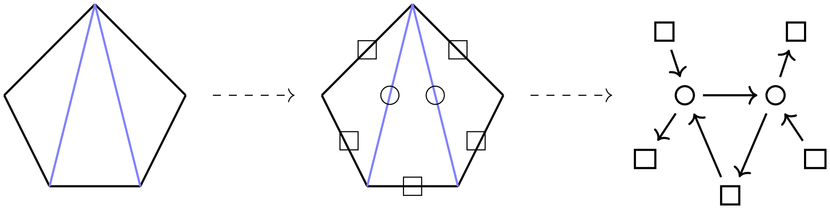


Definition

Given a marked surface (S, M), construct the associated quiver Q_T as above. Assign an indeterminant x_i to each vertex i, so that $\tilde{\mathbf{x}}_T = (x_1, \ldots, x_m)$ is an extended cluster. Then $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ is the **cluster algebra associated to the marked surface** (S, M).

The Cluster Algebra Associated to a Triangulated Surface

We may associate a quiver to a triangulation as follows:



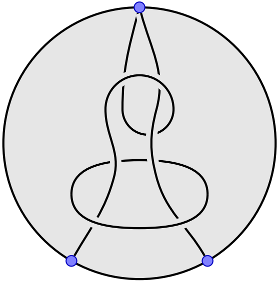
Definition

Given a marked surface (S, M), construct the associated quiver Q_T as above. Assign an indeterminant x_i to each vertex i, so that $\tilde{\mathbf{x}}_T = (x_1, \ldots, x_m)$ is an extended cluster. Then $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ is the **cluster algebra associated to the marked surface** (S, M).

Definition

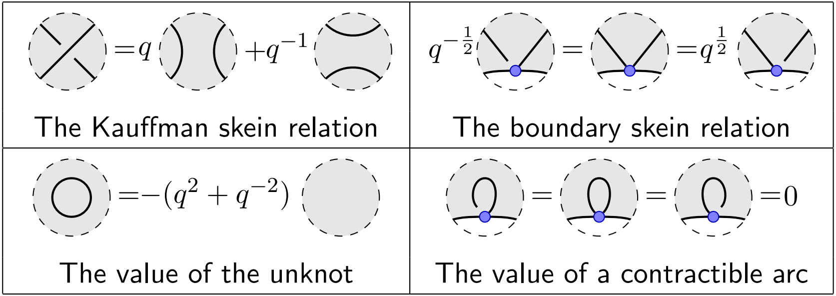
A **link diagram** (or link) L is a collection of curves in S, such that all intersections are simple and transverse, along with an ordering of curves at each intersection.

Link diagrams generalize the projection of a knot in $S \times [0, 1]$ onto S, where the orderings keep track of how the strands are passing over each other.



Some Relations

Let $\mathbb{Z}_q$ denote the ring $\mathbb{Z}[q^{\pm 1/2}]$ of Laurent polynomials in the indeterminant $q^{1/2}$. For any marked surface (S, M), let $\mathbb{Z}_q^{\text{Links}}$ denote the free $\mathbb{Z}_q$-module with basis given by equivalence classes of links in S. Let I be the submodule generated by the four relations pictured below.

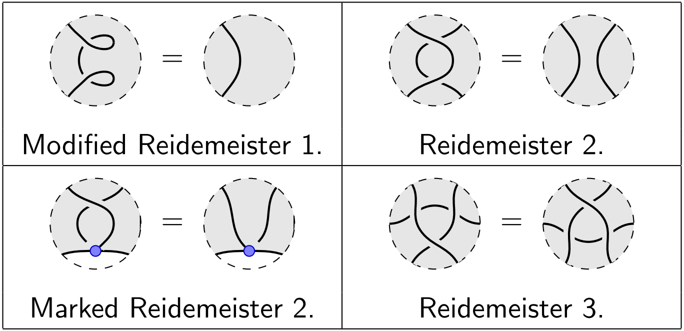


Definition

The **skein algebra** $\mathrm{Sk}_q(S)$ of a marked surface (S, M) is the the $\mathbb{Z}_q$-module $\mathrm{Sk}_q(S) = \mathbb{Z}_q^{\mathsf{Links}}/I$, along with an associative product given by superposition of link diagrams and unit given by the (class of the) empty link.

