I LIKE MY ALGEBRAS THE SAME AS MY CEREAL: WITH CLUSTERS

A NOTE TO MY PREVIOUS SELF

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1. Introduction

The purpose of this note is to provide an overview of cluster algebras for someone that is new to the area. An emphasis is placed on intuition and big picture ideas rather than on technical details. We try to provide adequate sources containing the details and proofs that are omitted. The material contained in this note is based on content explored during a semester-long reading course between myself and Dr. Greg Muller at the University of Oklahoma. Therefore, it is *not* to be treated as a comprehensive treatment of the subject. Rather, one should view this note more as an attractive advertisement to a junior mathematician.

Cluster algebras are a class of algebras which contain an abundance of interesting and deep structure, and which have long-reaching ties with other fields of mathematics. One studying these algebras can quickly find themselves knee-deep in geometry, topology, representation theory, physics, mirror symmetry, etc. Therefore, studying cluster algebras may be attractive to a wide variety of people with different backgrounds and interests. There are also extremely difficult open problems which require a seemingly insurmountable amount of background knowledge and insight, as well as easily-stated concrete problems that a first-year graduate student could dive right into. Whether you just like looking at cool new algebras you haven't seen before, or you wish to explore the connection between algebra and geometry, or something entirely different, cluster algebras are worth a look.

Another aspect of cluster algebras that may be enticing is that it is a relatively new field in mathematics. There are still fundamental questions that remain unanswered. The full implications of these objects are not fully or well known. It's not even clear if cluster algebras, as currently defined, are the "right" algebra to be studying (here we are alluding to the connection between a cluster algebra, the upper cluster algebra, and the so-called "canonical algebra" defined by Gross, Hacking, Keel, and Kontsevich which sits between them). There is plenty of room for someone to start researching in this area and to make significant contributions.

Cluster algebras were first introduced by Fomin and Zelevisnky in [FZ02]. The story goes that the two were studing total positivity and Luztig's theory of canonical bases when they formulated cluster algebras. They realized they had quite and interesting object on their hands and published four papers (aptly named "Cluster Algebras I-IV") fleshing out the foundations of the theory. Since then, there have been many refinements to definitions and

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ideas, and there have been multitudes of conjectures and theorems proven and disproven. We aim to touch on a few of the main areas that have been developed and which are being developed actively.

Throughout the note, we focus on cluster algebras which arise from quivers for the sake of clarity and concreteness. It should be noted that cluster algebras were originally defined in greater generality than this. We organize the note as follows:

- Section 2: We begin by introducing the preliminary definitions and examples and move on to some of the basic foundational results. These include the Laurent phenominon and various classification theorems. The aim of this first sections is to provide a glimpse into how a cluster algebra is put together.
- Section 3: We switch gears and explore a connection between cluster algebras and topological surfaces. In addition to providing a wealth of examples where cluster algebras appear in other areas of mathematics, this connection also gives us an efficient and hands-on way of perfoming computations in our cluster algebra. It also plays a role in the classification of mutation-finite cluster algebras, as well as providing many examples of *locally acyclic cluster algebras* which we define in Section 2.4.
- Section 4: Our last section dives into the search for a canonical basis for all cluster algebras. This area still has many open questions, and our goal is to provide a brief and nontechnical introduction to the machinery involved.

2. What Makes a Cluster Algebra?

Cluster Algebras are a class of commutative algebra with interesting combinatorial properties. Each algebra is generated by a set of elements called *cluster variables*. Moreover, there are distinguished overlapping subsets of cluster variables (each of the same cardinality) called *clusters*. One may pass from cluster to cluster via *mutation*. One may study the relationships between clusters via the *cluster complex*, a simplicial complex with cluster variables as vertices and faces determined by variables appearing together in a cluster. One may also study the dual graph, the *exchange graph*. In addition to combinatorial questions, one may, of course, study the algebraic structure. In this section, we aim to provide the relevant definitions and provide some nice combinatorial and structural results which should give an idea of how these algebras are put together. Much of the exposition follows [FWZ16].

In order to define these algebras, we often begin with a quiver and *mutate* away. This mutation process will generate a cluster algebra for us.

2.1. **Quivers**

Definition 2.1. A **quiver** Q is a an oriented graph. We only consider finite quivers with no loops nor oriented 2-cycles. We do allow multiple edges.

Definition 2.2. An **ice quiver** is a richer notion of a quiver where some of the vertices are considered *frozen* and the others are considered *mutable*.

In what follows, we never consider ice quivers with edges between frozen variables. We will often refer to ice quivers as just quivers. Finally, frozen vertices will be denoted by a square, and mutable vertices will be denoted by a circle.

Definition 2.3. Let k be a mutable vertex of a quiver Q. The **quiver mutation** μ_k at the vertex k transforms the quiver Q into a new quiver $Q' = \mu_k(Q)$ via a sequence of three steps:

(1) For each oriented two-arrow path $i \to k \to j$, add a new arrow $i \to j$ (unless both i and j are frozen, in which case, do nothing).

(2) Reverse the direction of all arrows incident to the vertex k.

(3) Remove all oriented 2-cycles that may have been created.

Example 2.4. Figure 1 below shows an example of the three step quiver mutation process. Here, we are mutating the quiver Q at vertex 1. The resulting quiver is $\mu_1(Q)$.

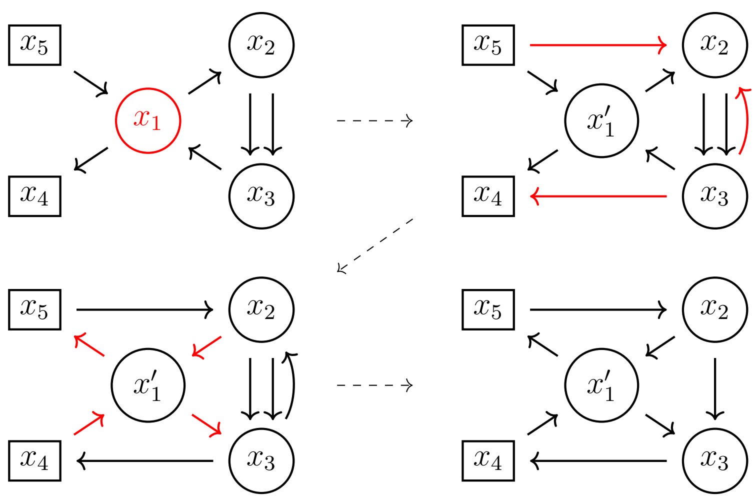


FIGURE 1. An example of the seed mutation process.

Let $m, n \in \mathbb{Z}_+$ with $m \geqslant n$. We consider a field $\mathcal{F}$ isomorphic to the field of rational functions over $\mathbb{Q}$ in m independent variables as an *ambient field* for a cluster algebra.

Definition 2.5. A **labeled seed** in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}}, Q)$ where

- $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ is an m-tuple of elements in $\mathcal{F}$ forming a free generating set; that is, $x_1, \ldots, x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m)$.
- Q is an (ice) quiver on vertices $1, \ldots, m$ where we associate x_i with vertex i. The vertices $1, \ldots, n$ are called *mutable*, and the vertices $n+1, \ldots, m$ are called *frozen*.

We utilize the following terminology:

- $\tilde{\mathbf{x}}$ is the (labeled) *extended cluster* of a labeled seed $(\tilde{\mathbf{x}}, Q)$.
- $\mathbf{x} = (x_1, \ldots, x_n)$ is the (labeled) *cluster* of this seed.
- The elements $x_1, \ldots, x_n$ are its *cluster variables*.

- The remaining elements $x_{n+1}, \dots, x_m$ are its *frozen variables*.

Definition 2.6. Let Q be an (ice) quiver with m vertices, n of which are mutable. Label the vertices so that the mutable vertices correspond to $1, \dots, n$, and the frozen vertices correspond to the remaining $n+1, \dots, m$. The **extended exchange matrix** of Q is the $m \times n$ matrix $\widetilde{B}(Q) = (b_{ij})$ define by[1]

$$b_{ij} = \begin{cases} \ell & \text{if there are } \ell \text{ arrows from vertex } i \text{ to vertex } j \text{ in } Q \\ -\ell & \text{if there are } \ell \text{ arrows from vertex } j \text{ to vertex } i \text{ in } Q \\ 0 & \text{else} \end{cases}$$

The **exchange matrix** $B(Q)$ is the $n \times n$ skew-symmetric submatrix of $\widetilde{B}(Q)$ occupying the first n rows[2].

Definition 2.7. Let $(\tilde{\mathbf{x}}, Q)$ be a labeled seed in $\mathcal{F}$, and let $k \in \{1, \dots, n\}$. The **seed mutation** μ_k in direction k transforms $(\tilde{\mathbf{x}}, Q)$ into a new labeled seed $\mu_k(\tilde{\mathbf{x}}, Q) = (\tilde{\mathbf{x}}', \mu_k(Q))$ where $\tilde{\mathbf{x}}' = (x'_1, \dots, x'_m)$ such that $x'_j = x_j$ for $j \neq k$ and $x'_k \in \mathcal{F}$ is given by the following *exchange relation*:

$$x'_k x_k = \prod (\text{upstream}) + \prod (\text{downstream})$$

where "upstream" accounts for all cluster variabes attached to vertices with arrows coming into vertex k, and "downstream" accounts for all cluster variables attached to vertices with arrows coming out of vertex k.

