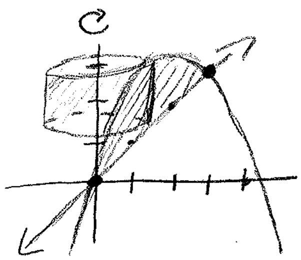
Homework 4

5.3 # 6, 8, 10, 30

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

6) $y = 4x - x^2, \; y = x$



Intersection points

$4x - x^2 = x$

$0 = x^2 - 3x$

$x = 0, 3$

$(0,0), (3,3)$

height of cylinder: $(4x - x^2) - (x)$

$= 3x - x^2$

radius of cylinder: x

Volume of rotation $= \int_0^3 2\pi x\,(3x - x^2)\,dx$

$= 2\pi \int_0^3 (3x^2 - x^3)\,dx$

$= 2\pi \left[x^3 - \frac{x^4}{4} \right]_0^3$

$= 2\pi \left[\left(3^3 - \frac{3^4}{4}\right) - 0 \right]$

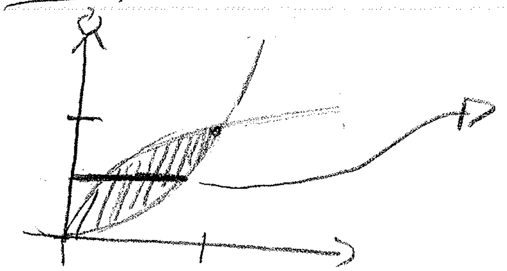
$= 2\pi \left[27 - \frac{81}{4} \right]$

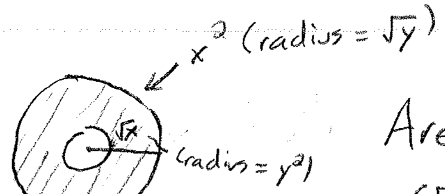
$= \pi \left[54 - \frac{81}{2} \right]$

$= \boxed{\frac{27\pi}{2}}$

8) Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y=\sqrt{x}$ and $y=x^2$. Find V both by slicing and by cylindrical shells.

<u>Slicing</u>





Area of washer:

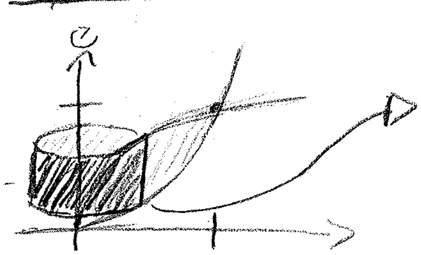
$$\pi(\sqrt{y})^2 - \pi(y^2)^2$$
$$= \pi y - \pi y^4$$

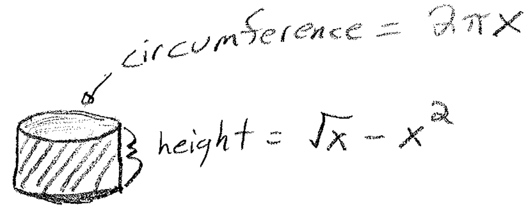
Volume of rotation:

$$\int_0^1 (\pi y - \pi y^4)\,dy$$

$$= \left[\pi \frac{y^2}{2} - \pi \frac{y^5}{5}\right]_0^1 = \left(\frac{\pi}{2} - \frac{\pi}{5}\right) - 0 = \boxed{\frac{3\pi}{10}}$$

<u>Cylindrical Shells</u>





Area of cylindrical shell:

$$2\pi x(\sqrt{x} - x^2)$$

Volume of rotation:

$$\int_0^1 2\pi x(\sqrt{x} - x^2)\,dx$$

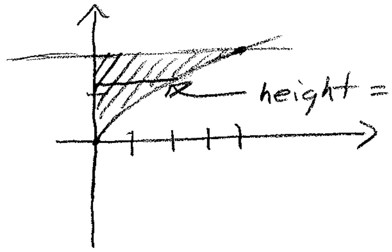
$$= 2\pi \int_0^1 x(x^{\frac{1}{2}} - x^2)\,dx = 2\pi \int_0^1 (x^{\frac{3}{2}} - x^3)\,dx$$

$$= 2\pi \left[\frac{2}{5} x^{\frac{5}{2}} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi \left[\left(\frac{2}{5} - \frac{1}{4}\right) - 0\right] = 2\pi \left[\frac{3}{20}\right] = \boxed{\frac{3\pi}{10}}$$