Some More Relations (Just For Fun)

The relations imposed in $\mathrm{Sk}_q(S)$ imply other important relations that are useful when doing computations. These are the modified Reidemeister moves from knot theory, along with an additional relation coming from the addition of marked endpoints. These relations are shown below.



Definition

The **localized skein algebra** $\mathrm{Sk}_q^\circ(S)$ is the Ore localization at the boundary arcs in S.

Since the skein algebra is generated by simple curves, $\mathrm{Sk}_q^\circ(S)$ is generated by simple curves along with inverses to boundary curves.

Definition

The **quantum cluster algebra** $\mathcal{A}_q(S)$ of S is the $\mathbb{Z}_q$-subalgebra of $\mathrm{Sk}_q^\circ(S)$ generated by simple arcs and the inverse to boundary arcs.

The Cluster Algebra is the Localized Skein Algebra

Theorem (Muller 2016)

We have $\mathcal{A}_q(S) \subset \mathrm{Sk}_q^\circ(S)$ for any triangulable marked surface (S, M). Moreover, if (S, M) has at least two marked points in each boundary component, $\mathcal{A}_q(S) = \mathrm{Sk}_q^\circ(S)$.

We note that the simple loops in $\mathrm{Sk}_q^\circ(S)$ define elements of $\mathcal{A}_q(S)$ that are not cluster variables and that considering these elements simplifies computations and provides a free $\mathbb{Z}_q$-basis of $\mathcal{A}_q(S)$.

Finally, if one specializes $q^{1/2} = 1$, we obtain $\mathcal{A}(S) = \mathrm{Sk}_1^\circ(S)$. Here, $\mathcal{A}(S)$ is such that cluster variables correspond to arcs without self-intersections, and the set of arcs in a triangulation is a cluster. That is, $\mathcal{A}(S) = \mathcal{A}(\widetilde{\mathrm{x}}_T, Q_T)$ as defined earlier.

Theorem (Muller 2016)

We have $\mathcal{A}_q(S) \subset \mathrm{Sk}_q^{\circ}(S)$ for any triangulable marked surface (S, M). Moreover, if (S, M) has at least two marked points in each boundary component, $\mathcal{A}_q(S) = \mathrm{Sk}_q^{\circ}(S)$.

We note that the simple loops in $\mathrm{Sk}_q^{\circ}(S)$ define elements of $\mathcal{A}_q(S)$ that are not cluster variables and that considering these elements simplifies computations and provides a free $\mathbb{Z}_q$-basis of $\mathcal{A}_q(S)$.

Finally, if one specializes $q^{1/2} = 1$, we obtain $\mathcal{A}(S) = \mathrm{Sk}_1^{\circ}(S)$. Here, $\mathcal{A}(S)$ is such that cluster variables correspond to arcs without self-intersections, and the set of arcs in a triangulation is a cluster. That is, $\mathcal{A}(S) = \mathcal{A}(\widetilde{\mathrm{x}}_T, Q_T)$ as defined earlier.

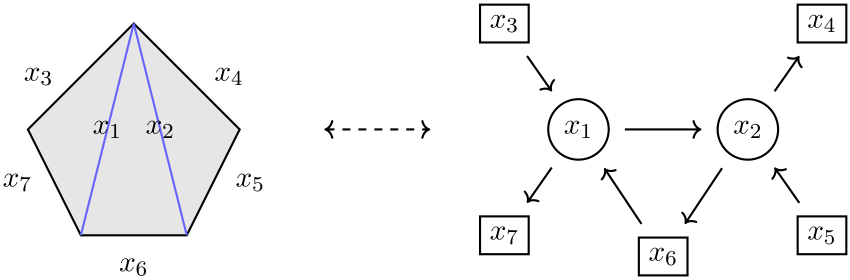
The Cluster Algebra is the Localized Skein Algebra

Theorem (Muller 2016)

