An Algebra So Nice, It Will Make You Quiver!

An Introduction to Cluster Algebras of Geometric Type

Joey Randich

University of Oklahoma

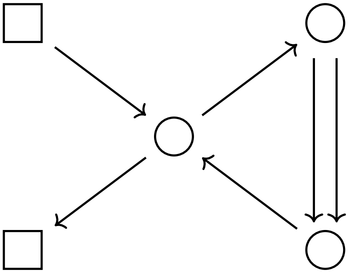
Graduate Algebra Symposium, April 21, 2018

Cluster Algebras are an interesting class of commutative algebras, introduced in a paper from 2002 by Fomin and Zelevinsky, whose development was initially motivated by total positivity and G. Lusztig's theory of canonical bases.

Cluster Algebras occur in the coordinate rings of important spaces such as semisimple Lie groups, Grassmanians, and decorated Teichmüller spaces. They've also found applications in other areas such as topology, discrete dynamical systems, and mathematical physics.

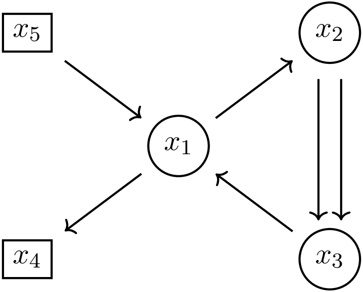
We consider finite quivers with no loops nor oriented 2-cycles. We do allow multiple edges. Moreover, we also distinguish two types of vertices: *frozen* vertices are drawn with squares, and *mutable* vertices are drawn with circles. Such a quiver is called an **Ice quiver**.

For example:

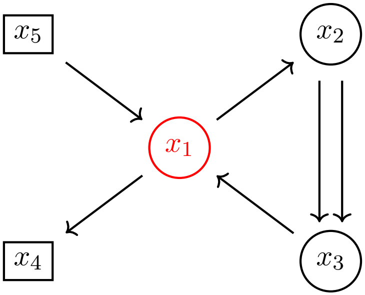


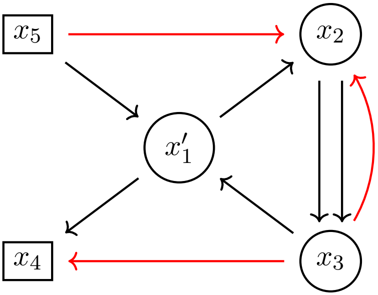
Let $m, n \in \mathbb{Z}_+$ with $m \geqslant n$. We consider a field $\mathcal{F}$ isomorphic to the field of rational functions over $\mathbb{Q}$ in m independent variables as an *ambient field* for a cluster algebra. Let $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ be an m-tuple of elements in $\mathcal{F}$ forming a free generating set; that is, $x_1, \ldots, x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m)$.

An example of a labeled seed is given below:

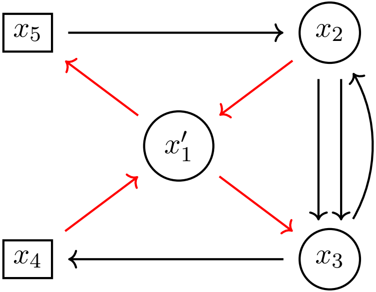


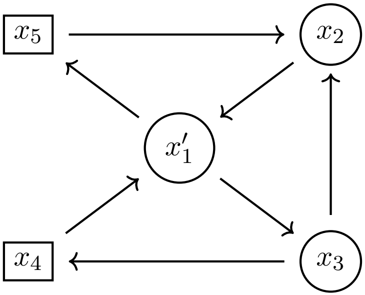
Seed Mutation




x_5
x_2
x_1'
x_4
x_3

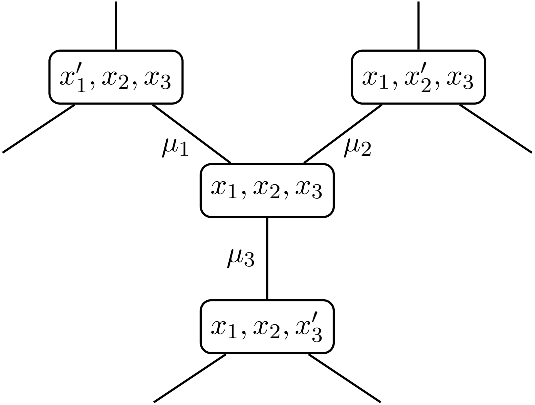
Seed Mutation





where $x'_k \in \mathcal{F}$ is given by the following *exchange relation*:

$$x'_k x_k = \prod(\text{upstream}) + \prod(\text{downstream})$$



Remark

A seed pattern is uniquely determined by any one of its seeds.