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

10) $y = \sqrt{x}$, $x = 0$, $y = 2$



Area of a cylindrical shell:

$$2\pi y\,(y^2)$$
$$= 2\pi y^3$$

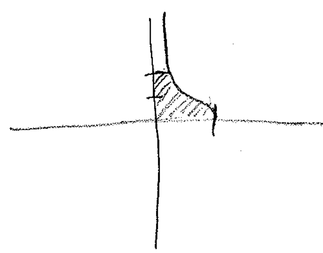
Volume of the rotation:

$$\int_0^2 2\pi y^3\,dy = \left[2\pi \frac{y^4}{4}\right]_0^2$$
$$= \left[\pi \frac{y^4}{2}\right]_0^2 = \left(\frac{2^4\pi}{2} - 0\right) = \boxed{8\pi}$$

Each integral represents the volume of a solid. Describe the solid.

30) $2\pi \int_0^2 \frac{y}{1+y^2}\,dy$

$$= \int_0^2 (2\pi y)\left(\frac{1}{1+y^2}\right)dy$$



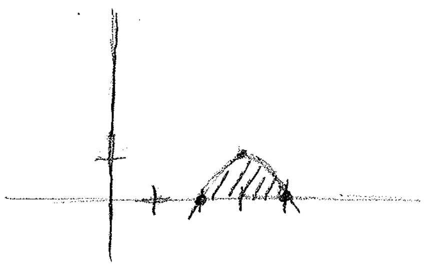
This is the volume of the region $0 \le x \le \frac{1}{1+y^2}$, $0 \le y \le 2$ rotated around the x-axis.

5.3 # 38, 5.4 # 3, 5, 14, 21

The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

38) $y = -x^2 + 6x - 8$, $y = 0$; about the x-axis.

Slicing



Area of a disc:

$$\pi(-x^2+6x-8)^2$$
$$= \pi(x^4 - 12x^3 + 52x^2 - 96x + 64)$$

Volume: $\displaystyle\int_2^4 \pi(x^4 - 12x^3 + 52x^2 - 96x + 64)\,dx$

$$= \pi\left[\frac{x^5}{5} - 3x^4 + \frac{52}{3}x^3 - 48x^2 + 64x\right]_2^4$$

$$= \pi\left[\left(\frac{4^5}{5} - 3(4)^4 + \frac{52}{3}(4)^3 - 48(4)^2 + 64(4)\right) - \left(\frac{2^5}{5} - 3(2)^4 + \frac{52}{3}(2)^3 - 48(2)^2 + 64(2)\right)\right]$$

$$= \boxed{\frac{16\pi}{15}}$$

Cylindrical Shells

$-y = x^2 - 6x + 8$

$-y = x^2 - 6x + 9 - 1$

$-y = (x-3)^2 - 1$

$1 - y = (x-3)^2$

$\pm\sqrt{1-y} = x - 3$

$x = 3 \pm \sqrt{1-y}$

Area of cylindrical shell:

$$2\pi y\left((3+\sqrt{1-y}) - (3-\sqrt{1-y})\right)$$
$$= 2\pi y(2\sqrt{1-y})$$
$$= 4\pi y\sqrt{1-y}$$

Volume:

$\displaystyle\int_0^1 4\pi y\sqrt{1-y}\,dy$ $\quad u = 1-y$, $du = -dy$ | $y = 1-u$, $dy = -du$

$$= -\int_1^0 4\pi(1-u)\sqrt{u}\,du$$

$$= -4\pi\int_1^0 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) = -4\pi\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^0$$

$$= -4\pi\left[0 - \left(\frac{2}{3} - \frac{2}{5}\right)\right] = \boxed{\frac{16}{15}\pi}$$

3) A variable force of $5x^{-2}$ lbs moves an object along a straight line when it is x feet from the origin. Calculate the work done in moving the object from x=1 ft to x=10 ft.