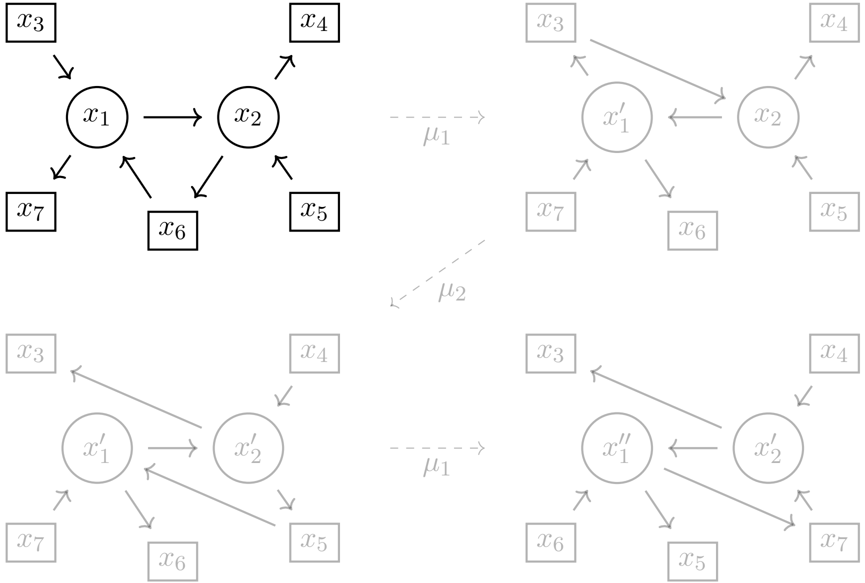
We have $\mathcal{A}_q(S) \subset \mathrm{Sk}_q^\circ(S)$ *for any triangulable marked surface* (S, M). *Moreover, if* (S, M) *has at least two marked points in each boundary component,* $\mathcal{A}_q(S) = \mathrm{Sk}_q^\circ(S)$.

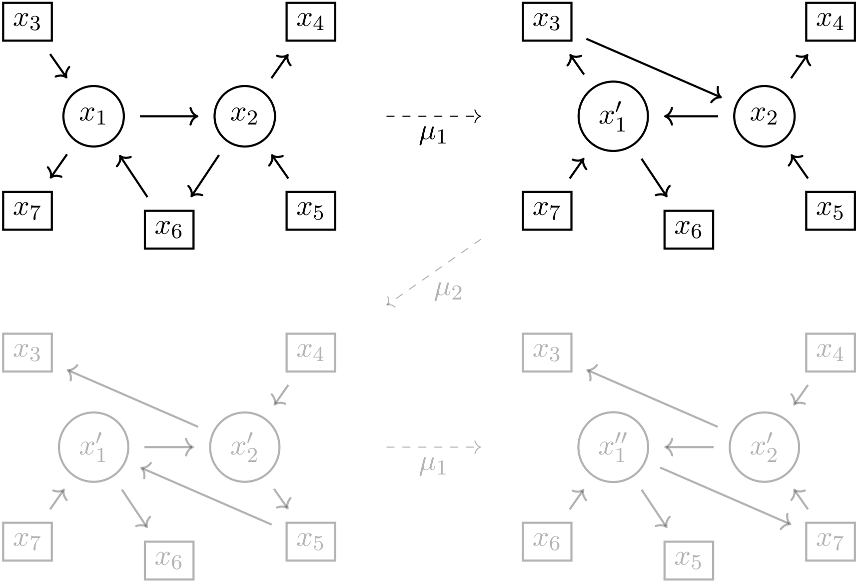
We note that the simple loops in $\mathrm{Sk}_q^\circ(S)$ define elements of $\mathcal{A}_q(S)$ that are not cluster variables and that considering these elements simplifies computations and provides a free $\mathbb{Z}_q$-basis of $\mathcal{A}_q(S)$.

Finally, if one specializes $q^{1/2} = 1$, we obtain $\mathcal{A}(S) = \mathrm{Sk}_1^\circ(S)$. Here, $\mathcal{A}(S)$ is such that cluster variables correspond to arcs without self-intersections, and the set of arcs in a triangulation is a cluster. That is, $\mathcal{A}(S) = \mathcal{A}(\widetilde{\mathbf{x}}_T, Q_T)$ as defined earlier.