Definition

Let $\{\Sigma_t\}_{t\in\mathbb{T}_n}$ be a seed pattern, and let

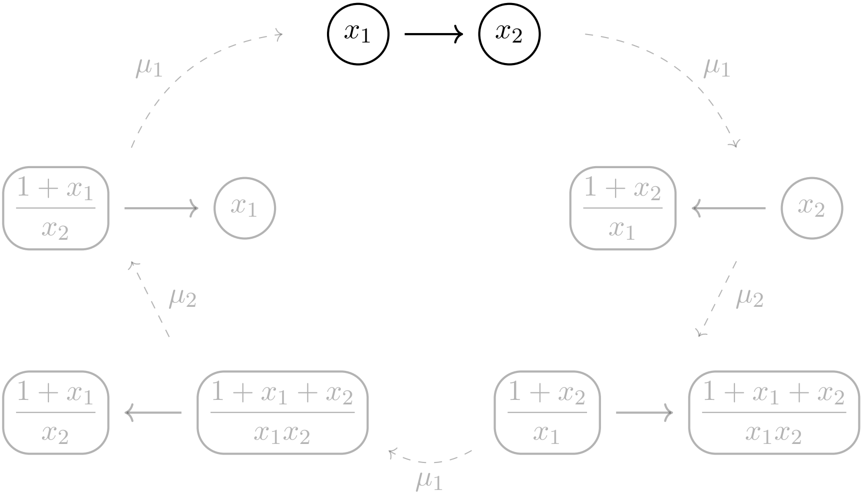
$$\mathcal{X} = \bigcup_{t\in\mathbb{T}_n} \mathbf{x}_t$$

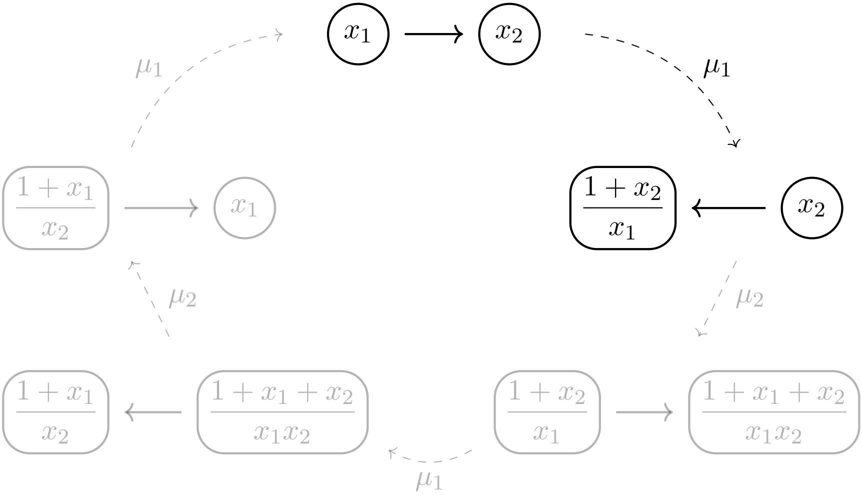
be the set of all cluster variables appearing in any of the seeds. Let $R = \mathbb{Q}[x_{n+1}, \ldots, x_m]$ be the polynomial ring generated by the frozen variables. The **cluster algebra** $\mathcal{A}$ (of geometric type) associated with the given seed pattern is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = R\,[\mathcal{X}]$. We say that $\mathcal{A}$ has *rank* n because every cluster in the underlying seed pattern has n cluster variables.

Theorem (Fomin, Zelevinsky 2002)