A more formal definition of seed mutation can be given in terms of the exchange matrix where we consider a labelled seed of the form $(\widetilde{\mathbf{x}}, \widetilde{B})$. It can be shown that the extended exchange matrix $\widetilde{B}(\mu_k(Q)) = (b'_{ij})$ of the mutated quiver $\mu_k(Q)$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$$

Then the exchange relation can be written as

$$x'_k x_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}$$

We immediately get the impression that this formality is cumbersome, and in practice, it's more useful to use the definition in terms of quivers.

[1]some people use the opposite convention where evey entry is multiplied by negative 1.

[2]It is possible to define everything in terms of skew-symmetric matrices without ever thinking of a quiver. In fact, one generalization of cluster algebras allows for *skew-symmetrizable* matrices which don't have as nice of a translation into quivers. We stick to skew-symmetric matrices and quivers for the sake of clarity, and we note that much of the main theory holds in this case. We also note that there are other generalizations of the definition we will give, but again, this won't interfere with the goal of this paper.

Example 2.8. In Figure 1, the new cluster variable we obtain is $x_1' = \dfrac{x_2x_4 + x_3x_5}{x_1}$, where the extended exchange matrix for the initial labelled seed is

$$\widetilde{B}(Q) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Next, consider the n-regular tree $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$ so that the n edges incident to each vertex all have distinct labels.

Definition 2.9. A **seed pattern** is defined by assigning a labeled seed $\Sigma_t = (\tilde{\mathbf{x}}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$ such that the seeds assigned to the end-points of any edge $t \overset{k}{-} t'$ are obtained from each other by the seed mutation in direction k.

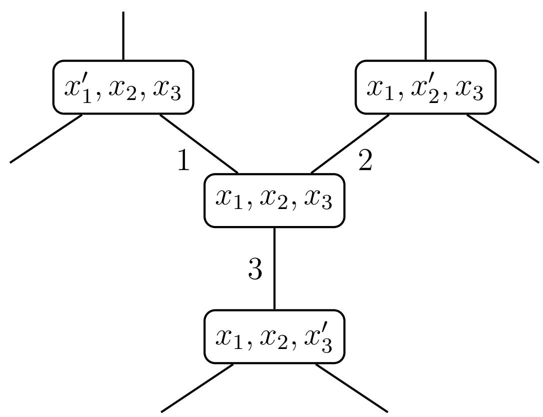


FIGURE 2. An example of a seed pattern for $n = 3$.

Remark 2.10. A seed pattern is uniquely determined by any one of its seeds.

We are finally ready to state our working definition of a cluster algebra.

Definition 2.11. Let $\{\Sigma_t\}_{t\in\mathbb{T}_n}$ be a seed pattern, and let

$$\mathcal{X} = \bigcup_{t\in\mathbb{T}_n} \mathbf{x}_t$$

be the set of all cluster variables appearing in any of the seeds. Let $R = \mathbb{Q}[x_{n+1}, \ldots, x_m]$ be the polynomial ring generated by the frozen variables. The **cluster algebra** $\mathcal{A}$ (of geometric type) associated with the given seed pattern is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = R[\mathcal{X}]$.

We say that $\mathcal{A}$ has *rank* n because every cluster in the underlying seed pattern has n cluster variables.

Remark 2.12. It is common to describe a cluster algebra by picking an initial labeled seed $\Sigma_0 = (\tilde{\mathbf{x}}_0, Q_0)$ in $\mathcal{F}$ and build a seed pattern from it. The corresponding cluster algebra is denoted $\mathcal{A}(\Sigma_0) = \mathcal{A}(\tilde{\mathbf{x}}_0, Q_0)$, and it is generated over R by all cluster variables appearing in the seeds which are mutation equivalent to Σ_0.

2.2. **Laurent Phenomenon**

If one starts mutating an initial seed, they may notice that the new cluster variables that are obtained seem to always be Laurent polynomials in the original cluster with positive integral coefficients. They may wonder if this is just a coincidence. If they keep mutating, they will quickly find themselves drowning in a sea of horrendous rational expressions which are anything but quick and easy to simplify by hand. Even the most tame quivers can become unruly quite quickly. However, as remarkable as it is, this is no coincidence.

Theorem 2.13 ([FZ02])**.** *Given any cluster* $\mathbf{x} = (x_1, \ldots, x_n)$*, we have* $\mathcal{A} \subset \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$. *That is, each cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any cluster.*

Fomin and Zelevinsky also proved that frozen variables never appear in the denominator of these Laurent polynomials. The proof of the so-called "positivity conjecture" – that the coefficients of these Laurent polynomials are always positive – took a while longer to get to. The result was proven in the rank 2 case first, and then for cluster algebras corresponding to surfaces [MSW11]. Next came acyclic cluster algebras (those whose underlying quiver has no cycles) [KQ14], and then Lee and Schiffler proved the result for all skew-symmetric cluster algebras (which is the class we are considering in this note) [LS15].

Theorem 2.14 ([LS15])**.** *Every skew-symmetric cluster algebra* $\mathcal{A}$ *has the property that each cluster variable can be expressed as a Laurent polynomial with positive integer coefficients in the elements of any cluster.*

Example 2.15. Figure 3 shows an example for $\mathcal{A}(\{x_1, x_2\}, A_2)$ exhibiting the Laurent phenomenon.

2.3. **Some Notions of Finiteness for a Cluster Algebra**

Another aspect of cluster algebras one is bound to consider when mutating an intial labelled seed is the notion of finiteness. For example, some labelled seeds only admit finitely many cluster variabels no matter how many mutations are performed. Others admit only finitely many distinct quivers, while others still have both properties. And of course, there are plenty of examples exhibiting neither property.

Definition 2.16. A cluster algebra is said to be of **finite type** if it has only finitely many clusters.

Interestingly, it turns out that the classification of finite type cluster algebras exactly mirrors the Cartan-Killing classification of complex simple Lie algebras, proven by Fomin and Zelevinsky.

Theorem 2.17 ([FZ03]). *A cluster algebra is of finite type if and only it can be obtained from a quiver that is some orientation of a finite type Dynkin Diagram.*

While you may not be fortunate enough to be working with a cluster algebra of finite type, you may be fortunate enough to only have to work with finitely many quivers.

Definition 2.18. A cluster algebra is said to have **finite mutation type** if there are only finitely many quivers (up to isomorphism) that appear in any seed.

The following result was proven by Felikson, Shapiro, and Tumarkin.

Theorem 2.19 ([FST12]). *Cluster algebras of geometric type*[3] *have finite mutation type if and only if one of the following holds:*

- *It has rank* $\leqslant 2$.
- *It is associated to the triangulation of a bordered two-dimensional surface*[4].
- *The underlying quiver is of the one of the types:* $E_6, E_7, E_8, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.

Example 2.20. Again, Figure 3 gives an example of the A_2 quiver which gives rise to a cluster algebra which is both of finite type and is mutation-finite.

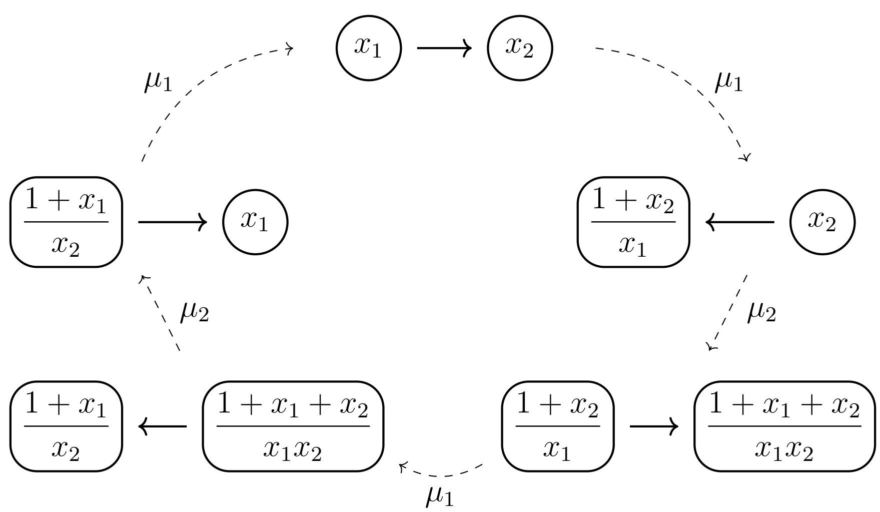


FIGURE 3. The labeled seed $(\{x_1, x_2\}, A_2)$ and its mutations. The cluster variables are clearly laurent polynomials with positive coefficients in the initial cluster variables. Moreover, after only finitely many mutations (five), one ends up with the initial seed. This gives an example of a finite type and mutation-finite cluster algebra.