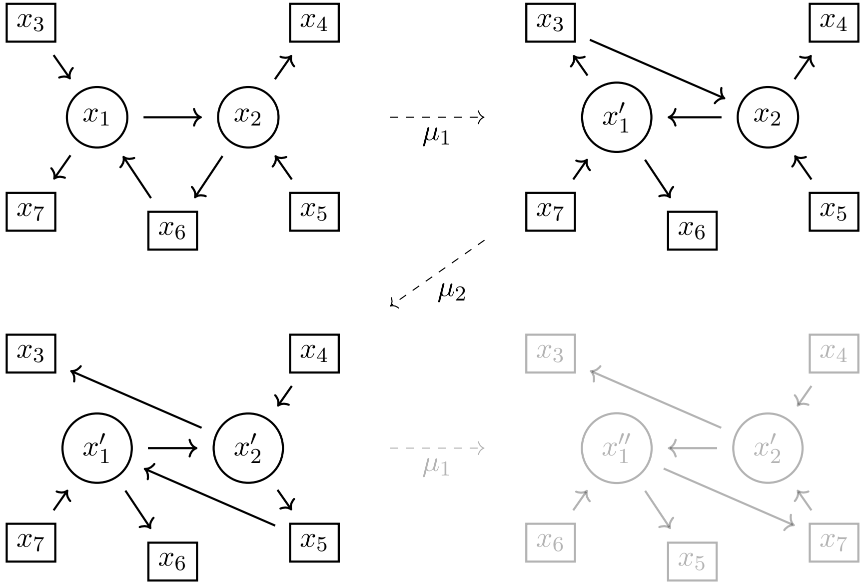

x_3
x_4
x_1
x_2
x_7
x_5
x_6
x_3
x_4
x_1
x_2
x_7
x_6
x_5

An Example

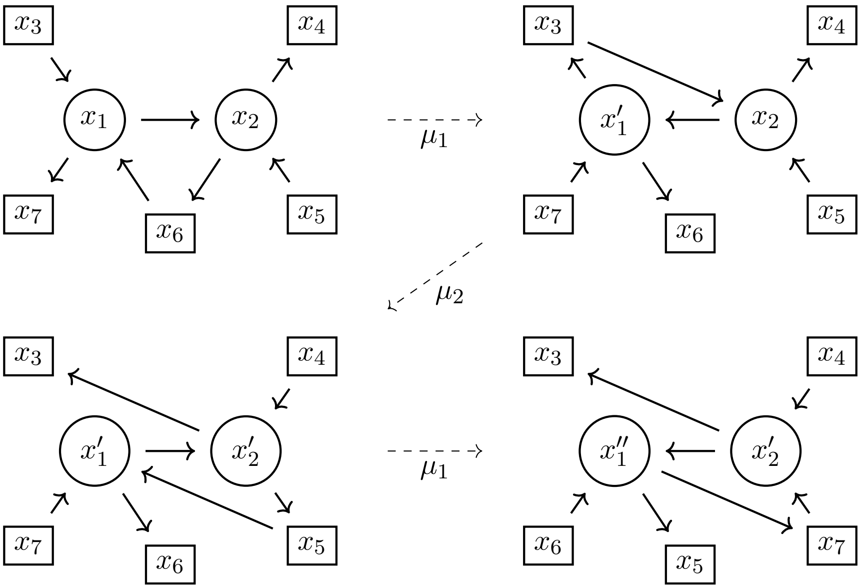



x_3
x_4
x_1
x_2
x_7
x_6
x_5
μ_1
x_3
x_4
x_1'
x_2
x_7
x_6
x_5
μ_2
x_3
x_4
x_1'
x_2'
x_7
x_6
x_5
μ_1
x_3
x_4
x_1''
x_2'
x_6
x_5
x_7

An Example


x_3
x_4
x_1
x_2
x_7
x_6
x_5
μ_1
x_3
x_4
x'_1
x_2
x_7
x_6
x_5
μ_2
x_3
x_4
x'_1
x'_2
x_7
x_6
x_5
μ_1
x_3
x_4
x''_1
x'_2
x_6
x_5
x_7

An Example



x_3
x_4
x_1
x_2
x_7
x_6
x_5
μ_1
x_3
x_4
x'_1
x_2
x_7
x_6
x_5
μ_2
x_3
x_4
x'_1
x'_2
x_7
x_6
x_5
μ_1
x_3
x_4
x''_1
x'_2
x_6
x_5
x_7