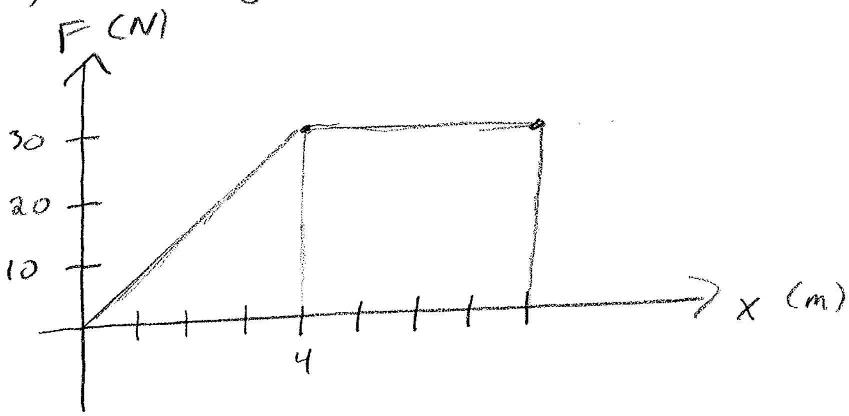
$$W = \int_1^{10} 5x^{-2}\,dx$$

$$= -5x^{-1}\Big|_1^{10}$$

$$= -5\left(\tfrac{1}{10}\right) + 5(1)$$

$$= 5 - \tfrac{1}{2} = \boxed{\tfrac{9}{2} \text{ ft-lb}}$$

5) Shown is a graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done by the force in moving an object a distance of 8 m?



Area of triangle $= \frac{1}{2}(30\,N)(4\,m)$
$= 60\,J$

Area of rectangle $= (30\,N)(4\,m)$
$= 120\,J$

Total Work $= 120 + 60 = \boxed{180\,J}$

14) A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6m?

The Force is weight.

Weight = m · gravity

∴ "density" is $8 \frac{kg}{m}$

$W = 8 \frac{kg}{m} \cdot (\text{length off ground}) \cdot 9.8\ m/s^2$

$= 78.4\ N/m \cdot (\text{length off ground})$

$W = \int_0^6 78.4(6-x)\,dx$

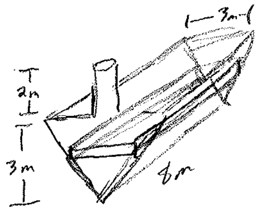
$= 78.4 \int_0^6 (6-x)\,dx$

$= 78.4\left[6x - \frac{1}{2}x^2\right]_0^6$

$= 78.4(18)$

$= 1411.2\ J$

21) A tank is full of water. Find the work required to pump the water out of the spout.

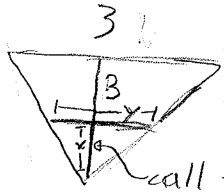


The Force here is weight.

$$W = F \cdot d$$
$$= m \cdot a \cdot d$$
$$= \rho \cdot V \cdot a \cdot d$$

ρ = density = $1000 \ kg/m^3$ (This is a constant for water).

a = gravity = $9.8 \ m/s^2$



here we have similar triangles.

We want y, so we can use ratios: $\frac{3}{x} = \frac{3}{y}$

Hence, $x = y$.

So, the Volume $= 8x\Delta x$

Distance is the height lifted, or $5-x$.

So $W = \int_0^3 1000(8x\,dx)(9.8)(5-x)$

$$= 9800\int_0^3 (40x - 8x^2)\,dx$$

$$= 9800\left[20x^2 - \frac{8}{3}x^3\right]_0^3$$

$$= 9800(180-72) = 9800(108)$$

$$= \boxed{1058400 \ J \approx 1.06 \times 10^6 \ J}$$

5.5 #4, 6, 10, 14 ; Rev Ch5 #4, 7, 12, 31

1) Find the average value of the function on the given interval.