Example 2.21. Figure 4 shows a quiver which is neither finite type nor mutation-finite.

[3]This can be extended to general cluster algebras.

[4]See section 3.2.

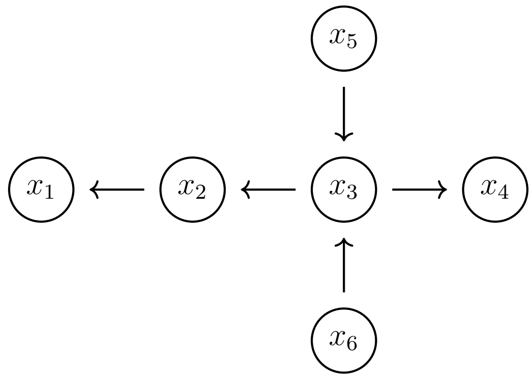


FIGURE 4. A labelled seed which yields a cluster algebra that is neither finite type, nor mutation finite.

Example 2.22. Figure 5 shows the Markov quiver, which gives a cluster algebra that is not finite type but is mutation-finite (it may be associated to a once-punctured torus).

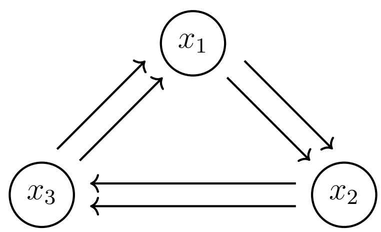


FIGURE 5. The Markov quiver.

2.4. **The Upper Cluster Algebra**

We've now seen some foundational results regarding the structure of a cluster algebra. In particular, we now know that every cluster variable can be written as a Laurent polynomial in any cluster with positive integral coefficients. However, we must take care not to make the mistake of viewing the cluster algebra as the collection of all such elements, as this is not true in general.

Definition 2.23 ([BFZ05])**.** For a given labelled seed, the **upper cluster algebra** $\mathcal{U}$ is the set of all rational functions that satisfy the Laurent phenomenon, which forms a subalgebra of $\mathcal{F}$. Precisely,

$$\mathcal{U} := \bigcap_{\substack{\text{clusters} \\ \mathbf{x}=(x_1,\ldots,x_n)}} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

Now, at first glance, it would appear that $\mathcal{U}$ is actually the algebra we'd like to study. Often times it is "too big," however, and we only have $\mathcal{A} \subset \mathcal{U}$. The Markov quiver gives rise to a cluster algebra where this containment is proper; see Figure 5.

There is a large class of cluster algebras for which we have equality, however. Muller introduced the notion of a **locally acyclic cluster algebra** in [Mul13]. Briefly, a locally acyclic cluster algebra is a cluster algebra which admits a finite cover (in a geometric sense) by acyclic cluster algebras. Many important results about acyclic cluster algebras extend to locally acyclic cluster algebras, including the following.

Theorem 2.24 ([Mul13])**.** *If a cluster algebra $\mathcal{A}$ is locally acyclic, then $\mathcal{A} = \mathcal{U}$.*

As the name might suggest, all acyclic cluster algebras are locally acyclic. But there are also many more. In particular, a wide range of cluster algebras associated to surfaces are locally acyclic[5], which we introduce in the next section.

Example 2.25. Figure 6 gives an example of a quiver giving rise to a locally acyclic cluster algebra which is not acyclic. The Markov quiver in Figure 5 gives rise to a cluster algebra which is not locally acyclic.

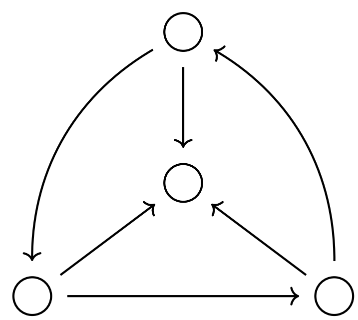


FIGURE 6. A quiver giving rise to a locally acyclic cluster algebra which is not acyclic.

3. A Topological Interlude

A large subclass of cluster algebras may be associated to certain triangulated surfaces. This association arises by constructing a quiver from a given triangulation. Mutating the quiver is equivalent to mutating the triangulation (a process involving "flips" of arcs). In addition to the incarnation of cluster algebras into another field of mathematics, this neat topological translation gives us an effective and efficient way of performing certain computations in the cluster algebra once we pass to the Kauffman Skein algebra (see Examples 3.25 and 3.26, below). Moreover, we have already seen these types of cluster algebras play a role in the classification of mutation-finite cluster algebras, as well as being an example of locally acyclic cluster algebras.

3.1. **Marked Surfaces & Ideal Triangulations**

These first two subsections contain results from a paper by Fomin, Shapiro, and Thurston [FST08]. Schiffler has a batch of lecture notes [Sch16] which contain a nice summary of the ideas from the paper, and a portion of our exposition parallels his.

[5]This holds whenever the surface has at least two marked points in each boundary component [Mul13]. See the next section for details on associating marked surfaces to cluster algebras.

Definition 3.1. Let S be a connected oriented Riemann surface with boundary. Let $M \subset S$ denote a finite set of *marked points* with at least one point on each connected boundary component. The pair (S, M) is referred to as a **marked surface** or just a surface.

Points in the interior of S are called *punctures*. For our purposes, we do not consider pairs (S, M) of the following forms:

- Spheres with 1, 2, or 3 punctures;
- an unpunctured or once-punctured monogon;
- an unpuctured digon;
- an unpunctured triangle.

Definition 3.2. An **arc** γ in (S, M) is a curve in S, considered up to isotopy, such that

- the endpoints of γ are in M;
- γ does not cross itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from M and from ∂S,
- γ does not cut out an unpunctured monogon or an unpunctured digon (so γ is not contractible into M or onto the boundary of S).

Definition 3.3. A curve β between two marked points which lies entirely in the boundary of S and which does not pass through any other marked point is called a **boundary arc**.

Definition 3.4. Two arcs are called **compatible** if there are representatives of their respective isotopy classes that do not intersect in the interior of S.

Definition 3.5. An **ideal triangulation** is a maximal collection of distinct pairwise compatible arcs $\{\gamma_1, \ldots, \gamma_n\}$, together with all possible boundary arcs $\{\beta_1, \ldots, \beta_m\}$. The arcs of a triangulation cut the surface into *ideal triangles*. Triangles that have only two distinct sides are called *self-folded triangles*[6].

Example 3.6. For a once-punctured triangle, there are 10 ideal triangulations which can easily be verified by direct inspection. Figure 7 shows three possible ideal triangulations.

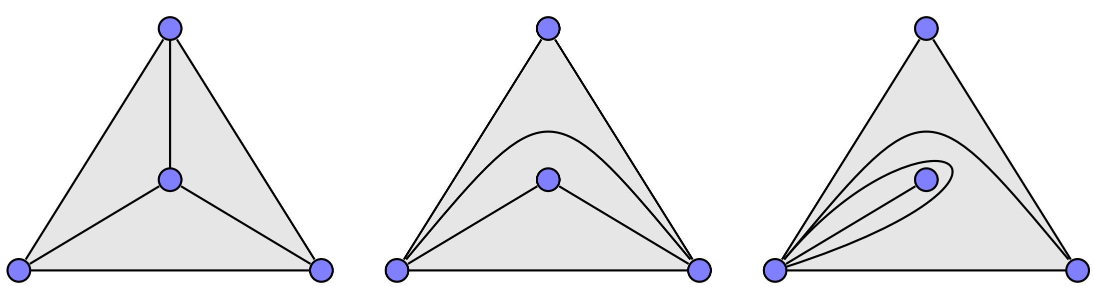


FIGURE 7. Three different ideal triangulations of a once-punctured triangle.

[6]Figure 7 shows an example of a self-folded triangle in the right-most triangulation.

Proposition 3.7. *Each ideal triangulation consists of*

$$n = 6g + 3b + 3p + c - 6$$

arcs where g is the genus of S, b is the number of boundary components, p is the number of punctures, and c is the number of marked points on the boundary.

Definition 3.8. A **flip** is a transformation of an ideal triangulation T that removes an arc γ from T and replaces it with a (unique) different arc γ' such that $T' = T \setminus \{\gamma\} \cup \{\gamma'\}$ forms an ideal triangulation.

Example 3.9. Figure 7 actually shows two different flips. Beginning with the left-most triangulation, consider the vertical arc. Flipping this arc yields the middle triangulation. From this triangulation, flipping the arc connecting the puncture and the bottom right marked point yields the right-most triangulation.

Definition 3.10. The **arc complex** $\Delta^\circ(S, M)$ is the simplicial complex on the set of all arcs in (S, M) whose simplices are collections of distinct mutually compatible arcs, and whose maximal simplices are the ideal triangulations.

Definition 3.11. The **exchange graph** $E^\circ(S, M)$ is the dual graph of the arc complex. That is, $E^\circ(S, M)$ is a graph with vertices labelled by ideal triangulations of (S, M) whose edges correspond to flips of a triangulation.

