Irreducible Representations of $Sp(\mathbb{C}^{2\ell}, \Omega)$ on $\bigwedge \mathbb{C}^{2\ell}$

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March, 2016

We fix $G \subset GL_{\mathbb{C}}(N)$ to be a reductive linear algebraic group.

Definition

- By $[G]$ we denote the *set of equivalence classes of irreducible representations of* G.
- On the other hand, $\widehat{G}$ will denote the *subset of* $[G]$ *of equivalence classes of finite-dimensional irreducible representations of* G.
- The corresponding sets of equivalence classes of representations of an associative algebra $\mathcal{A}$ will be denoted by $[\mathcal{A}]$ or $\widehat{\mathcal{A}}$.

Remark

We write $\rho^\lambda : G \to \mathrm{End}(F^\lambda)$
for a representative of the class λ, for each $\lambda \in [G]$ and denote this representative by $(\rho^\lambda, F^\lambda)$.

Definition

By $\mathcal{A}(G)$ (or, by $\mathbb{C}[G]$) we denote the group algebra associated with the group G.

Remark

Every G-module is considered as an $\mathcal{A}(G)$-module and vice-versa.

Example

Fix $\{e_i\}$ to be a basis of $V = \mathbb{C}^{2\ell}$.
Then define $\{\varphi^i\}$ to be a basis of V^* such that $\Omega(e_i, \varphi^j) = \delta_{ij}$, where Ω is a non-degenerate skew-symmetric bilinear form.

Definition

On $\bigwedge V$, define the *exterior product* $\varepsilon : \wedge^k \mathbb{C}^{2\ell} \to \wedge^{k+1} \mathbb{C}^{2\ell}$ and the *interior product* $\iota : \wedge^k \mathbb{C}^{2\ell} \to \wedge^{k-1} \mathbb{C}^{2\ell}$.

Remark

Then we have the following relations:

$$\{\varepsilon(x), \varepsilon(y)\} = 0,$$
$$\{\iota(x^*), \iota(y^*)\} = 0,$$
$$\{\varepsilon(x), \iota(x^*)\} = \Omega(x^*, x) Id_{\wedge^k \mathbb{C}^{2\ell}}.$$

Example Continued

Definition

Let $E = \sum_{i=1}^{2\ell} \varepsilon(e_i)\iota(\varphi^i)$ denote the skew-symmetric Euler operator on $\bigwedge V$.

Remark

For $u \in \wedge^k V$, $Eu = ku$.

Definition

Let $Y = \varepsilon(\frac{1}{2} Id)$, $X = -Y^*$, and $H = \ell Id - E$.

Remark

$$[E, X] = -2X, \ [E, Y] = 2Y, \ [Y, X] = E - \ell Id$$

Recall that for any vector space V,
$\mathrm{End}(V)$ is an associative algebra with unity I_V,
the identity map on V.

Definition

For any subset $U \subset \mathrm{End}(V)$,
let $\mathrm{Comm}(U) := \{T \in \mathrm{End}(V) | TS = ST \text{ for any } S \in U\}$
denote the *commutant* of U.

Remark

The set $\mathrm{Comm}(U)$ forms an associative algebra with unity I_V.

Example Continued

Theorem

On $\wedge\mathbb{C}^{2\ell}$,

$$\mathrm{Comm}(Sp(\mathbb{C}^{2\ell})) = \mathrm{Span}_{\mathbb{C}}\{X, H, Y\} \cong \mathfrak{sl}_{\mathbb{C}}(2).$$

Definition

A k-vector $u \in \wedge^k\mathbb{C}^{2\ell}$ is called Ω-*harmonic* when $Xu = 0$.
The k-homogeneous space of Ω-harmonic elements is denoted by $\mathcal{H}(\wedge^k\mathbb{C}^{2\ell}) = \{u \in \wedge^k\mathbb{C}^{2\ell} | Xu = 0\}$.
The space of Ω-harmonic is denoted by $\mathcal{H}(\bigwedge\mathbb{C}^{2\ell}, \Omega)$.