4) $g(t) = \frac{t}{\sqrt{3+t^2}}$, $[1,3]$

$$g_{ave} = \frac{1}{3-1}\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt$$

$u = 3+t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$3+(3)^2 = 12$

$3+1^2 = 4$

$$= \frac{1}{2}\int_4^{12} \frac{1}{\sqrt{u}}\frac{1}{2}du$$

$$= \frac{1}{4}\int_4^{12} u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\,2u^{\frac{1}{2}}\Big|_4^{12}$$

$$= \frac{1}{2}\left(\sqrt{12}+\sqrt{4}\right) = \frac{1}{2}\left(2\sqrt{3}+2\right) = \boxed{\sqrt{3}+1}$$

6) $f(\theta) = \sec^2\left(\frac{\theta}{2}\right)$, $[0, \frac{\pi}{2}]$

$$f_{ave} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{\theta}{2}\right)d\theta$$

$u = \frac{\theta}{2}$

$du = \frac{1}{2}d\theta$

$2du = d\theta$

$\frac{0}{2} = 0$

$\frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$

$$= \frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} 2\sec^2(u)\,du$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \sec^2(u)\,du$$

$$= \frac{4}{\pi}\tan u\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi}\left(\tan\left(\frac{\pi}{4}\right) - \tan(0)\right) = \frac{4}{\pi}(1-0) = \boxed{\frac{4}{\pi}}$$

10) $f(x) = \sqrt{x}$, $[0,4]$

a) Find the average value of f on the given interval.

$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx$$

$$= \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}\,x^{\frac{3}{2}}\Big|_0^4$$

$$= \frac{1}{6}\left(4^{\frac{3}{2}}\right)$$

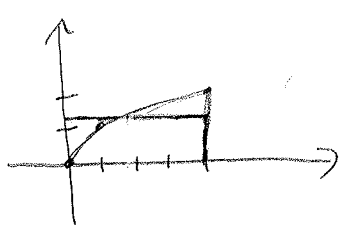
$$= \frac{1}{6}(8) = \boxed{\frac{4}{3}}$$

b) Find c such that $f_{ave} = f(c)$

$$\frac{4}{3} = \sqrt{x}$$

$$\boxed{\frac{16}{9} = x}$$

c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.



14) Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

$$f_{ave} = \frac{1}{b-0}\int_0^b (2+6x-3x^2)\,dx$$

$$= \frac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b dx$$

$$= \frac{1}{b}\left[2b + 3b^2 - b^3\right]$$

$$= 2 + 3b - b^2$$

$\therefore f_{ave} = 3$

$$2 + 3b - b^2 = 3$$

$$0 = b^2 - 3b + 1$$

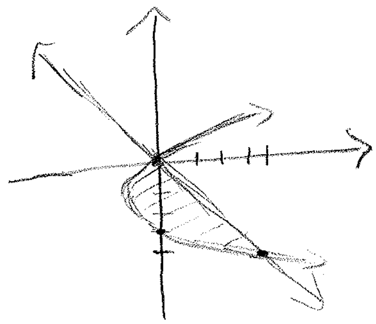
$$b = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$

Both roots are positive and real, so both are valid.

$$\boxed{b = \frac{3 \pm \sqrt{5}}{2}}$$

4) Find the area of the region bounded by the given curves.

$x + y = 0 \quad , \quad x = y^2 + 3y$



$x = -y$

$x = y^2 + 3y$

$-y = y^2 + 3y$

$0 = y^2 + 4y$

$y = 0, -4$

$$\int_{-4}^{0} (-y - y^2 - 3y)\, dy$$

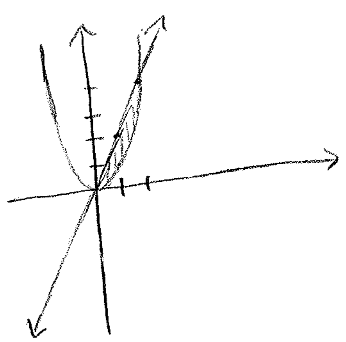
$$= \int_{-4}^{0} (-y^2 - 4y)\, dy = -\frac{1}{3}y^3 - 2y^2 \Big|_{-4}^{0} = -\left[-\frac{1}{3}(-4)^3 - 2(-4)^2\right]$$