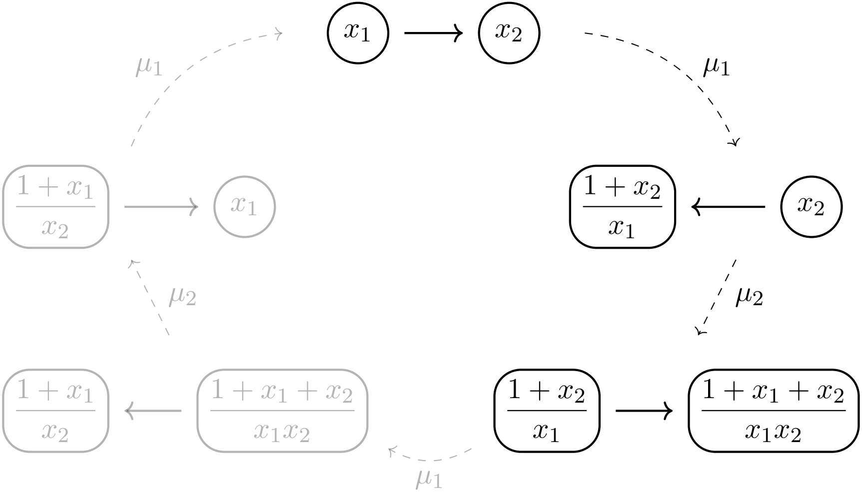
Given any cluster $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathcal{A} \subset \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$. *That is, each cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any cluster.*