Proposition 3.12 ([FST08]). *The exchange graph $E^\circ(S, M)$ is connected, hence any two ideal triangulations are related by a sequence of flips.*

3.2. **The Cluster Algebra Associated to a Triangulated Surface**

Given any marked surface (S, M) and an ideal triangulation $T = \{\tau_1, \ldots, \tau_m\} = \{\gamma_1, \ldots, \gamma_n\} \cup \{\beta_1, \ldots, \beta_{m-n}\}$ of this surface, we may construct an associated ice quiver $Q_{(S,M,T)}$ in the following way:

- For each arc $\gamma_k \in T$, assign a mutable vertex k.
- For each boundary arc $\beta_h \in T$, assign a vertex $n + h$.
- For every ideal triangle Δ which is formed by T, add an arrow from $i \to j$ whenever τ_i and τ_j are sides of Δ such that τ_j follows τ_i in clockwise order.

See Figure 8 for an example. Whenever it is clear which marked surface we are dealing with, we will denote $Q_{(S,M,T)}$ simply by Q_T.

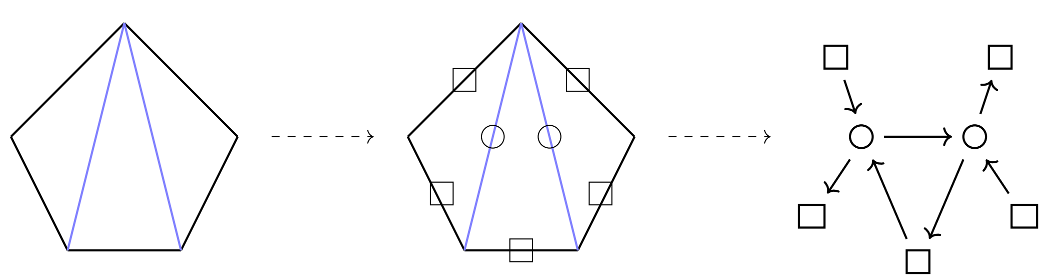


Figure 8. Associating a quiver to a triangulation.

Definition 3.13. Given a marked surface (S, M), construct the associated quiver $Q_{(S,M,T)} = Q_T$ as above. Assign an indeterminant $x_i \in \mathcal{F}$ to each vertex i, so that $\tilde{\mathbf{x}}_T = (x_1, \ldots, x_m)$ is an extended cluster. Then $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ is the **cluster algebra associated to the marked surface** (S, M).

Definition 3.14. The **cluster complex** of a cluster algebra $\mathcal{A}$ (not necessarily associated to a marked surface), is the possibly infinite simplicial complex on the set $\mathcal{X}$ of all cluster variables whose maximal simplices are the clusters.

Definition 3.15. The **exchange graph** of a cluster algebra is the dual graph of the cluster complex. That is, it is a graph with vertices labeled by clusters (or labelled seeds), and whose edges correspond to mutations.

Fomin, Shapiro, and Thurston showed that $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ does not depend on the initial choice of ideal triangulation T [FST08]. Moreover, for an ideal triangulation T' obtained from T via a flip replacing arc k, we have that $\mu_k(Q_T) = Q_{T'}$. Moreover, they prove the following:

Theorem 3.16 ([FST08])**.** *The cluster variables of $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ are in bijection with the arcs of (S, M), and the clusters of $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$ are in bijection with the ideal triangulations of (S, M).*

Theorem 3.17 ([FST08])**.** *The arc complex $\Delta^\circ(S, M)$ is a subcomplex of the cluster complex of $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$, and the exchange graph $E^\circ(S, M)$ is a subgraph of the exchange graph of $\mathcal{A}(\tilde{\mathbf{x}}_T, Q_T)$.*

Now, in general, we only get subgraphs and subcomplexes because some arcs are not flippable (for example, the radius of a self-folded triangle[7]). In order to circumvent this issue, Fomin, Shapiro, and Thurston introduce a generalized notion of an arc, called a *tagged arc*. This leads to a *tagged arc complex* with an associated tagged dual graph. We omit the details of this generalization[8], but emphasize that this gives us the desired isomorphisms between the complexes as well as between graphs, and that this topological construction was a new addition to the field whose sole motivation comes from the study of cluster algebras.

There are many cluster algebras that may be associated to marked surfaces. In particular, cluster algebras of type A_n ($n \geqslant 1$), B_n ($n \geqslant 4$), or E_n ($6 \leqslant n \leqslant 8$) may be realized in this way, with type A_n coming from polygons and type B_n coming from once-punctured polygons (ignoring boundary arcs). Another example is the Markov quiver, which arises from the once-punctured torus.

3.3. The (Kauffman) Skein Algebra

A skein algebra is a certain algebra associated to an unmarked surface, first introduced in [Tur89] and [Prz91] to generalize Kauffman's bracket for computing the Jones polynomial. This algebra has become important in knot theory. In [Mul16b], Muller generalizes this to marked surfaces and shows (under appropriate conditions) that a certain localization of this algebra sits between the cluster algebra and the upper cluster algebra associated to the same

[7] Try flipping the self-folded triangle in Figure 7, and see how far you get.

[8] All of these details may be found in [FST08].

marked surface. In fact, he works on the level of *quantum* cluster algebras, which include the aforementioned case as a specialization at $q^{1/2} = 1$, which we explain later.

In this section, we restrict our attention to marked surfaces (S, M) whose marked points only occur on the boundary of S. This restriction is not necessary for the following theory to hold, but things get more complicated, otherwise. When discussing "curves" in S, we allow for arcs, as defined above, as well as loops (closed curves without endpoints).

Now, in the quantum case, we keep track of the order in which curves cross over one another. In these diagrams, the crossings are drawn in the natural way. We mention that when specializing the quantum case to $q^{1/2} = 1$, we obtain the commutative version of the algebras which falls in line with the previous subsections. Here, we lose track of the crossings. See Figure 9 for an example of the specialized case in which we do not care about crossings. See Figure 10 for an example where we do keep track of crossings.

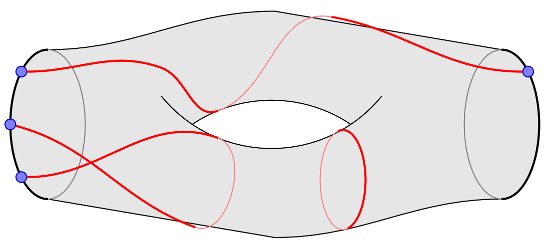


FIGURE 9. An example of a marked surface with boundary and some curves.

Definition 3.18. A **link diagram** (or link) L is a collection of curves in S, such that all intersections are simple and transverse, along with an ordering of curves at each intersection.

Link diagrams generalize the projection of a knot in $S \times [0, 1]$ onto S, where the orderings keep track of how the strands are passing over each other. We keep track of the order in our diagrams in the natural way by drawing the curve "on top" to appear as though it's actually on top (see Figure 10).

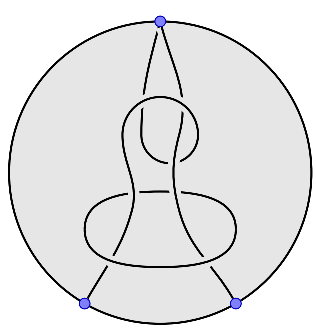


FIGURE 10. An example of a link diagram in Sk_q.

Let $\mathbb{Z}_q$ denote the ring $\mathbb{Z}[q^{\pm 1/2}]$ of Laurent polynomials in the indeterminant $q^{1/2}$. For any marked surface (S, M), let $\mathbb{Z}_q^{\mathrm{Links}}$ denote the free $\mathbb{Z}_q$-module with basis given by equivalence classes of links in S. Let I be the submodule generated by the four relations pictured in Figure 11.

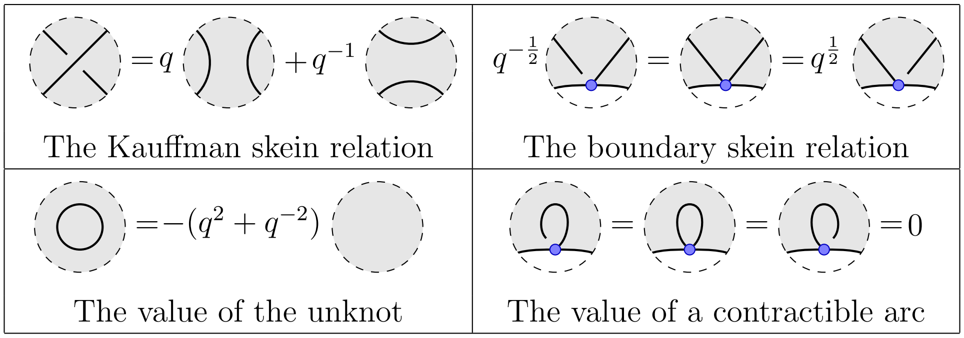


FIGURE 11. The defining relations of $\mathrm{Sk}_q(S)$ (and the generators of I).

