

Similarly, the number of hours that machine  $Y$  will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of  $x_1$ ,  $x_2$ , and  $x_3$  so that

$$2x_1 + 3x_2 + 4x_3 = 80$$

$$2x_1 + 2x_2 + 3x_3 = 60.$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$x_1 = \frac{20 - x_3}{2}$$

$$x_2 = 20 - x_3$$

$$x_3 = \text{any real number such that } 0 \leq x_3 \leq 20,$$

since we must have  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ . When  $x_3 = 10$ , we have

$$x_1 = 5, \quad x_2 = 10, \quad x_3 = 10$$

while

$$x_1 = \frac{13}{2}, \quad x_2 = 13, \quad x_3 = 7$$

when  $x_3 = 7$ . The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

As you have probably already observed, the method of elimination has been described, so far, in general terms. Thus we have not indicated any rules for selecting the unknowns to be eliminated. Before providing a very systematic description of the method of elimination, we introduce in the next section the notion of a matrix. This will greatly simplify our notational problems and will enable us to develop tools to solve many important applied problems.

## Key Terms

Linear equation  
Solution of a linear equation  
Linear system  
Unknowns  
Inconsistent system

Consistent system  
Homogeneous system  
Trivial solution  
Nontrivial solution  
Equivalent systems

Unique solution  
No solution  
Infinitely many solutions  
Manipulations on linear systems  
Method of elimination

## 1.1 Exercises

In Exercises 1 through 14, solve each given linear system by the method of elimination.

1.  $x + 2y = 8$   
 $3x - 4y = 4$

2.  $2x - 3y + 4z = -12$   
 $x - 2y + z = -5$   
 $3x + y + 2z = 1$

3.  $3x + 2y + z = 2$   
 $4x + 2y + 2z = 8$   
 $x - y + z = 4$

5.  $2x + 4y + 6z = -12$   
 $2x - 3y - 4z = 15$   
 $3x + 4y + 5z = -8$

4.  $x + y = 5$   
 $3x + 3y = 10$

6.  $x + y - 2z = 5$   
 $2x + 3y + 4z = 2$

$$\begin{array}{ll} 7. & x + 4y - z = 12 \\ & 3x + 8y - 2z = 4 \end{array} \quad \begin{array}{ll} 8. & 3x + 4y - z = 8 \\ & 6x + 8y - 2z = 3 \end{array}$$

$$\begin{array}{ll} 9. & x + y + 3z = 12 \\ & 2x + 2y + 6z = 6 \end{array} \quad \begin{array}{ll} 10. & x + y = 1 \\ & 2x - y = 5 \\ & 3x + 4y = 2 \end{array}$$

$$\begin{array}{ll} 11. & 2x + 3y = 13 \\ & x - 2y = 3 \\ & 5x + 2y = 27 \end{array} \quad \begin{array}{ll} 12. & x - 5y = 6 \\ & 3x + 2y = 1 \\ & 5x + 2y = 1 \end{array}$$

$$\begin{array}{ll} 13. & x + 3y = -4 \\ & 2x + 5y = -8 \\ & x + 3y = -5 \end{array} \quad \begin{array}{ll} 14. & 2x + 3y - z = 6 \\ & 2x - y + 2z = -8 \\ & 3x - y + z = -7 \end{array}$$

15. Given the linear system

$$\begin{array}{l} 2x - y = 5 \\ 4x - 2y = t, \end{array}$$

- Determine a particular value of  $t$  so that the system is consistent.
- Determine a particular value of  $t$  so that the system is inconsistent.
- How many different values of  $t$  can be selected in part (b)?

16. Given the linear system

$$\begin{array}{l} 3x + 4y = s \\ 6x + 8y = t, \end{array}$$

- Determine particular values for  $s$  and  $t$  so that the system is consistent.
- Determine particular values for  $s$  and  $t$  so that the system is inconsistent.
- What relationship between the values of  $s$  and  $t$  will guarantee that the system is consistent?

17. Given the linear system

$$\begin{array}{l} x + 2y = 10 \\ 3x + (6 + t)y = 30, \end{array}$$

- Determine a particular value of  $t$  so that the system has infinitely many solutions.
- Determine a particular value of  $t$  so that the system has a unique solution.
- How many different values of  $t$  can be selected in part (b)?