$$= -\left[\frac{64}{3} - 32\right]$$

$$= \boxed{\frac{32}{3}}$$

7) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

$y = 2x \; , \; y = x^2$ about the x-axis



$$\int_{0}^{2} \left[\pi (2x)^2 - \pi (x^2)^2\right] dx$$

$$= \pi \int_{0}^{2} \left[4x^2 - x^4\right] dx$$

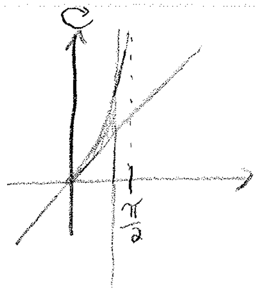
$$= \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)_0^2$$

$$= \pi \left(\frac{4}{3}(8) - \frac{1}{5}(32)\right)$$

$$= \boxed{\frac{64}{15}\pi}$$

12) Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

$y = \tan x$, $y = x$, $x = \frac{\pi}{3}$; about the y-axis



Cylindrical Shells are easier here.

height: $\tan x - x$

$$\boxed{V = \int_0^{\frac{\pi}{3}} 2\pi x (\tan x - x)\, dx}$$

31) If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x+h]$?

Let $F(t) = \int f(t)\, dt$. (Hence $F'(t) = f(t)$)

$$\lim_{h \to 0} \frac{1}{x+h-x} \int_x^{x+h} f(t)\, dt$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$

$$= \lim_{h \to 0} \frac{1}{h} \left[F(x+h) - F(x)\right]$$

$$= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= F'(x)$$

$$= f(x)$$

6.1 # 14, 18, 19, 20, 22, 26, 37, 45, 50

14) Determine whether $h(x) = 1 + \cos x$, $0 \le x \le \pi$ is one-to-one.

$h'(x) = -\sin x$

On $0 < x < \pi$, $-\sin x < 0$, hence $h(x)$ is strictly decreasing on $0 \le x \le \pi$.

Thus, $h(x)$ is one-to-one.

18) If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$

$f'(x) = 5x^4 + 3x^2 + 1 > 0$,

so f is one-to-one, and it has an inverse.

$f^{-1}(3)$ If $x = 1$,

$(1)^5 + (1)^3 + (1) = 3$

So, $\boxed{f^{-1}(3) = 1}$

Since f is one-to-one, $\boxed{f(f^{-1}(2)) = 2}$

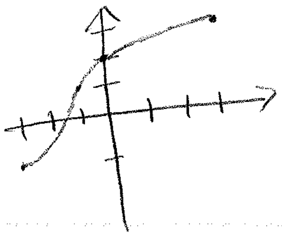
19) If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

$h'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$, so $h(x)$ is one-to-one and it has an inverse.

If $x = 4$, $4 + \sqrt{4} = 6$

So $\boxed{h^{-1}(6) = 4}$

20) The graph of f is given



a) Why is f one-to-one?

f passes the Horizontal Line Test

b) What are the domain and range of f^{-1}?

Domain: $[-1, 3]$

Range: $[-3, 3]$

c) What is the value of $f^{-1}(2)$?

$f^{-1}(2) = 0$

d) Estimate the value of $f^{-1}(0)$.

$f^{-1}(0) \approx -1.7$

22) In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

$$m = \frac{m_0}{\sqrt{1+\frac{v^2}{c^2}}}$$

$$\sqrt{1+\frac{v^2}{c^2}} = \frac{m_0}{m}$$

$$1+\frac{v^2}{c^2} = \frac{m_0^2}{m^2}$$

$$\frac{v^2}{c^2} = \frac{m_0^2}{m^2} - 1$$

$$v^2 = c^2\left(\frac{m_0^2}{m^2} - 1\right)$$

$$v = c\sqrt{\frac{m_0^2}{m^2} - 1}$$

This formula gives the speed v of the particle in terms of its mass, m.