Definition 3.19. The **skein algebra** $\mathrm{Sk}_q(S)$ of a marked surface (S, M) is the the $\mathbb{Z}_q$-module $\mathrm{Sk}_q(S) \coloneqq \mathbb{Z}_q^{\mathrm{Links}}/I$, along with an associative product given by superposition of link diagrams and unit given by the (class of the) empty link.

One must actually show that this gives a well-defined algebra; see [Mul16b] for details. The relations imposed in $\mathrm{Sk}_q(S)$ imply other important relations that are useful when doing computations. These are the modified Reidemeister moves from knot theory, along with an additional relation coming from the addition of marked endpoints. These relations are shown in Figure 12.

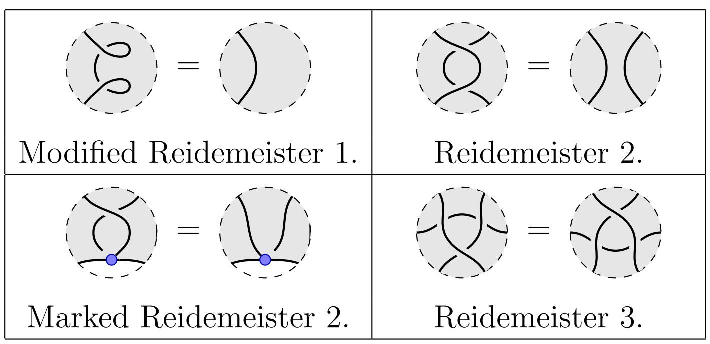
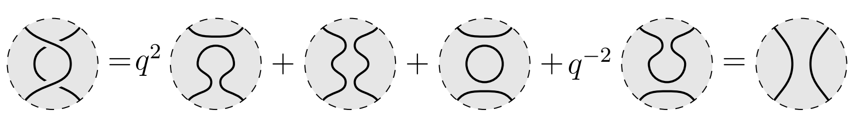


FIGURE 12. The Reidemeister moves for links with endpoints.

These additional relations can be shown via direct computation using the relations in I. We show one such calculation for the Reidemeister 2 move as an example of how to work in this algebra:



Muller also shows that $\mathrm{Sk}_q(S)$ is a domain, is finitely generated, and has a $\mathbb{Z}_q$-basis parameterized by simple multicurves (links without crossing data and without interior intersectoins or contractible curves).

In order to relate skein algebras to cluster algebras, Muller considers *quantum cluster algebras*, introduced in [BZ05], which are non-commutative deformations of cluster algebras, in which two cluster variables in the same cluster quasi-commute; that is, $x_i x_j = q^m x_j x_i$ for some $m \in \mathbb{Z}$. A remarkable amount of the theory generalizes to the quantum setting. In particular, there is a quantum cluster algebra $\mathcal{A}_q$ and a quantum upper cluster algebra $\mathcal{U}_q$. In [Mul16b], Muller introduces two corresponding algebras associated to marked surfaces, $\mathcal{A}_q(S)$ and $\mathcal{U}_q(S)$.

Before we define this, we must introduce the *localized skein algebra*. We note that the set of boundary arcs is always contained in a triangulation T of a marked surface, so that the localization $\mathrm{Sk}_q(S)[T^{-1}]$ contains the inverse to each boundary arc. In particular, the boundary arcs form an Ore set in $\mathrm{Sk}_q(S)$.

Definition 3.20. The **localized skein algebra** $\mathrm{Sk}_q^\circ(S)$ is the Ore localization at the boundary arcs in S.

Since the skein algebra is generated by simple curves, $\mathrm{Sk}_q^\circ(S)$ is generated by simple curves along with inverses to boundary curves.

Definition 3.21. The **quantum cluster algebra** $\mathcal{A}_q(S)$ of S is the $\mathbb{Z}_q$-subalgebra of $\mathrm{Sk}_q^\circ(S)$ generated by simple arcs and the inverse to boundary arcs.

Definition 3.22. The **quantum upper cluster algebra** $\mathcal{U}_q(S)$ of S is the $\mathbb{Z}_q$-algebra consisting of elements in the skew-field $\mathcal{F}$ which can be written as a skew-Laurent polynomial in each triangulation.

Theorem 3.23 ([Mul16b])**.** *We have $\mathcal{A}_q(S) \subset \mathrm{Sk}_q^\circ(S) \subset \mathcal{U}_q(S)$ for any triangulable marked surface (S, M).*

Theorem 3.24 ([Mul16b])**.** *For any triangulable marked surface (S, M) which has at least two marked points in each boundary component, $\mathcal{A}_q(S) = \mathrm{Sk}_q^\circ(S) = \mathcal{U}_q(S)$.*

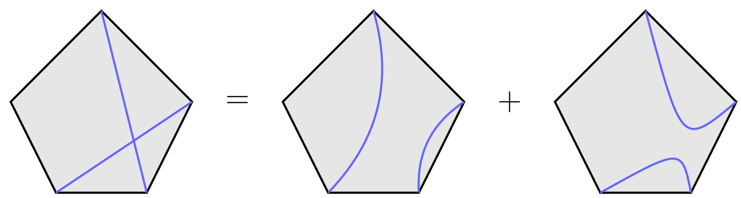
We note that the simple loops in $\mathrm{Sk}_q^\circ(S)$ define elements of $\mathcal{A}_q(S)$ that are not cluster variables and that considering these elements simplifies computations and provides a free $\mathbb{Z}_q$-basis of $\mathcal{A}_q(S)$.

Finally, we point out that if one specializes $q^{1/2} = 1$, the skein algebra becomes commutative since the order of crossings no longer matters. The diagrams in this case are then drawn without crossings, and we obtain $\mathcal{A}(S) = \mathrm{Sk}_1^\circ(S) = \mathcal{U}(S)$. Here, $\mathcal{A}(S)$ is such that cluster variables correspond to arcs without self-intersections, and the set of arcs in a triangulation is a cluster. That is, $\mathcal{A}(S) = \mathcal{A}(\widetilde{\mathbf{x}}_T, Q_T)$ as defined in the previous subsection.

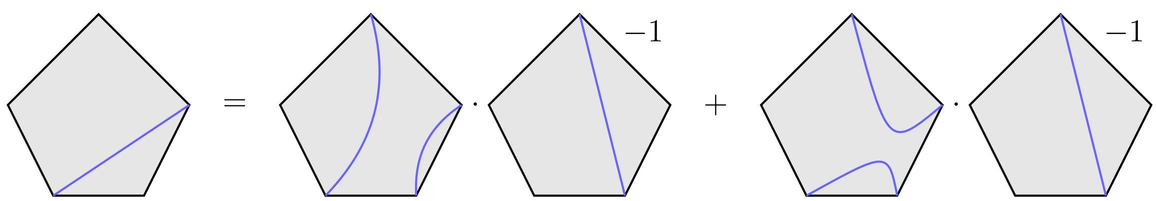
To finish this section, we wish to show an example of how the diagrammatics vastly simplify calculations, such as when working with elements which may be in distant clusters. Before doing so, we comment that it can be shown that in order to write a curve in terms of a given triangulation, one needs only to consider the link which consists of the arcs in the triangulation superimposed (multiplied) with the curve in question. Simplifying this link using the relations allows one to write the curve in terms of the arcs of the triangulation.

Example 3.25. Here, we will work in the specialized $q^{1/2} = 1$ case so that we may ignore the order of crossings. Let (S, M) be a pentagon with marked points on the corners. Let's write the arc shown below on the left in terms of the triangulation shown below on the right:

We can superimpose our desired arc with the arcs of the triangulation it crosses, and the reduce these crossings using the appropriate relations. In this case, we only have one crossing to deal with:

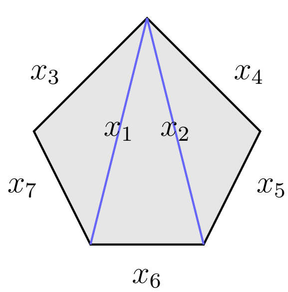


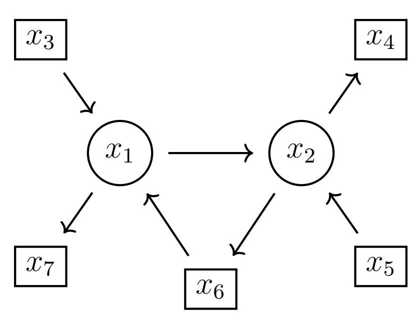
Hence, we have



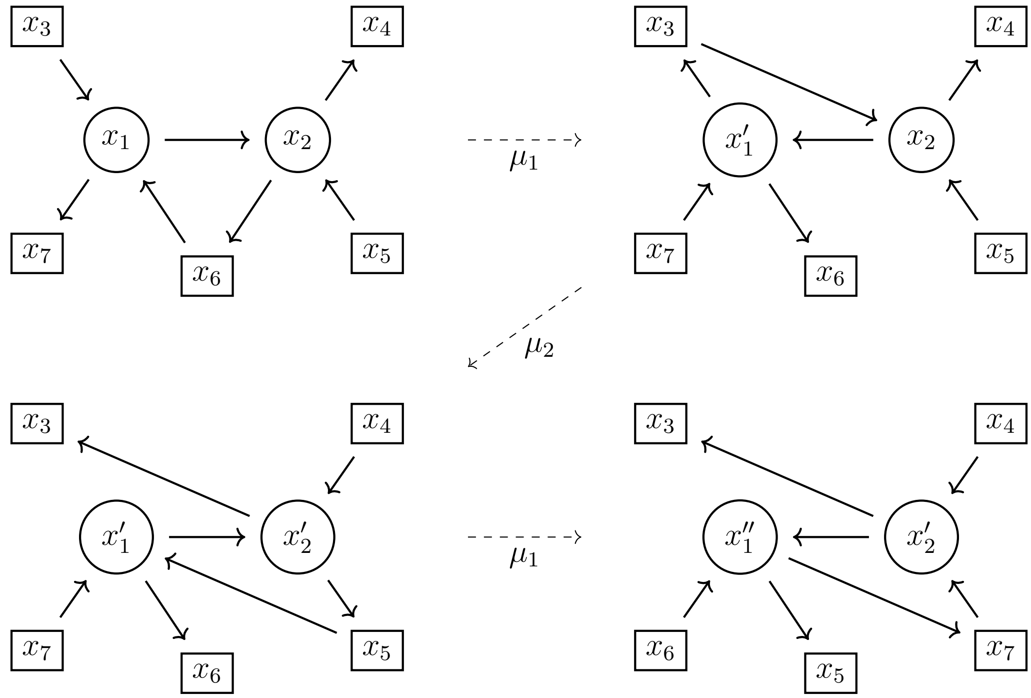
where the arcs in the third term and one of the arcs in the first term on the right hand side of the equation are isotopic to the boundary arcs, which are part of the triangulation.

Example 3.26. Working with the same space as in Example 3.25, let $\widetilde{\mathbf{x}} = (x_1, x_2, x_3, \ldots, x_7)$ where x_1 and x_2 are the mutable variables. Then $\widetilde{\mathbf{x}}_T$ is given by the following diagram on the bottom left, where T is the initial ideal triangulation of (S, M). The corresponding quiver (labelled seed) is shown on the right.

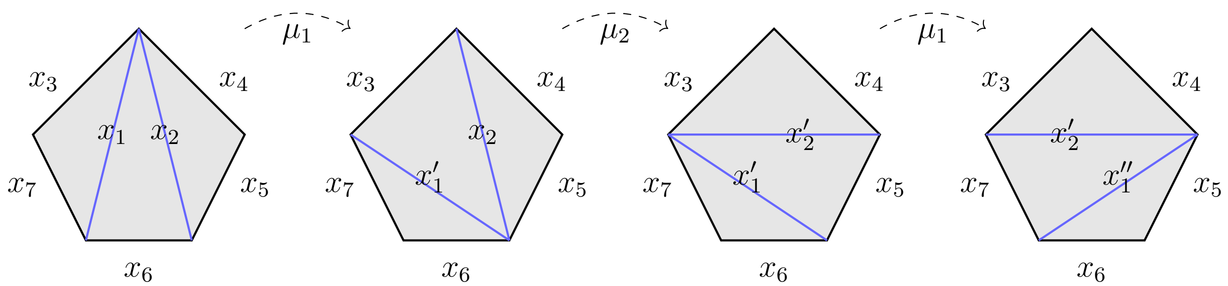




The following sequence of seed mutations



corresponds to the following sequence of flips:



Now, we have that $x_1' = \dfrac{x_3x_6 + x_7x_2}{x_1}$, $x_2' = \dfrac{x_3x_5 + x_4x_1'}{x_2} = \dfrac{x_3x_5x_1 + x_3x_6x_4 + x_7x_4x_2}{x_1x_2}$, and

$$x_1'' = \frac{x_7x_5 + x_6x_2'}{x_1'} = \frac{x_7x_5 + x_6\left(\dfrac{x_3x_5x_1 + x_3x_6x_4 + x_7x_4x_2}{x_1x_2}\right)}{\dfrac{x_3x_6 + x_7x_2}{x_1}}.$$

After only three mutations, computing the resulting cluster variable is pretty ugly. But, observe that x_1'' corresponds to the arc that was considered in example 3.25. We see immediately from the resulting equation of that example that $x_1'' = \dfrac{x_1x_5 + x_4x_6}{x_2}$. It can be verified that this matches the above formula, after simplification. Even in the simplest examples, the computational savings are immense when working with the diagrammatics in the skein algebra.

4. A Canonical Basis for Cluster Algebras

In this section, we discuss the so-called "theta basis" for a cluster algebra which consists of elements called theta functions. This "basis" is attractive as it contains the cluster monomials and has positive structure constants. Moreover, these coefficients essentially count objects called broken lines. It seems that many questions about cluster algebras should thus be able to be reduced to counting problems. Unfortunately, these theta functions don't always give a basis of the cluster algebra in question; the cluster algebra may only be contained in the algebra spanned by the theta functions, hence justifying the usage of quotations, above. We will discuss this a little later. Moreover, the machinery involved in even defining theta functions is heavy, and quite frankly, opressive. Therefore, we attempt to outline the intuition behind theta functions without diving too far into the deep end.

4.1. *g*-vectors

In practice, working with the cluster variables can be overwhelming. If one is not interested in the actual Laurent polynomial itself, but just in keeping track of the variables and their relation with the rest of the cluster algebra, one may wish to focus on g-vectors. For each cluster variable in a rank d cluster algebra, there is a unique vector in $\mathbb{Z}^d$ one may associate to it, called a g-vector. We can view this vector as a "serial number" for the cluster variable. One way in which this assignment can be made is as follows[9].

Definition 4.1. A ***g*-seed** is a quiver Q (say of rank d), along with a basis of $\mathbb{Z}^d$ indexed by the vertices of Q. The basis element g_k associated to a vertex k is called its ***g*-vector**.

Given any $v \in \mathbb{Z}^d$, v may be expressed as $v = \sum_{\substack{\text{vertices} \\ k \text{ of } Q}} c_k(v) g_k$ where $c_k(v) \in \mathbb{Z}$.

Definition 4.2. A vertex k is called **green** if for all $v \in \mathbb{N}^d \subset \mathbb{Z}^d$, the coefficient $c_k(v) \geqslant 0$. On the other hand, a vertex k is called **red** if for all $v \in \mathbb{N}^d \subset \mathbb{Z}^d$, the coefficient $c_k(v) \leqslant 0$.

Definition 4.3. Given a quiver Q, the g-seed in which the g-vector of the i^{th} vertex is the standard basis element $e_i = (0, \ldots, 1, \ldots, 0)$ is called the **initial *g*-seed**.

Obviously, the initial g-seed has all green vertices.

Definition 4.4. Given a quiver Q and a corresponding g-seed, we may **mutate** the g-seed at any red or green vertex, k. The quiver is mutated as usual, and the g-vectors all remain the same except for at vertex k, which becomes

$$\mu_k(g_k) = \begin{cases} -g_k + \sum_{\substack{\text{arrows} \\ j \to k}} g_j & \text{if } k \text{ is green} \\ -g_k + \sum_{\substack{\text{arrows} \\ k \to j}} g_j & \text{if } k \text{ is red} \end{cases}$$

[9] Much of this exposition is taken from [Mul16a].

Notice that mutating a green vertex turns it red, and vice versa. So mutating at the same vertex yields the orginal g-seed (as we have come to expect from quiver mutation).

There is an alternative (but equivalent) formulation of the notion of green and red vertices. This involves "framed quivers" and can be more useful in practice when determining whether your vertex is red or green, though the quiver mutations are a little uglier. This formulation is as follows[10]:

Definition 4.5. Let Q be a quiver. We define the **framed quiver** $\widehat{Q}$ to be the (ice) quiver Q along with a frozen vertex k' for each vertex k in Q and an arrow from $k \to k'$.

Definition 4.6. For any quiver R mutation equivalent to a framed quiver $\widehat{Q}$, we call a vertex k in R **green** if there are no arrows from a frozen vertex to vertex k. We call a vertex k in R **red** if there are no arrows from vertex k to any frozen vertices.

So instead of looking at the coefficients $c_k(v)$ to determine if a vertex k is green or red, one may simply consider the arrows between k and the frozen vertices in the framed quiver.

The following result is equivalent to the *sign coherence* result, and assures us that our vertices will always be either green or red.

Theorem 4.7 ([DWZ10])**.** *After any sequence of mutations at an initial g-seed, every vertex in the resulting quiver is either green or red.*

This means that given any g-seed, one may mutate arbitrarily and obtain new g-seeds. In particular, if one starts with the initial g-seed, one may obtain all possible g-vectors which correspond to cluster variables. This is because every cluster variable may be obtained by mutating an intial labelled seed, and these mutations agree.

What's more interesting is that this assignment is well-defined. That is, if a cluster variable appears in two different clusters, which correspond to two different g-seeds, the g-vector associated to the cluster variable in question is always the same. This result seems slightly magical; it was proven in a different setting in [FZ07] where it's more straightforward.