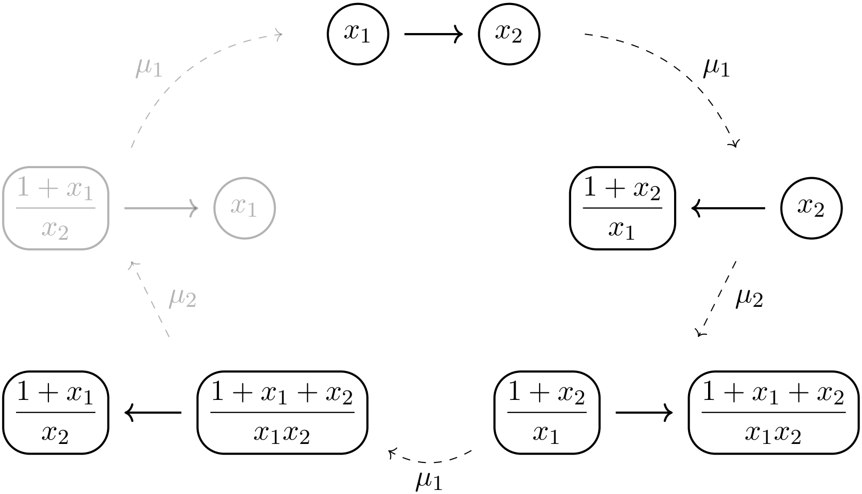
Our First Example

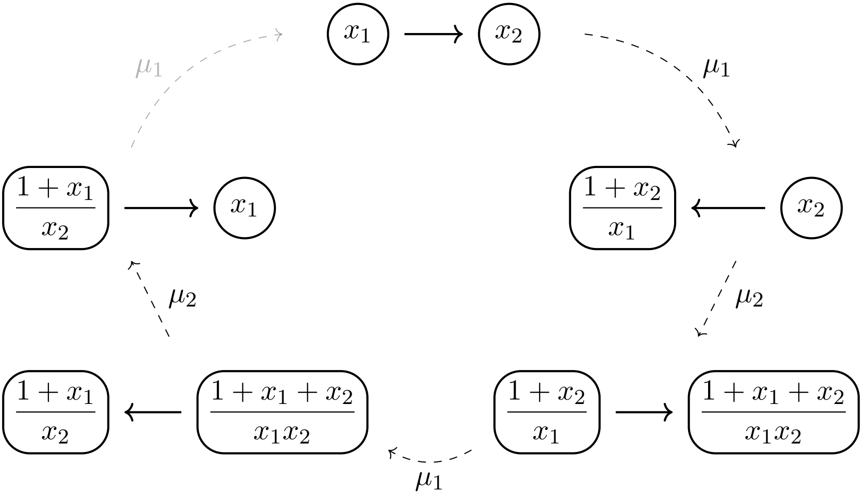


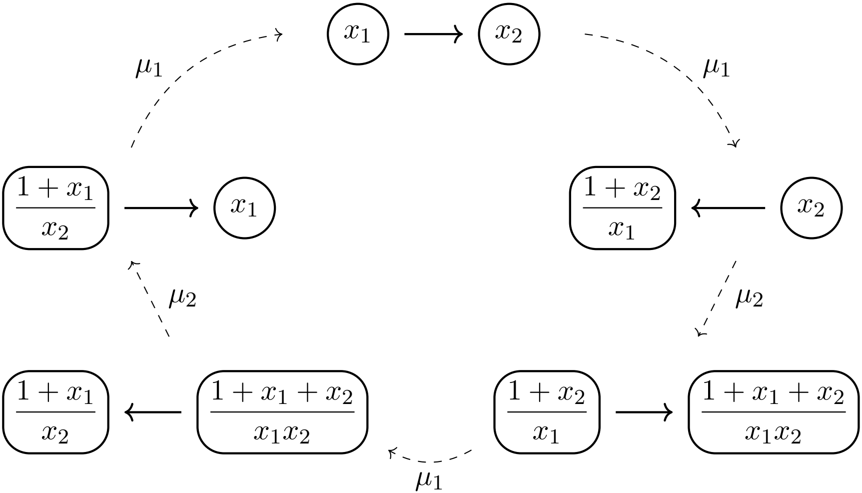


x_1
x_2
μ_1
μ_1
$\frac{1+x_1}{x_2}$
x_1
$\frac{1+x_2}{x_1}$
x_2
μ_2
μ_2
$\frac{1+x_1}{x_2}$
$\frac{1+x_1+x_2}{x_1x_2}$
$\frac{1+x_2}{x_1}$
$\frac{1+x_1+x_2}{x_1x_2}$
μ_1


x_1
x_2
μ_1
$\frac{1+x_1}{x_2}$
x_1
$\frac{1+x_2}{x_1}$
x_2
μ_2
μ_2
μ_1
$\frac{1+x_1}{x_2}$
$\frac{1+x_1+x_2}{x_1x_2}$
$\frac{1+x_2}{x_1}$
$\frac{1+x_1+x_2}{x_1x_2}$
μ_1


x_1
x_2
μ_1
μ_1
$\frac{1+x_1}{x_2}$
x_1
$\frac{1+x_2}{x_1}$
x_2
μ_2
μ_2
$\frac{1+x_1}{x_2}$
$\frac{1+x_1+x_2}{x_1x_2}$
$\frac{1+x_2}{x_1}$
$\frac{1+x_1+x_2}{x_1x_2}$
μ_1


x_1
x_2
μ_1
μ_1
$\frac{1+x_1}{x_2}$
x_1
$\frac{1+x_2}{x_1}$
x_2
μ_2
μ_2
$\frac{1+x_1}{x_2}$
$\frac{1+x_1+x_2}{x_1x_2}$
$\frac{1+x_2}{x_1}$
$\frac{1+x_1+x_2}{x_1x_2}$
μ_1


x_1
x_2
μ_1
μ_1
$\frac{1+x_1}{x_2}$
x_1
$\frac{1+x_2}{x_1}$
x_2
μ_2
μ_2
$\frac{1+x_1}{x_2}$
$\frac{1+x_1+x_2}{x_1x_2}$
$\frac{1+x_2}{x_1}$
$\frac{1+x_1+x_2}{x_1x_2}$
μ_1

This last example has the property that there are only finitely many distinct cluster variables that appear, AND there are only finitely many quivers that appear!

One may wonder how often this situation occurs.

Interestingly, it turns out that the classification of finite type cluster algebras exactly mirrors the Cartan-Killing classification of complex simple Lie algebras.

Theorem (Fomin, Zelevinsky 2003)

A cluster algebra is of finite type if and only it can be obtained from a quiver that is some orientation of a finite type Dynkin Diagram.

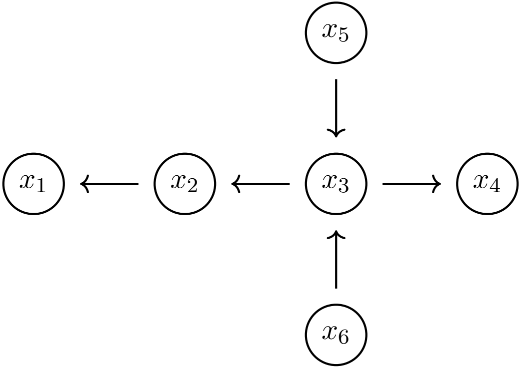
Theorem (Felikson, Shapiro, Tumarkin 2009)

Cluster algebras of geometric type[1] *have finite mutation type if and only if one of the following holds:*

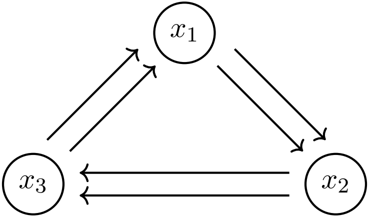
- *It has rank* $\leqslant 2$.
- *It is associated to the triangulation of a bordered two-dimensional surface.*
- *The underlying quiver is of the one of the types:* $E_6, E_7, E_8, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.

[1] This can be extended to general cluster algebras.

An Infinite Type Quiver With Infinite Mutation Type



The Markov Quiver



THANK YOU!