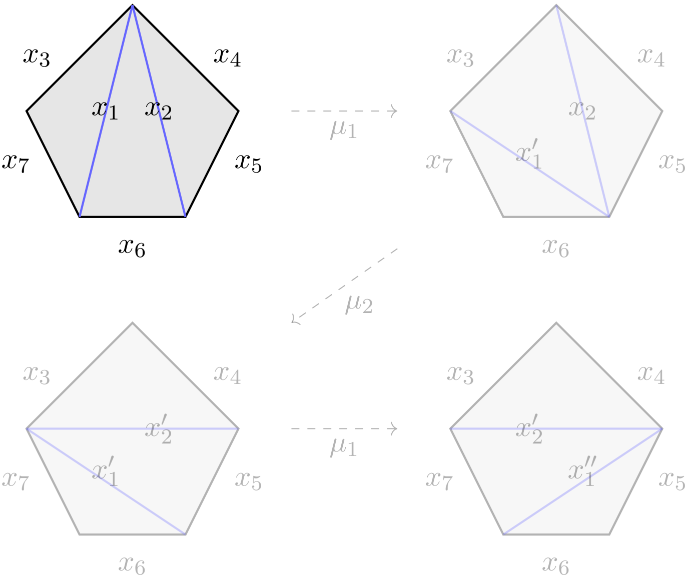
We have that

$$x_1' = \frac{x_3x_6 + x_7x_2}{x_1}$$

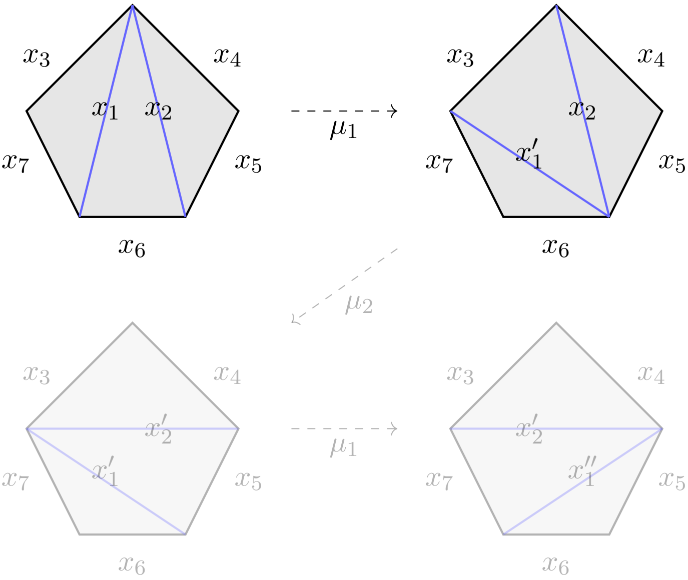
$$x_2' = \frac{x_3x_5 + x_4x_1'}{x_2} = \frac{x_3x_5 + x_4\left(\frac{x_3x_6 + x_7x_2}{x_1}\right)}{x_2}$$

$$x_1'' = \frac{x_7x_5 + x_6x_2'}{x_1'} = \frac{x_7x_5 + x_6\left(\frac{x_3x_5 + x_4\left(\frac{x_3x_6 + x_7x_2}{x_1}\right)}{x_2}\right)}{\frac{x_3x_6 + x_7x_2}{x_1}}$$

This expression for x_1'' is gross! Now, we know that the quiver mutations we just saw are equivalent to the following flips of triangulations:

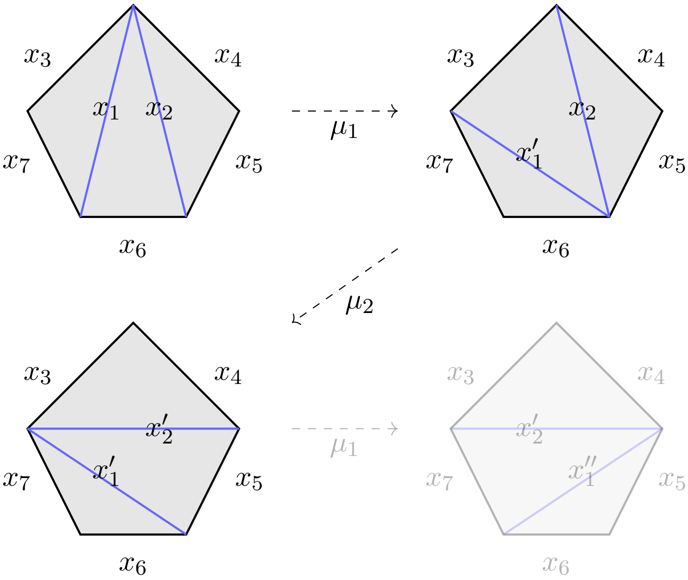


x_3
x_4
x_1
x_2
x_7
x_5
x_6
μ_1
x_3
x_4
x_2
x_1'
x_7
x_5
x_6
μ_2
x_3
x_4
x_2'
x_1'
x_7
x_5
x_6
μ_1
x_3
x_4
x_2'
x_1''
x_7
x_5
x_6