Furthermore, it was shown several years later in [GHKK18] that this assignment gives an *injection* from the set of all cluster variables to the set of all g-vectors. So, we have a unique way of associating a g-vector to every cluster variable. We may also associate an appropriate linear combination of g-vectors to every cluster monomial, and it's true that the g-vectors of cluster monomials of a fixed cluster are the lattice points in a cone which follows from the fact that each g-seed is a $\mathbb{Z}$-basis for $\mathbb{Z}^d$.

Now, in general, the cluster variables don't account for all g-vectors (we only have an injection).

Example 4.8. Consider the Kronecker 2-quiver (the quiver with two mutable vertices and two arrows from vertex 1 to 2). It can be shown that every g-vector which is *not* of the form $(a, -a)$ for $a > 0$ may be associated to a cluster variable[11]. Even in this simple example, we

[10] This formulation is taken from [BDP14].

[11] The proof of Lemma 2.2.1 in [Mul16a] contains the necessary computations for g-seeds; one then only needs to show that the union of the corresponding cones is the complement of the $(a, -a)$ ray.

don't get all of the g-vectors. In section 4.4, we introduce theta functions which, in this case, assign the so-called "loop element" of the cluster algebra[12] to the g-vector $(1,-1)$. Then any g-vector of the form $(a,-a)$ is a power of this element. With this assignment, we have every g-vector corresponding to an element of the cluster algebra, and in fact, the theta functions we mention later on will form a basis of this cluster algebra.

Theorem 4.9 ([BDP14]). *Let $(Q,\{g_k\})$ be a g-seed which is mutation equivalent to an initial g-seed $(Q_{in},\{e_k\})$.*

(1) If every vertex of $(Q,\{g_k\})$ is green, then there is a quiver isomorphism $f: Q_{in} \to Q$ such that $g_{f(k)} = e_k$.

(2) If every vertex of $(Q,\{g_k\})$ is red, then there is a quiver isomorphism $f: Q_{in} \to Q$ such that $g_{f(k)} = -e_k$.

This means that if your g-seed is all green, it must actually be the initial g-seed, and if your seed is all red, you have the same quiver but with the negative standard basis elements. Obviously an all green seed always exists (the initial g-seed). But the all red seed may not; that is, one may never be able to acheive an all red seed via g-seed mutation.

This poses an interesting question: when is it possible to find a sequence of mutations resulting in an all red final seed? It turns out that this question has profound implications for theta functions. Before we discuss the implications, we further examine such a sequence.

4.2. **Maximal Green Sequences & Green to Red Sequences**

Definition 4.10. A **maximal green sequence** for a quiver Q is a sequence of mutations beginning at the initial g-seed such that each mutation is at a green vertex and every vertex in the final g-seed is red.

Example 4.11. Figure 13 demonstrates a maximal green sequence.

As Muller points out in [Mul16a], there are several families of well-behaved cluster algebras which exhibit maximal green sequences. Therefore, it had been conjectured that the existence of such a sequence was equivalent to the cluster algebra having certain nice properties such as being equal to its upper cluster algebra. In the same paper, Greg dashed the hopes and dreams of the conjecturers by providing a counter example to the conjecture that if a quiver admits a maximal green sequence, then so do all quivers mutation equivalent to it.

Definition 4.12. For a,b,c non-negative integers, let $Q_{(a,b,c)}$ denote the rank three quiver which has a-many arrows from vertex 1 to vertex 2, b-many arrows from 2 to 3, and c-many arrows from 3 to 1.

[12] In this case, the loop element is $\dfrac{x_1^2+1+x_2^2}{x_1x_2}$. Moreover, we mention that the loop element is so named because it corresponds to a noncontractible loop in the skein algebra corresponding to the Kronecker 2-quiver (whose marked surface is an annulus).

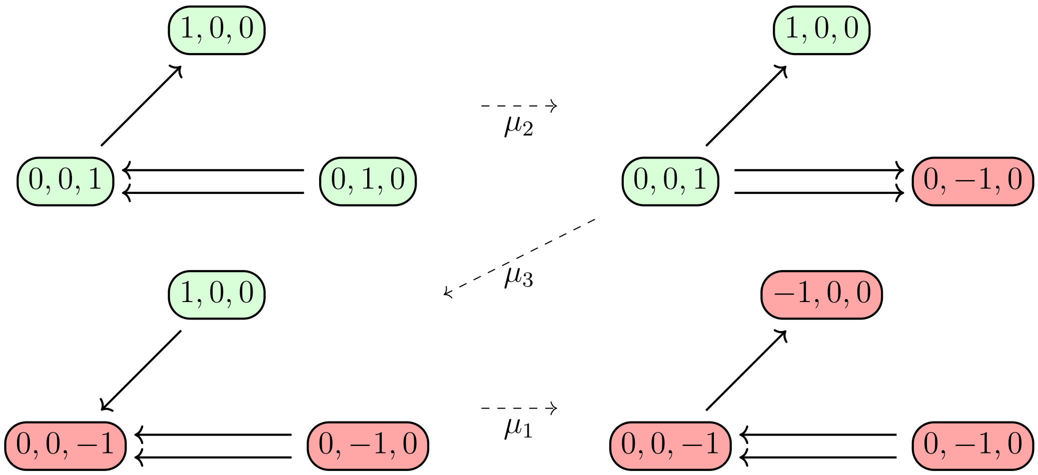


FIGURE 13. The quiver $Q_{(0,2,1)}$ exhibits a maximal green sequence.

The counter example[13] Greg provides is the $Q_{(2,2,3)}$ quiver. This quiver does not exhibit a maximal green sequence, but is mutation equivalent to $Q_{(0,2,1)}$ (see Figure 14), which does exhibit a maximal green sequence (see Figure 13).

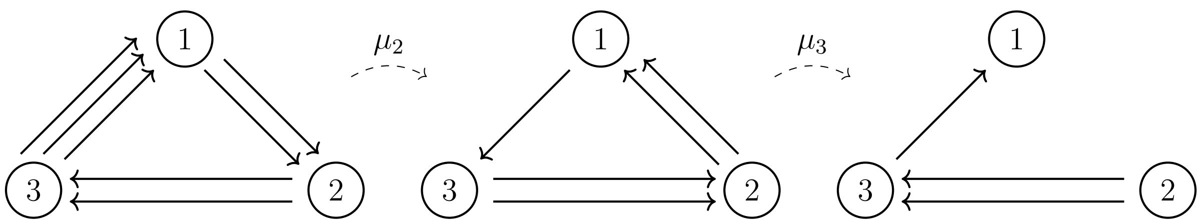


FIGURE 14. $Q_{(2,2,3)}$ is mutation equivalent to $Q_{(0,2,1)}$.

So, the lofty conjectures were disproved. However, one may weaken the notion of a maximal green sequence and study green-to-red sequences, which have some of the same properties. In particular, the existence of a green-to-red sequence ensures that the all red g-seed exists, which was our initial question posed before this subsection.

Definition 4.13. A **green-to-red sequence** for a quiver Q is a sequence of mutations beginning at the initial g-seed and ending at a g-seed in which every vertex is red.

Of course, a maximal green sequence is a green-to-red sequene, but there are quivers which admit a green-to-red sequence but not a maximal green sequence. The $Q_{(2,2,3)}$ quiver is such an example - See [Mul16a] for details.

[13] Actually, a whole family of counterexamples is given. These come from quivers of the form $Q_{(a,b,c)}$ with $a, b, c \geqslant 2$ and $abc - a^2 - b^2 - c^2 + 4 < 0$.

4.3. **Scattering Diagrams**

The cones of g-vectors in a cluster algebra are part of a richer object called a *scattering diagram*, which fills all of affine space with a wall-and-chamber structure[14]. Unfortunately, formally defining these objects can be cumbersome, so we opt for nontechnical definitions which will allow us to deliver the appropriate intuition without getting bogged down in details.

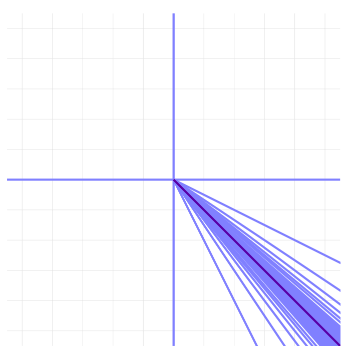
Definition 4.14. Given a cluster algebra with a fixed initial cluster, the associated **scattering diagram** is a (typically infinite) collection of *walls* in $\mathbb{R}^d$. Here, a *wall* is a codimension 1 polyhedral cone, along with some "scattering data."

Now, for a given labelled seed $(\mathbf{x}, Q)$, one may associate a so-called *consistent* scattering diagram, which we won't get into here, but just comment that consistent diagrams are the type we need for the following theory to hold [GHKK18].