18. Is every homogeneous linear system always consistent? Explain.

19. Given the linear system

$$\begin{array}{l} 2x + 3y - z = 0 \\ x - 4y + 5z = 0, \end{array}$$

(a) Verify that  $x_1 = 1, y_1 = -1, z_1 = -1$  is a solution.

(b) Verify that  $x_2 = -2, y_2 = 2, z_2 = 2$  is a solution.

(c) Is  $x = x_1 + x_2 = -1, y = y_1 + y_2 = 1$ , and  $z = z_1 + z_2 = 1$  a solution to the linear system?

(d) Is  $3x, 3y, 3z$ , where  $x, y$ , and  $z$  are as in part (c), a solution to the linear system?

20. Without using the method of elimination, solve the linear system

$$\begin{array}{l} 2x + y - 2z = -5 \\ 3y + z = 7 \\ z = 4. \end{array}$$

21. Without using the method of elimination, solve the linear system

$$\begin{array}{l} 4x = 8 \\ -2x + 3y = -1 \\ 3x + 5y - 2z = 11. \end{array}$$

22. Is there a value of  $r$  so that  $x = 1, y = 2, z = r$  is a solution to the following linear system? If there is, find it.

$$\begin{array}{l} 2x + 3y - z = 11 \\ x - y + 2z = -7 \\ 4x + y - 2z = 12. \end{array}$$

23. Is there a value of  $r$  so that  $x = r, y = 2, z = 1$  is a solution to the following linear system? If there is, find it.

$$\begin{array}{l} 3x - 2z = 4 \\ x - 4y + z = -5 \\ -2x + 3y + 2z = 9. \end{array}$$

24. Show that the linear system obtained by interchanging two equations in (2) is equivalent to (2).

25. Show that the linear system obtained by adding a multiple of an equation in (2) to another equation is equivalent to (2).

26. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.2.

27. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.3.

28. Let  $C_1$  and  $C_2$  be circles in the plane. Describe the number of possible points of intersection of  $C_1$  and  $C_2$ . Illustrate each case with a figure.

29. Let  $S_1$  and  $S_2$  be spheres in space. Describe the number of possible points of intersection of  $S_1$  and  $S_2$ . Illustrate each case with a figure.

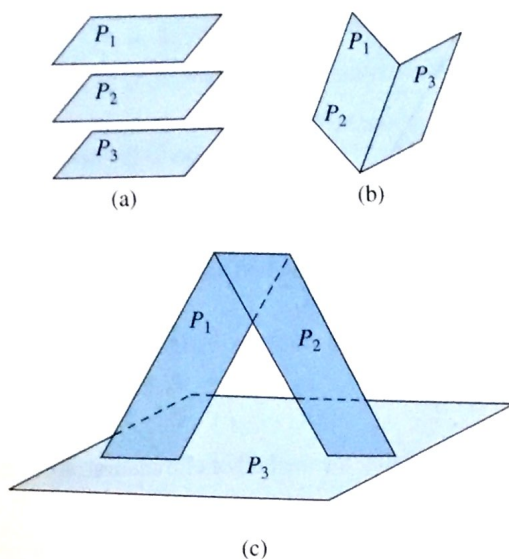


FIGURE 1.3

30. An oil refinery produces low-sulfur and high-sulfur fuel. Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant; each ton of high-sulfur fuel requires 4 minutes in the blending plant and 2 minutes in the refining plant. If the blending plant is available for 3 hours and the refining plant is available for 2 hours, how many tons of each type of fuel should be manufactured so that the plants are fully used?
31. A plastics manufacturer makes two types of plastic: regular and special. Each ton of regular plastic requires 2 hours in plant A and 5 hours in plant B; each ton of special plastic requires 2 hours in plant A and 3 hours in plant B. If plant A is available 8 hours per day and plant B is available 15 hours per day, how many tons of each type of plastic can be made daily so that the plants are fully used?
32. A dietician is preparing a meal consisting of foods A, B, and C. Each ounce of food A contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food B contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food C contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?
33. A manufacturer makes 2-minute, 6-minute, and 9-minute film developers. Each ton of 2-minute developer requires 6 minutes in plant A and 24 minutes in plant B. Each ton of 6-minute developer requires 12 minutes in plant A and 12 minutes in plant B. Each ton of 9-minute developer requires 12 minutes in plant A and 12 minutes in plant B. If plant A is available 10 hours per day and plant B is

available 16 hours per day, how many tons of each type of developer can be produced so that the plants are fully used?