Definition

- Let $\mathcal{R}$ be a subalgebra of $\mathrm{End}(W)$ such that
  1. $\mathcal{R}$ acts irreducibly on W.
  2. If $g \in G$ and $T \in \mathcal{R}$, then $(g, T) \mapsto \rho(g)T\rho(g^{-1}) \in \mathcal{R}$ defines an action of G on $\mathcal{R}$.
- Then we denote by

$$\mathcal{R}^G = \{T \in \mathcal{R} | \rho(g)T = T\rho(g) \text{ for all } g \in G\}$$

the *commutant of* $\rho(G)$ *in* $\mathcal{R}$.

Remark

Since elements of $\mathcal{R}^G$ commute with elements from $\mathcal{A}(G)$, we may define a $\mathcal{R}^G \otimes \mathcal{A}(G)$-module structure on W. Alternatively, we may consider W as a $(\mathcal{R}^G, \mathcal{A}(G))$-bimodule.

Definition

Let $E^\lambda = \mathrm{Hom}_G(F^\lambda, W)$ for $\lambda \in \widehat{G}$.

Remark

Then E^λ is an $\mathcal{R}^G$-module satisfying

$$Tu(\pi^\lambda(g)v) = T\rho(g)u(v) = \rho(g)(Tu(v)),$$

where $u \in E^\lambda$, $v \in F^\lambda$, $T \in \mathcal{R}^G$, and $g \in G$.

Theorem

As an $\mathcal{R}^G \otimes \mathcal{A}(G)$-bimodule, the space W decomposes as

$$W \cong \bigoplus_{\lambda \in \widehat{G}} E^\lambda \boxtimes F^\lambda. \tag{1}$$

In the above theorem $E \boxtimes F$ stands for the outer (external) tensor product of the $\mathcal{R}^G$-module E and of the $\mathcal{A}(G)$-module F.

Example Continued

Let $F^{(\ell-k)}$ denote the irreducible representation of $\mathfrak{sl}_{\mathbb{C}}(2)$ with dimension $\ell - k + 1$.

Theorem

Then, there exists a canonical decomposition of $\bigwedge \mathbb{C}^{2\ell}$ *as a* $(\mathfrak{sl}_{\mathbb{C}}(2), Sp(\mathbb{C}^{2\ell}))$*-bimodule,*

$$\bigwedge \mathbb{C}^{2\ell} \cong \bigoplus_{k=0}^{\ell} F^{(\ell-k)} \boxtimes \mathcal{H}(\wedge^k \mathbb{C}^{2\ell}, \Omega).$$

Theorem (Duality)

Each multiplicity space E^λ is an irreducible $\mathcal{R}^G$-module. Further, if $\lambda, \mu \in \widehat{G}$ and $E^\lambda \cong E^\mu$ as an $\mathcal{R}^G$-module, then $\lambda = \mu$.

Theorem (Duality Correspondence)

Let σ be the representation of $\mathcal{R}^G$ on W and let $\widehat{G}$ denote the set of equivalence classes of the irreducible representation $\{E^\lambda\}$ of the algebra $\mathcal{R}^G$ occurring in W. Then the following hold:

- *The representation (σ, W) is a dirrect sum of irreducible $\mathcal{R}^G$-modules, and each irreducible submodule E^λ occurs with finite mulitplicity, $\dim(F^\lambda)$.*
- *The map $F^\lambda \to E^\lambda$ is a bijection.*

Example Continued

Corollary (Duality)

- *As a G-module,*

$$\bigwedge V \cong \bigoplus_{\lambda \in \widehat{G}} F^{(\lambda)} \otimes \mathrm{Hom}_G \left(F^{(\lambda)}, \bigwedge V \right).$$

- $F^{(\lambda)} \otimes \mathrm{Hom}_G \left(F^{(\lambda)}, \bigwedge V \right)$ *is an irreducible* $\mathrm{End}_G \left(\bigwedge V \right)$*-module.*
- $\left(\rho, \bigwedge V \right)$ *is the direct sum of irreducible* $\mathrm{End}_G(\bigwedge V)$*-module*

Example Continued

Corollary

- $$\bigwedge^{k} \mathbb{C}^{2\ell} = \bigoplus_{i=0}^{[k/2]} Id^{i} \wedge \mathcal{H}\left(\bigwedge^{k-2i} \mathbb{C}^{2\ell}, \Omega\right)$$
- *The space* $\mathcal{H}(\bigwedge^{j} \mathbb{C}^{2\ell}, \Omega)$ *has dimension* $\binom{2\ell}{j} - \binom{2\ell}{j-2}$, *for* $j = 1, \ldots, \ell$.

Thank you.