Theorem 4.15 ([GHKK18])**.** *For every quiver Q, say of rank d, there is a consistent scattering diagram $\mathfrak{D}(Q)$, unique up to equivalence, such that*

- *for each $i \in \{1, \dots, d\}$, there is a wall of the form $(e_i, e_i^\perp)$, and*
- *every other wall (n, W) in $\mathfrak{D}(Q)$ is such that $Bn \notin W$.*

Example 4.16. The scattering diagram for the Kronecker 2-quiver, where $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ is depicted in Figure 15 below.



Walls:
$(e_1, e_1^\perp) = ((1,0), (0, \mathbb{R}))$
$(e_2, e_2^\perp) = ((0,1), (\mathbb{R}, 0))$
$\forall k \geq 1, ((k, k+1), \mathbb{R}_{\geq 0} \cdot (k+1, -k))$
$\forall k \geq 1, ((k+1, k), \mathbb{R}_{\geq 0} \cdot (k, -k-1))$
For each $k \geq 0$, two copies of
$((2^k, 2^k), \mathbb{R}_{\geq 0} \cdot (1, -1))$

FIGURE 15. An infinite consistent scattering diagram

Definition 4.17. A **chamber** in a scattering diagram $\mathfrak{D}$ is a path-connected component of $\mathbb{R}^d \setminus \mathfrak{D}$.

Note that the all-positive chamber and all-negative chambers are always chambers of any scattering diagram associated to a quiver. For example, in Figure 15 above, the all-positive chamber is the first quadrant, and the all-negative chamber is the third quadrant.

[14]One basic utilization of this richer object is that it allows one to visualize a maximal green sequence or a green-to-red sequence by considering scattering diagrams and paths in these diagrams.

Definition 4.18. A chamber in a scattering diagram $\mathfrak{D}(Q)$ is **reachable** if it can be connected to the all-positive chamber by a finite transverse path.

Theorem 4.19 ([GHKK18])**.** *Let $\{g_1, \ldots, g_d\}$ be the g-vectors in a g-seed which is mutation equivalent to the initial g-seed with quiver Q. Then*

$$\mathbb{R}_{>0}g_1 + \cdots + \mathbb{R}_{>0}g_d$$

is a reachable chamber in $\mathfrak{D}(Q)$. This induces a bijection between g-seeds mutation equivalent to the initial g-seed on Q, and reachable chambers of $\mathfrak{D}(Q)$.

As Muller explains in [Mul16a], we obtain the following statements:

- A vertex k in a g-seed is green if and only if the g-vector g_k is on the green side of the wall spanned by the other g-vectors.
- Two g-seeds are related by mutation if and only if the corresponding reachable chambers share a facet. The green mutation goes from the green side to the red, and the red mutation goes from the red side to the green.
- An all-green g-seed must correspond to the all-positive chamber, and it must be the initial g-seed.
- An all-red g-seed must correspond to the all-negative chamber. Hence, the existence of an all-red seed is equivalent to the all-negative chamber being reachable.

There is a way to associate sequences of mutations of g-seeds exactly with certain finite transverse paths in $\mathfrak{D}(Q)$. Hence,

- A maximal green sequence is equivalent to a finite transverse path from the all-positive chamber to the all-negative chamber, which always crosses a wall from the green side to the red side.
- A green-to-red sequence is equivalent to a finite tranverse path from the all-positive chamber to the all-negative chamber.

So we now have a nice characterization of our question of when the all-red seed exists.

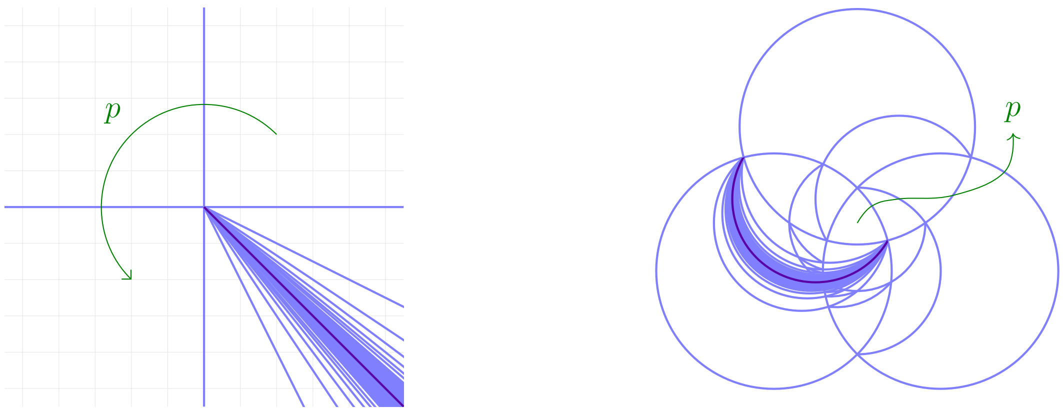
Example 4.20. [Mul16a] For the Kronecker 2-quiver, there is a unique maximal green sequence which can be realized by the path p in Figure 16a.

A finite transverse path cannot cross the purple ray, since it supports an infinite number of walls.[15] If a maximal green sequence could begin by mutating at vertex 2, the corresponding path would start by crossing the ray $(\mathbb{R}_{\geq 0}, 0)$. The path cannot cross the purple wall and it cannot recross the ray $(\mathbb{R}_{\geq 0}, 0)$, because that would correspond to a red mutation. Hence, the path is trapped between the two, and can never reach the all-negative chamber.

Example 4.21. [Mul16a] For $Q_{(1,1,2)}$ which has $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, there are many maximal green sequences, one of which can be realized by the path p depicted in Figure 16b.

[15]Additionally, it would have to cross an infinite number of walls to get to the purple ray.

Figure 16b shows a stereographic projection of the scattering diagram and has been chosen so that the point $(1, 1, 1)$ maps to the origin, and so the green side of the stereographic projection of a wall is always the concave side. A maximal green sequence for Q corresponds to a finite transverse path in $\mathfrak{D}(Q)$ which travels from the inner-most chamber (which is the all-positive chamber) to the exterior (the all-negative chamber) and only crosses walls from the concave side to the convex side.



(A) A path (B) Stereographic projection of a path

FIGURE 16. Paths corresponding to maximal green sequences.

4.4. **Theta Functions & The Theta Basis**

Now, g-seeds correspond to reachable chambers in a scattering diagram. So the cluster variables that appear in these seeds are associated to g-vectors. But these only account for some the the possible g-vectors. It would be attractive to find other elements in the cluster algerba which may be associated to these remaining g-vectors. But how may one go about this?

Definition 4.22. A **broken line** in a scattering diagram is a piecewise linear map from $[0, \infty) \to \mathbb{R}^d$ which is non-linear only at the walls and is allowed to bend at a wall only in certain ways determined by the scattering data associated to each wall.

Definition 4.23. Each lattice point $m \in \mathbb{Z}^d \subset \mathbb{R}^d$ in a scattering diagram determines a **theta function** $\Theta_q(m)$ given by a certain formal sum over all broken lines with a fixed starting point q and initial derivative m. Products between theta functions can be also be computed by counting certain collections of broken lines.

Now, it is not clear that these summations should be finite, so the theta function may not even exist. Luckily, there are some conditions which may be found in Proposition 0.14 of [GHKK18] that guarantee the existence of all our theta functions. In particular, we have the following result.

Theorem 4.24 ([GHKK18])**.** *If a cluster algebra exhibits a green-to-red sequence (such as a maximal green sequence), then the theta function* $\Theta_q(m)$ *exists for all* $m \in \mathbb{Z}^d$.

So, we see that our basic question of when an all-red g-seed exists actually plays an important role in the existence of theta functions. And the existence of theta functions provides an even bigger role in the problem of finding a so-called **canonical basis** for a cluster algebra.

Now, in many cases, the theta functions from such a basis of the corresponding cluster algebra which contains the cluster monomials and which has positive structure constants. In general, it has been shown that the span of the theta functions which do exist (which may not be all of them), call this algebra Can, sits between the cluster algebra and its upper cluster algebra [GHKK18]. That is, $\mathcal{A} \subset \text{Can} \subset \mathcal{U}$. In this case, the theta functions spanning Can are indeed linearly independent, and the basis has been shown to have positive structure constants.

So, in particular, whenever $\mathcal{A} = \mathcal{U}$, it must be that the theta functions form a basis of the cluster algebra (for example for locally-acyclic cluster algebras). However, these inclusions can be proper, with the Markov quiver providing such an example.

It should be noted that the general feeling is that often times, $\mathcal{A}$ is "too small" and that $\mathcal{U}$ is "too big." So in the years to come, we may realize that Can is the "right" algebra to work with, rather than the cluster algebra.

While these results are positively in favor of the theta functions, these functions are still difficult to deal with. In general, determining all possible broken lines requires a fair bit of analysis. Moreover, when working with rank higher than 2, the scattering diagrams are difficult to work with - stereographic projections won't be helpful because the broken lines are very dependent on their position in space, and we cannot draw higher-dimensional pictures that are enlightening.

Hence, there is still *plenty* of work to be done in this area, including computing explicit formulas, even in the most basic cases.

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