34. Suppose that the three points  $(1, -5)$ ,  $(-1, 1)$ , and  $(2, 7)$  lie on the parabola  $p(x) = ax^2 + bx + c$ .
- Determine a linear system of three equations in three unknowns that must be solved to find  $a$ ,  $b$ , and  $c$ .
  - Solve the linear system obtained in part (a) for  $a$ ,  $b$ , and  $c$ .

35. An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest annually at the rates of 9%, 10%, and 6%, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?

36. For the software you are using, determine the command that “automatically” solves a linear system of equations.

37. Use the command from Exercise 36 to solve Exercises 3 and 4, and compare the output with the results you obtained by the method of elimination.

38. Solve the linear system

$$x + \frac{1}{2}y + \frac{1}{3}z = 1$$

$$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = \frac{11}{18}$$

$$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = \frac{9}{20}$$

by using your software. Compare the computed solution with the exact solution  $x = \frac{1}{2}$ ,  $y = \frac{1}{3}$ ,  $z = 1$ .

39. If your software includes access to a computer algebra system (CAS), use it as follows:

- For the linear system in Exercise 38, replace the fraction  $\frac{1}{2}$  with its decimal equivalent 0.5. Enter this system into your software and use the appropriate CAS commands to solve the system. Compare the solution with that obtained in Exercise 38.
- In some CAS environments you can select the number of digits to be used in the calculations. Perform part (a) with digit choices 2, 4, and 6 to see what influence such selections have on the computed solution.

40. If your software includes access to a CAS and you can select the number of digits used in calculations, do the following: Enter the linear system

$$0.71x + 0.21y = 0.92$$

$$0.23x + 0.58y = 0.81$$

into the program. Have the software solve the system with digit choices 2, 5, 7, and 12. Briefly discuss any variations in the solutions generated.

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \quad \text{and} \quad E^T = [2 \quad -1 \quad 3].$$

## Key Terms

Matrix  
Rows  
Columns  
Size of a matrix  
Square matrix  
Main diagonal  
Element or entry of a matrix

Equal matrices  
 $n$ -vector (or vector)  
 $R^n, C^n$   
 $\mathbf{0}$ , zero vector  
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Matrix addition  
Scalar multiple

Difference of matrices  
Summation notation  
Index of summation  
Linear combination  
Coefficients  
Transpose

## 1.2 Exercises

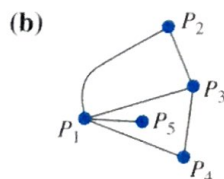
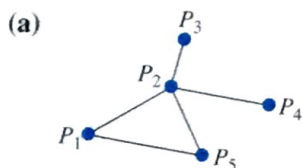
1. Let

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}.$$

- (a) What is  $a_{12}$ ,  $a_{22}$ ,  $a_{23}$ ?  
(b) What is  $b_{11}$ ,  $b_{31}$ ?  
(c) What is  $c_{13}$ ,  $c_{31}$ ,  $c_{33}$ ?
2. Determine the incidence matrix associated with each of the following graphs:



3. For each of the following incidence matrices, construct a graph. Label the vertices  $P_1, P_2, \dots, P_5$ .

(a)  $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

(b)  $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

4. If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find  $a$ ,  $b$ ,  $c$ , and  $d$ .

5. If

$$\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix},$$

find  $a$ ,  $b$ ,  $c$ , and  $d$ .

