11.10: Taylor and Maclaurin Series

We're going to look at writing $f(x) = \sin x$ as a series (which will be called its Taylor series). Let's suppose $\sin x = \sum_{n=0}^{\infty} c_n x^n$. How can we find c_0? Find c_0.

How can we find c_1, c_2, and c_3? Find c_1, c_2, and c_3.

Looking at the previous work, what is c_n in general?

Write the definition of a **Taylor series centered at 0** (this is also called a *Maclaurin series*).

Write the definition of a **Taylor series centered at** a.

1. Find the Maclaurin series of $f(x) = xe^x$.

Important Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R = 1$$

2. Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$ when $x = 3$.

3. Use a series to find the limit:

$$\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2}$$

Further problems: 1, 3, 4, 6, 8, 14, 16, 35, 36, 37, 38, 40, 53, 54, 62, 63, 74, 75, 79

Review

More Geometric Series

It's possible to evaluate geometric series even if they don't start at 1! In these examples, our a is the first term in the series and r is the second term divided by the first term. In these problems, our series converges if $|r| < 1$.

1. Find the sum $\sum_{i=4}^{\infty} \frac{(-3)^{i+1}}{4^i}$

2. Find the sum $\sum_{n=3}^{\infty} \frac{1 + (-3)^{n+1}}{2^{(2n)}}$

The Integral Test

If our summand is a decreasing, positive, and continuous function AND we can integrate it, then $\sum_{n=k}^{\infty} f(n)$ converges or diverges whenever $\int_{k}^{\infty} f(x)\, dx$ converges or diverges. Notice that this does not tell us what the sum is, but only if the series converges or diverges.

1. Determine whether $\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4}$ is convergent or divergent.

2. Determine whether $\sum_{n=1}^{\infty} ke^{-k^2}$ is convergent or divergent.

3. Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent or divergent.

Important Series 4: The p-series

The p-series is any series $\sum_{n=0}^{\infty} \frac{1}{n^p}$. Notice n is our index and p is a number! If this was switched, it would be a geometric series. This series converges when $p > 1$ and diverges when $p \leq 1$.

1. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ is convergent or divergent.

2. Determine whether $\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$ is convergent or divergent.

Further Problems: 1, 2, 3, 4, 5, 6, 7, 8, 17, 21, 23