26) Find a formula for the inverse of the function.

$$y = x^2 - x \,,\ x \geq \frac{1}{2}$$

$$y + \frac{1}{4} = x^2 - x + \frac{1}{4}$$

$$y + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$$

$$\pm\sqrt{y + \frac{1}{4}} = x - \frac{1}{2}$$

$\sqrt{y + \frac{1}{4}} = x - \frac{1}{2}$ because $x \geq \frac{1}{2}$ in the original function.

$$\sqrt{y + \frac{1}{4}} + \frac{1}{2} = x$$

$$\boxed{f^{-1}(x) = \sqrt{y + \frac{1}{4}} + \frac{1}{2}}$$

37) $f(x) = 9 - x^2$, $0 \leq x \leq 3$, $a = 8$

a) Show f is one-to-one.

Option 1) $f'(x) = -2x$

On $0 < x \leq 3$, $f'(x) < 0$,

So f is one-to-one on $0 \leq x \leq 3$

Option 2) $x_1 \neq x_2$

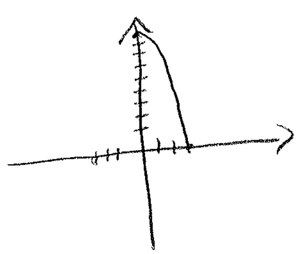
$\Rightarrow x_1^2 \neq x_2^2$ (since $x \geq 0$)

$\Rightarrow 9 - x_1^2 \neq 9 - x_2^2$

$\Rightarrow f(x_1) \neq f(x_2)$

So f is one-to-one for $x \geq 0$, in particular, for $0 \leq x \leq 3$.

Option 3)



By the Horizontal Line Test, f is one-to-one.

b) Use Theorem 7 to find $(f^{-1})'(a)$.

Thm 7: $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

① $f'(x) = -2x$

② $f^{-1}(8) = 1$

since $9-(1)^2 = 8$

③ $(f^{-1})'(8) = \dfrac{1}{-2(1)} = \boxed{-\frac{1}{2}}$

c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

$y = 9 - x^2$

$x^2 = 9 - y$

$x = \pm\sqrt{9-y}$

$x = \sqrt{9-y}$ + since $0 \le x \le 3$

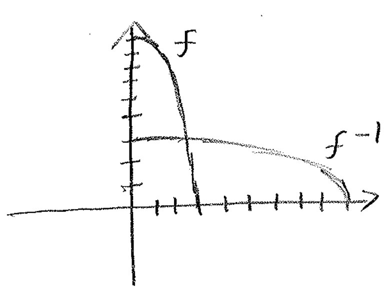
$$\boxed{\begin{array}{l} f^{-1}(x) = \sqrt{9-x} \\ \text{Domain: } [0,9] \\ \text{Range: } [0,3] \end{array}}$$

d) Calculate $(f^{-1})'(a)$ using $f^{-1}(x)$.

$(f^{-1})'(x) = -\frac{1}{2}(9-x)^{-\frac{1}{2}}$

$(f^{-1})'(8) = -\frac{1}{2}(9-8)^{-\frac{1}{2}} = \boxed{-\frac{1}{2}}$

e) Sketch f and f^{-1}



45) If $f(x) = \int_3^x \sqrt{1+t^3}\,dt$, find $(f^{-1})'(0)$

① $f'(x) = \sqrt{1+x^3}$

② $f^{-1}(0) = 3$ (since $\int_3^3 \sqrt{1+t^3}\,dt = 0$)

③ $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(3)} = \dfrac{1}{\sqrt{1+3^3}} = \boxed{\dfrac{1}{\sqrt{28}}}$

50) a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

Proof

$g(x) = f^{-1}(x)$

So, $g'(x) = \dfrac{1}{f'(g(x))}$

By the quotient rule, $g''(x) = \dfrac{-f''(g(x))\,g'(x)}{[f'(g(x))]^2}$

$$= -\frac{f''(g(x))\left[\frac{1}{f'(g(x))}\right]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

▨

b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

f is increasing, so $f' > 0$

f is concave upward, so $f'' > 0$

Hence $$-\frac{f''(g(x))}{[f'(g(x))]^3} < 0$$

or $g'' < 0$

Thus $f^{-1} = g$ is concave downward.