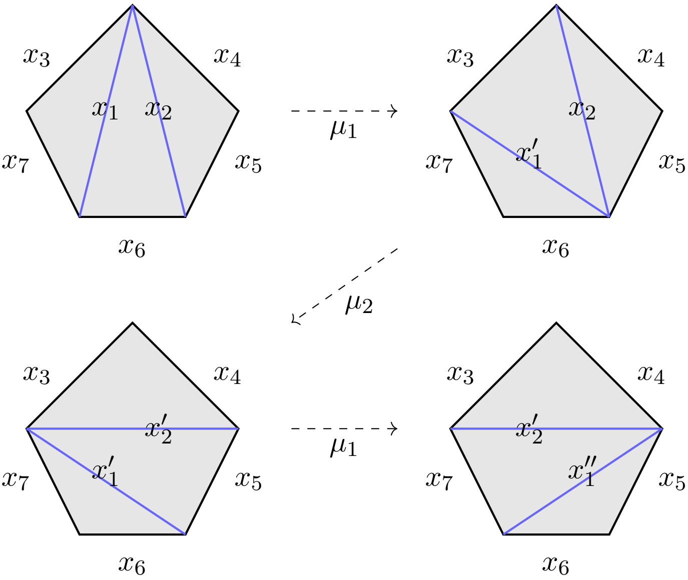
x_3
x_4
x_1
x_2
x_7
x_5
x_6
μ_1
x_3
x_4
x_2
x_1'
x_7
x_5
x_6
μ_2
x_3
x_4
x_2'
x_1'
x_7
x_5
x_6
μ_1
x_3
x_4
x_2'
x_1''
x_7
x_5
x_6

An Example



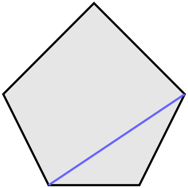
x_3
x_4
x_1
x_2
x_7
x_5
x_6
μ_1
x_3
x_4
x_2
x_1'
x_7
x_5
x_6
μ_2
x_3
x_4
x_2'
x_1'
x_7
x_5
x_6
μ_1
x_3
x_4
x_2'
x_1''
x_7
x_5
x_6

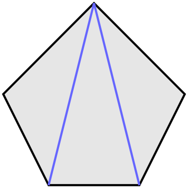
An Example



x_3
x_4
x_1
x_2
x_7
x_5
x_6
μ_1
x_3
x_4
x_2
x_1'
x_7
x_5
x_6
μ_2
x_3
x_4
x_2'
x_1'
x_7
x_5
x_6
μ_1
x_3
x_4
x_2'
x_1''
x_7
x_5
x_6

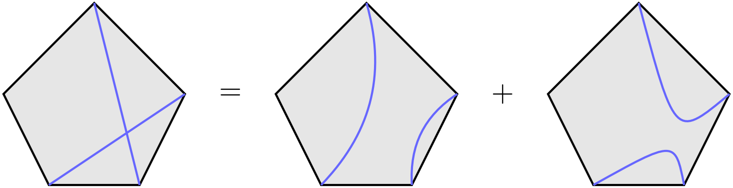
Now, if we consider this variable in the skein algebra, writing x_1'' in terms of the initial cluster amounts to writing the arc on the left in terms of the triangulation on the right:



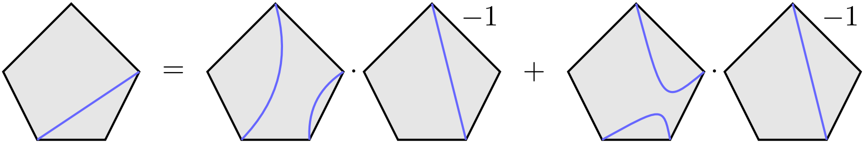


An Example

Note that



Hence, we have



Therefore, it must be that $x_1'' = \dfrac{x_1 x_5 + x_4 x_6}{x_2}$.

THANK YOU!