In Exercises 6 through 9, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix},$$

$$\text{and } O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. If possible, compute the indicated linear combination:

(a)  $C + E$  and  $E + C$       (b)  $A + B$

(c)  $D - F$       (d)  $-3C + 5O$

(e)  $2C - 3E$       (f)  $2B + F$

7. If possible, compute the indicated linear combination:

(a)  $3D + 2F$       (b)  $3(2A)$  and  $6A$

(c)  $3A + 2A$  and  $5A$

(d)  $2(D + F)$  and  $2D + 2F$

(e)  $(2 + 3)D$  and  $2D + 3D$

(f)  $3(B + D)$

8. If possible, compute the following:

(a)  $A^T$  and  $(A^T)^T$

(b)  $(C + E)^T$  and  $C^T + E^T$

(c)  $(2D + 3F)^T$       (d)  $D - D^T$

(e)  $2A^T + B$       (f)  $(3D - 2F)^T$

9. If possible, compute the following:

(a)  $(2A)^T$       (b)  $(A - B)^T$

(c)  $(3B^T - 2A)^T$       (d)  $(3A^T - 5B^T)^T$

(e)  $(-A)^T$  and  $-(A^T)$       (f)  $(C + E + F^T)^T$

10. Is the matrix  $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  a linear combination of the matrices  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ? Justify your answer.

11. Is the matrix  $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$  a linear combination of the matrices  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ? Justify your answer.

12. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -2 & 3 \\ 5 & 2 & 4 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If  $\lambda$  is a real number, compute  $\lambda I_3 - A$ .

13. If  $A$  is an  $n \times n$  matrix, what are the entries on the main diagonal of  $A - A^T$ ? Justify your answer.
14. Explain why every incidence matrix  $A$  associated with a graph is the same as  $A^T$ .
15. Let the  $n \times n$  matrix  $A$  be equal to  $A^T$ . Briefly describe the pattern of the entries in  $A$ .
16. If  $\mathbf{x}$  is an  $n$ -vector, show that  $\mathbf{x} + \mathbf{0} = \mathbf{x}$ .

17. Show that the summation notation satisfies the following properties:

(a)  $\sum_{i=1}^n (r_i + s_i) a_i = \sum_{i=1}^n r_i a_i + \sum_{i=1}^n s_i a_i$

(b)  $\sum_{i=1}^n c(r_i a_i) = c \left( \sum_{i=1}^n r_i a_i \right)$

18. Show that  $\sum_{i=1}^n \left( \sum_{j=1}^m a_{ij} \right) = \sum_{j=1}^m \left( \sum_{i=1}^n a_{ij} \right)$ .

19. Identify the following expressions as true or false. If true, prove the result; if false, give a counterexample.

(a)  $\sum_{i=1}^n (a_i + 1) = \left( \sum_{i=1}^n a_i \right) + n$

(b)  $\sum_{i=1}^n \left( \sum_{j=1}^m 1 \right) = mn$

(c)  $\sum_{j=1}^m \left( \sum_{i=1}^n a_i b_j \right) = \left[ \sum_{i=1}^n a_i \right] \left[ \sum_{j=1}^m b_j \right]$

20. A large steel manufacturer, who has 2000 employees, lists each employee's salary as a component of a vector  $\mathbf{u}$  in  $R^{2000}$ . If an 8% across-the-board salary increase has been approved, find an expression involving  $\mathbf{u}$  that gives all the new salaries.

21. A brokerage firm records the high and low values of the price of IBM stock each day. The information for a given week is presented in two vectors,  $\mathbf{t}$  and  $\mathbf{b}$ , in  $R^5$ , showing the high and low values, respectively. What expression gives the average daily values of the price of IBM stock for the entire 5-day week?

22. For the software you are using, determine the commands to enter a matrix, add matrices, multiply a scalar times a matrix, and obtain the transpose of a matrix for matrices with numerical entries. Practice the commands, using the linear combinations in Example 13.

23. Determine whether the software you are using includes a computer algebra system (CAS), and if it does, do the following:

- (a) Find the command for entering a symbolic matrix. (This command may be different than that for entering a numeric matrix.)

- (b) Enter several symbolic matrices like

$$A = \begin{bmatrix} r & s & t \\ u & v & w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Compute expressions like  $A + B$ ,  $2A$ ,  $3A + B$ ,  $A - 2B$ ,  $A^T + B^T$ , etc. (In some systems you must