11.8: Power Series

1. Write the definition of a **power series**.

2. If we replace c_n with a constant, a, what type of series is this? When is this convergent? Hint: Plug in a constant r.

3. Using the ratio test, when is $\sum_{n=0}^{\infty} n!x^n$ convergent?

4. Write the definition of a **power series centered at** a.

5. Using the ratio test, when does the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converge? Sketch this interval on a number line.

Fact: If we have a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, then one of the following occurs:

- The series always converges.
- The series only converges when $x = 0$.
- The series converges when $|x-a| < R$.

Above, R is called the **radius of convergence**. When $x = a + R$ or $x = a - R$, we will have to test for convergence separately.

6. Find the radius of convergence and interval of convergence of $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Further problems: 1, 2, 3, 7, 8, 14, 15, 17, 27

11.9: Representations of Functions as a Power Series

1. What is $\sum_{n=0}^{\infty} r^n$ equal to if $r < 1$?

2. Using 1, what is $\sum_{n=0}^{\infty} x^n$ equal to? What is the interval of convergence?

3. Using 2, write $\frac{1}{1+x^2}$ as a power series. What is the interval of convergence?

4. Using 2, write $\dfrac{1}{x+2}$ as a power series. What is the interval of convergence?

5. Using 4, write $\dfrac{x^3}{x+2}$ as a power series. What is the interval of convergence?

Note: We can differentiate or integrate a power series by differentiating or integrating every term seperately.

6. Using the above note, write $\dfrac{1}{(1-x)^2}$ as a power series. What is the interval of convergence?

7. Using the above note, write $\ln(1+x)$ as a power series. What is the interval of convergence?

8. Evaluate $\int \frac{1}{1+x^7}\, dx$ by first writing it as a power series, then integrating.

Further problems: 1, 3, 4, 5, 8, 9, 15, 17, 19, 20, 25, 27

Review

What is a series?

A **series** is a sum. For example, $1 + 2 + 3 + 4 + 5 + \ldots$. Frequently, we will shorthand this as $\sum_{n=1}^{\infty} n$. We can see these are the same if we plug in $n = 1, 2, 3, 4$, and 5 into the summation.

We write an arbitrary sum as $\sum a_n$, or maybe $\sum_{n=1}^{\infty}$ if we know the indices.

The partial sum of a series s_n is adding up the first n terms, i.e. $s_4 = a_1 + a_2 + a_3 + a_4$.

Since we can't add up infinitely things by hand (that would take a really long time!), we say $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. In other words, the sum is the limit of the partial sums.

Often it is hard to tell what the sum of a series is, but we *can* tell if it is convergent or not using a variety of tests for convergence.

Important Series 1: Geometric Series

The **geometric series** is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$. For example, $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$.

Geometric series are really really nice! Why? Because we know exactly *when* they converge AND! *what* they converge to! It is often hard to find out when a series converges. For example, if you can tell me what values of s give $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^s = 0$, you would receive 1 million dollars![1]

Fact: If $|r| \geq 1$, then the series is divergent, i.e. it does not converge. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

Examples:

1. Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

2. Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{26} + \ldots$

[1] https://en.wikipedia.org/wiki/Riemann_hypothesis

Important Series 2: The Harmonic Series

The **harmonic series** is:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

.

This series diverges! This series frequently helps us show the divergence of other series.

****Important** fact:** A series is divergent if $\lim_{n\to\infty} a_n \neq 0$. However, if $\lim_{n\to\infty} a_n = 0$, we can't say whether a series converges or not. For example, notice $\lim_{n\to\infty} \frac{1}{n} = 0$, but the harmonic series is divergent!

Important Series 3: The Telescoping Series

A **telescoping series** is any series that cancels out in pairs.

- For example, find the sum $\sum_{i=1}^{\infty} \frac{1}{n(n+1)}$.

- Find the sum $\sum_{i=1}^{\infty} \frac{2}{n(n+2)}$.

Further problems: 1, 2, 4, 5, 9, 17, 21, 23, 27, 32, 37, 43, 45, 57, 61